Problems Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. **See the note at the end of this issue for new details concerning the submission of new proposals and solutions.** Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before
July 15, 2021

- **5637:** *Proposed by Kenneth Korbin, New York, NY*

  In triangle ABC three circles are tangent to the incircle, and all of these circles are tangent to two sides of the triangle.

  The radii of these three circles are $r_a = \dfrac{1}{4}$, $r_b = \dfrac{4}{9}$, and $r_c = \dfrac{16}{49}$. Find the sides of $\triangle ABC$.

- **5638:** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turna-Severin, Romania*

  Let a, b, c be real numbers such that $a, b, c \geq -1$, and $a + b + c = 3$. Then:

  $$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leq \frac{3}{4}.$$

- **5639:** *Proposed by Dorin Mărghidanu, Corabia, Romania*

  If $n \in N$, calculate the limit of:

  $$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{(2^k - 1)(2^{k+1} - 1)}.$$

- **5640:** *Proposed by Titu Zvonaru, Comănesti, Romania*

  Let a, b, c be real numbers such that $ab + bc + ca = 0$. Prove that

  $$\left(a^2 + b^2 + c^2\right)\left(a^2b^2 + b^2c^2 + c^2a^2\right) \geq 54a^2b^2c^2 - 162\max(a^4bc,\ ab^4c,\ abc^4).$$

- **5641:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Find all integer numbers $x_1, x_2, x_3, \ldots, x_n$ such that

  $$x_1^4 + 6x_2^2 \quad < \quad 4x_2^3 + 5x_2 - x_1,$$

$$\begin{array}{rcl} x_2^4+6x_3^2 & < & 4x_3^3+5x_3-x_2, \\ & \cdots & \\ x_{n-1}^4+6x_n^2 & < & 4x_n^3+5x_n-x_{n-1}, \\ x_n^4+6x_1^2 & < & 4x_1^3+5x_1-x_n. \end{array}$$

- **5642:** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

$$\sum_{n=1}^{\infty}(-1)^{n-1}H_n\left(\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}+\cdots\right),$$

where $H_n = 1+\frac{1}{2}+\cdots+\dfrac{1}{n}$ denotes the nth harmonic number.

Solutions

- **5619:** *Proposed by Kenneth Korbin, New York, NY*

If x, y and z are positive integers such that

$$x^2+xy+y^2=z^2$$

then there are two different Pythagorean triangles with area $K = xyz(x+y)$.

Find the sides of the triangles if $z = 61$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We start with the following

Lemma:

Let x, y and z be integers such that

$$x^2+xy+y^2=z^2.$$

Then there are integers a, b and c such that $z = c(a^2+ab+b^2)$ and one of the three options applies:

$$\begin{array}{ll} (i) & (x,y)=c(a^2-b^2, 2ab+b^2) \\ (ii) & (xy)=c(-a^2-2ab, a^2-b^2) \\ (iii) & (x,y)=c(2ab+b^2, -a^2-2ab). \end{array}$$

For all options we have

$$xyz(x+y) \;=\; ab(a-b)(a+b)(b+2a)(a^2+ab+b^2)(a+2b)c^4.$$

Proof:

I provide a proof that is based on properties of the Eisenstein integers $Z[\omega] = \{a+b\omega | a, b \in Z\}$, where $\omega = e^{\frac{2\pi i}{3}}$. Then (see for instance https://en.wikipedia.org/wiki/Eisenstein integer):

(i) $Z[\omega]$ forms a commutative ring of algebraic integers in the algebraic number field $Q(\omega)$.

(ii) $Z[\omega]$ is an Euclidean domain whose norm N is given by $N(a + b\omega) = a^2 - ab + b^2$. As a result of this $Z[\omega]$ is a factorial ring.

(iii) The group of units in $Z[\omega]$ is the cyclic group formed by the sixth roots of unity in the complex plane. Specifically, they are $\{\pm 1, \pm\omega \pm \omega^2\}$ These are just the Eisenstein integers of norm one.

(iv) An ordinary prime number (or rational prime) which is 3 or congruent to 1 (mod 3) is of the form $(x^2 - xy + y^2)$ for some integers x, y and may therefore be factored into $(x+y\omega)(x+y\omega^2)$ and because of that it is not prime in the Eisenstein integers. Ordinary primes congruent to 2 (mod 3) cannot be factored in this way and they are primes in the Eisenstein integers as well.

Let $c = \gcd(x, y)$. Then $\left(\frac{x}{c}\right)^2 + \left(\frac{x}{c}\right)\left(\frac{y}{c}\right) + \left(\frac{y}{c}\right)^2 = \left(\frac{z}{c}\right)$ and $\gcd\left(\frac{x}{c}, \frac{y}{c}\right) = 1$. Setting $x = cx', y = cy'$ and $z = cz'$ we may assume that $\gcd(x, y) = 1$. Assume that

$$(x - \omega y)(x - \omega^2 t) = x^2 + xy + y^2 = z^2. \tag{1}$$

We factor z into Eisenstein primes:

$$z = p_1 p_2 \cdots p_r q_1 \overline{q_1} q_2 \overline{q_2} \cdots q_s \overline{q_s},$$

where $p_1, p_2, .., p_r$ are ordinary primes congruent to 2 (mod 3) and $q_1, q_2, \cdots, q_s$ are Eisenstein primes of the form $a - \omega b$. Here $\overline{q_j}$ is the complex conjugate of q_j. From (1) we conclude that z has no primes factors that are congruent to 2 (mod 3), for if p_j divides $(x-\omega y)(x-\omega^2 y)$ then it either divides $x-\omega y$ or it divides $x-\omega^2 y$. In both cases p_j divides x and y which is not possible, since x and y are assumed to be relatively prime. For the same reason, if $q_j\overline{q}_j \neq 3$ and q_j divides $x-\omega y$ then $\overline{q_j}$ does not divide $x-\omega y$, for otherwise $q_j\overline{q_j}$ is an ordinary prime that divides $x - \omega y$ and thus divides x and y. So if $q_j\overline{q_j} \neq 3$ then $x-\omega y$ is divisible by either $(q_j)^2$ or by $(\overline{q}_j)^2$. If $q_j\overline{q_j} = 3$ then q_j equals $\omega - 1$, and $\overline{q_j}$ equals $\omega^2 - 1$ which is an associate of $\omega - 1$, since $\omega^2 - 1 = (\omega - 1)(\omega + 1) = -(\omega - 1)\omega^2$. But then x and y are divisible by 3 contrary to our assumption.

To sum up: the representation

$$(x - \omega y)(x - \omega^2 y) = z^2$$

implies that there are integers a and b, as well as a unit u such that

$$x - \omega y = u(a - \omega b)^2.$$

Clearly, $\omega = \omega^4$. So the units ω and ω^2 can be incorporated into the square and we conclude that there are numbers a and b such that either $x-\omega y = \pm(a-\omega b)^2$ or $x-\omega y = \pm(a\omega - \omega^2 b)^2 = \pm(b + (a + b)\omega)^2$ or $x - \omega y = \pm(a\omega^2 - b)^2 = \pm(a + b + a\omega)^2$. Thus there are numbers a and b such that

$(x, y) = \pm(a^2 - b^2, 2ab + b^2)$ or
$(x, y) = \pm(-a^2 - 2ab, a^2 + b^2)$ or

$(x, y) = \pm(2ab + b^2, -a^2 - 2ab)$.

The first part of the lemma follows. Its easy to verify that for all of the three options we have

$$xyz(x + y) = ab(a - b)(a + b)(b + 2a)(a^2 + ab + b^2)(a + 2b)c^4.$$

It is well known that if (u, v, w) is a Pythagorean triangle there are integers m and n such that

$$u = m^2 - n^2, \quad v = 2mn, \quad w = m^2 + n^2.$$

The area of the triangle equals $uv/2 = mn(m^2 - n^2)$. Hence we need to find all tuples (a, b, c, m, n) such that

$$xyz(x + y) = mn(m^2 - n^2) = ab(a - b)(a + b)(b + 2a)(a^2 + ab + b^2)(a + 2b)c^4.$$

This Diophantine equation has at least the solutions

$$\begin{array}{lcll}
m &=& c(a^2 + ab + b^2), & n = c(a - b)(a + b), \quad (2)\\
m &=& c(b - a)(a + b), & n = c(a^2 + ab + b^2), \quad (3)\\
m &=& c(a^2 + ab + b^2), & n = bc(2a + b), \quad (4)\\
m &=& -bc(2a + b), & n = c(a^2 + ab + b^2), \quad (5)\\
m &=& c(a^2 + ab + b^2), & n = -ac(a + 2b), \quad (6)\\
m &=& ac(a + 2b), & n = c(a^2 + ab + b^2). \quad (7)
\end{array}$$

From (2) to (7) we derive the following Pythagorean triangles:

$$(c^2(a^2 + ab + b^2)^2 - c^2(a - b)^2(a + b)^2)^2 + (2c^2(a^2 + ab + b^2)(a - b)(a + b))^2$$
$$= (c^2(a^2 + ab + b^2)^2 + c^2(a - b)^2(a + b)^2)^2, \quad (8)$$

$$(c^2(a^2 + ab + b^2)^2 - b^2c^2(2a + b)^2 + (2c^2(a^2 + ab + b^2)b(2a + b))^2$$
$$= (c^2(a^2 + ab + b^2)^2 + b^2c^2(2a + b)^2)^2, \quad (9)$$

$$(c^2(a^2 + ab + b^2)^2 - a^2c^2(a + 2b)^2)^2 + (2c^2(a^2 + ab + b^2)a(a + 2b))^2$$
$$= (c^2(a^2 + ab + b^2)^2 + a^2c^2(a + 2b)^2)^2. \quad (10)$$

If $z = 61$ then $(x, y) \in \{(9, 56), (56, 9\}$, and $xyz(x + y) = 1998360$. The equation $mn(m^2 - n^2) = mn(m + n)(m - n) = 1998360$ has the solutions $(m, n) \in \{\pm(61, 9), \pm(-9, 61), (61, 56), (-56, 61), \pm(65, 61), \pm(-61, 65)$ with the associated Pythagorean triangles $(1098, 3640, 3802), (3640, 1098, 3802), (585, 6832, 6857), (6832, 585, 6857),$ $(504, 7930, 7946), (7930, 504, 7946)$. They are found by an exhaustive search taking into account that $1998360 = 2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 61$ has only finitely many divisors.

This result can be obtained as well through the lemma and formulas (8), (9), (10). Indeed, $z = 61 = c(a^2 + ab + b^2)$ has the solutions $(a, b, c) \in \{(-9, 4, 1), (-9, 5, 1, (-5, -4, 1), (-5, 9, 1), (-4, -5, 1), (-4, 9, 1), (-1, 0, 61), (-1, 1, 61),$ $(0, -1, 61), (0, 1, 61), (1, -1, 61), (1, 0, 61), (4, -9, 1), (4, 5, 1), (5, -9, 1), (5, 4, 1), (9, -5, 1), (9, -4, 1)\}$. Inserting these values into (8), (9), (10) gives exactly the Pythagorean triangles identified above.

Remark The lemma is linked to integer triangles with an angle of $2\pi/3$. A reference is the Wikipedia article (https://en.wikipedia.org/wiki/Integer triangle#Integer triangles with120.C2.B0 angle) which cites the three papers:

- Burn, Bob, "Triangles with a 60° angle and sides of integer length", Mathematical Gazette 87, March 2003, 148-153.
- Read, Emrys, "On integer-sided triangles containing angles of 120° or 60°", Mathematical Gazette 90, July 2006, 299-305.
- Selkirk, K., "Integer-sided triangles with an angle of 120°", Mathematical Gazette 67, December 1983, 251-255.

Solution 2 by Michel Bataille, Rouen, France

Since $x(x+y) = z^2 - y^2$ and $y(x+y) = z^2 - x^2$, we have $K = yz(z^2 - y^2) = xz(z^2 - x^2)$. Therefore K is the area of the Pythagorean triangles with sides

$$2yz, z^2 - y^2, z^2 + y^2, \qquad \text{and} \qquad 2xz, z^2 - x^2, z^2 + x^2. \tag{1}$$

These triangles are distinct since their hypotenuses $z^2 + y^2, z^2 + x^2$ are. Indeed, we must have $x \neq y$ because of the impossible equality $3x^2 = z^2$ in the standard decomposition of z^2 the exponent of the prime 3 is even while it is odd in the decomposition of $3x^2$.

Now, suppose that $z = 61$. We show that for positive integers x, y, the equality $x^2 + xy + y^2 = 61^2$ holds if and only if $\{x, y\} = \{9, 56\}$. A short calculation shows that $x^2 + xy + y^2 = 61^2$ if $\{x, y\} = \{9, 56\}$. Conversely, let x, y be positive integers such that $x^2 + xy + y^2 = 61^2$

First, x, y are coprime: if p were a prime dividing x and y, say $x = px_1, y = py_1$, we would have $p^2(x_1^2 + x_1y_1 + y_1^2) = 61^2$, hence $p = 61$ and $x_1^2 + x_1y_1 + y_1^2 = 1$, a contradiction since x_1, y_1 are positive integers.

Second, we have $\frac{x}{61-y} = \frac{61+y}{x+y} = \frac{m}{n}$ where m, n are coprime positive integers such that $m > n$ (since $x < 61$). We readily deduce that

$$(m^2 - mn + n^2)x = 61m(2n - m) \quad \text{and} \quad (m^2 - mn + n^2)y = 61(m^2 - n^2). \tag{2}$$

Note that $m < 2n$ (since $m^2 - mn + n^2 > 0$) so that $n < m < 2n$.

Now, we prove that either $m^2 - mn + n^2 = 61$ or $m^2 - mn + n^2 = 3 \cdot 61$, and that the latter can occur only if $2n - m$ and $2m - n$ are multiple of 3.

If $m(2n - m)$ and $m^2 - n^2$ are coprime, then from (2)

$$m^2 - mn + n^2 = \gcd((m^2 - mn + n^2)x, (m^2 - mn + n^2)y) = \gcd(61m(2n - m), 61(m^2 - n^2)) = 61.$$

Otherwise, let p be a prime divisor of both $m(2n - m)$ and $m^2 - n^2$. Then, p cannot divide m (p must divide $m - n$ or $m + n$ and so it would also divide n), hence p divides $2n - m$ (and $m - n$ or $m + n$).

But p cannot divide $m-n$ since otherwise p would divide $(2n-m)+(m-n)=n$ and also $m=(m-n)+n$. Thus, p divides $m+n$ and also $(2n-m)+(m+n)=3n$, $2(m+n)-(2n-m)=3m$ and $2m-n=3m-(m+n)$. It follows that $p=3$, with $2n-m$ and $2m-n$ multiple of 3. A consequence is that $\gcd(m(2n-m),(m^2-n^2))=3^r$ for some positive integer r and so $m^2-mn+n^2=3^r\cdot 61$. But 3^2 divides $(2n-m)(2m-n)=3mn-2(m^2-mn+n^2)=3(mn-2\cdot 3^{r-1}\cdot 61)$ and so 3 divides $mn-2\cdot 3^{r-1}\cdot 61$. Since 3 does not divide mn (otherwise it would divide $m^2+n^2=(m^2-mn+n^2)+mn$, which is impossible since m,n are not both multiple of 3), we must have $r=1$ and so $m^2-mn+n^2=3\cdot 61$.

Let us examine the two possibilities.

• If $m^2-mn+n^2=61$, then $(2m-n)^2+3n^2=244$, hence $n^2\le 81$ and $(2m-n)^2=244-3n^2$. Successively trying the squares $1,4,9,16,25,36,49,64,81$ for n^2 and recalling that $n<m<2n$, we easily obtain that the only possibility is $m=9$ and $n=5$, which leads to $x=9$ and $y=56$.
• If $m^2-mn+n^2=3\cdot 61$, then $(2m-n)^2+3n^2=3\cdot 4\cdot 61$ and we know that $2m-n=3w$ for some positive integer w such that $\frac{n}{3}<w<n$. We deduce that $n^2+3w^2=244$. Proceding as in the previous case, we see that the only possibility is $w=5$, $n=13$ leading first to $m=14, n=13$ and then to $x=56, y=9$.
To conclude, we return to (1) and calculate the sides of the Pythagorean triangle when, say, $x=9, y=56, z=61$. We obtain the triples of sides

$$(6832,585,6857),\qquad (1098,3640,3802).$$

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

Solution. We show that there are three different Pythagorean triangles, given by:

$$a_1=xy,\quad b_1=2z(x+y),\quad c_1=2x^2+3xy+2y^2$$
$$a_2=x(x+y),\quad b_2=2yz,\quad c_2=x^2+xy+2y^2$$
$$a_3=y(x+y),\quad b_3=2xz,\quad c_3=2x^2+xy+y^2$$

It is straightforward to verify that $a_1b_1=a_2b_2=a_3b_3=2K$. Also, using $z^2=x^2+xy+y^2$, we check that $a_i^2+b_i^2=c_i^2$ for $i\in\{1,2,3\}$. When $z=61$, we may take $x=9$ and $y=56$, so that the sides of the three triangles are $(504,7930,7946)$, $(585,6832,6857)$, and $(3640,1098,3802)$.

In order to show in general that these are three different triangles, we first note that $a_i\neq a_j$ for $i\neq j$: We have $a_1<a_2$ and $a_1<a_3$, and if $a_2=a_3$, then $x=y$, which would imply $3x^2=z^2$ for positive integers x and z. Next, we note that $a_1<b_2$ and $a_1<b_3$, since $x<z$ and $y<z$. Finally, if $a_2=b_3$, then $x+y=2z$, so $x^2+2xy+y^2=4z^2$, which would contradict $x^2+xy+y^2=z^2$.

Solution 4 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

We first find the solutions of $x^2+xy+y^2=61^2$. Multiply the equation by 4 and complete the square to find $(2x+y)^2+3y^2=4\cdot 61^2$. We must have $1\le y\le 2\cdot 61/\sqrt{3}\approx 70.4$. Now a quick search shows the only positive integer solutions (x,y) are precisely $(56,9)$ and $(9,56)$. Thus

$$K=xyz(x+y)=1998360=2^3\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 61.$$

Now suppose there exists a Pythagorean triangle with sides a, b and hypotenuse c. Then the area of this triangle is $\frac{1}{2}ab$. Thus $ab=2K$. So we need $ab=2K$ with a^2+b^2 a square. Now $2K$ has only 240 divisors so a quick search yields (wlog $a<b$) only three possible triangles with sides (a,b,c) given by

$$(504,7930,7946),\quad (585,6832,6857),\quad (1098,3640,3802).$$

Solution 5 by The Eagle Problem Solvers, Georgia Southern University, Satesboro, GA and Savannah, GA

We show that there are actually three different Pythagorean triangles with area $K=xyz(x+y)$ for each triple (x,y,z) of positive integers satisfying $x^2+xy+y^2=z^2$.

Pythagorean triples have the well-known parametrization $(a,b,c)=(v^2-u^2,2uv,u^2+v^2)$, where u and v are positive integers with $u<v$. A Pythagorean triangle with lengths (a,b,c) has area $ab/2=uv(v^2-u^2)$. For each triple (x,y,z) of positive integers satisfying $x^2+xy+y^2=z^2$, we show there are three different Pythagorean triangles with area $K=xyz(x+y)$ by choosing (u,v) to be one of (x,z), (y,z), or $(z,x+y)$. Notice that $(x+y)^2=z^2+xy>z^2$, so that $x+y>z>\max\{x,y\}$.

If $(u,v)=(x,z)$, then $z^2-x^2=y(x+y)$ and $(a,b,c)=(y(x+y),2xz,x^2+z^2)$ is a Pythagorean triple. If $(u,v)=(y,z)$, then $z^2-y^2=x(x+y)$ and $(a,b,c)=(x(x+y),2yz,y^2+z^2)$ is a Pythagorean triple. If $(u,v)=(z,x+y)$, then $(x+y)^2-z^2=xy$ and $(a,b,c)=(xy,2z(x+y),z^2+(x+y)^2)$ is a Pythagorean triple. In all three cases, the area of the Pythagorean triangle is $ab/2=xyz(x+y)=K$.

The solutions to $x^2+xy+y^2=z^2$ may be viewed as points on a curve in the projective plane. Lines through the point $(-1:0:1)$ on the curve are given by $ty=s(x+z)$, where s and t are not both zero. This line intersects the curve in a second point $(x:y:z)=(t^2-s^2:s^2+2st:s^2+st+t^2)$, which gives a parametrization of positive integer solutions to $x^2+xy+y^2=z^2$ for positive integers s and t with $s<t$:

$$(x,y,z)=(t^2-s^2,s^2+2st,s^2+st+t^2).$$

If $z=61$, then the only pair of positive integers satisfying $s^2+st+t^2=61$ with $s<t$ is $(s,t)=(4,5)$, so $x=t^2-s^2=9$ and $y=s^2+2st=56$. Thus, the Pythagorean triangles with area $K=xyz(x+y)=1,998,360$ are

$$(y(x+y),2xz,x^2+z^2)=(3640,1098,3802),$$

$$(x(x+y),2yz,y^2+z^2)=(585,6832,6857),$$

and

$$(xy,2z(x+y),z^2+(x+y)^2)=(504,7930,7946).$$

Editor′s note: After this problem was posted its author Kenneth Korbin found a third solution to it. This third solution is also displayed in the solutions featured above.

Also solved by Dionne Bailey, Elsie Campbell,and Charles Diminnie, Angelo State University, San Angelo, TX; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Trey Smith, Angelo State University, San Angelo, TX; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Hyun Bin Yoo, South Korea and the proposer.

- **5620:** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania*

Prove: If $a, b, \in [0,1];\ a \leq b$, then

$$4\sqrt{ab} \leq a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \leq 2(a+b).$$

Solution 1 by Moti Levy, Rehovot, Israel

Let

$$\alpha := \sqrt{ab} \leq 1, \quad \beta := \frac{a+b}{2} \leq 1, \quad r := \frac{b}{a},$$

Then the original inequality can be reformulated as

$$\sqrt{r} \leq \frac{r^{\alpha} + r^{\beta} + r^{1-\alpha} + r^{1-\beta}}{4} \leq \frac{1}{2} + r.$$

Since $f(x) := r^x$ is convex function, then

$$\frac{r^{\alpha} + r^{\beta} + r^{1-\alpha} + r^{1-\beta}}{4} \geq r^{\frac{\alpha+\beta+(1-\alpha)+(1-\beta)}{4}} = \sqrt{r}.$$

The Bernoulli's inequality is

$$(1+x)^{\alpha} \leq 1 + \alpha x, \quad 0 \leq \alpha \leq 1, \quad x \geq -1.$$

Using the Bernoulli's inequality we get

$$r^{\alpha} \leq 1 + \alpha(r-1),$$
$$r^{1-\alpha} \leq 1 + (1-\alpha)(r-1)$$

hence

$$r^{\alpha} + r^{1-\alpha} \leq 1 + r.$$

It follows that

$$\frac{\left(r^{\alpha} + r^{1-\alpha}\right) + \left(r^{\beta} + r^{1-\beta}\right)}{4} \leq \frac{1}{2} + r.$$

Remark: The constraint $a \leq b$ is redundant.

Solution 2 by Michel Bataille, Rouen, France

We suppose $a, b \in (0,1]$ and do not use the hypothesis $a \leq b$.

Let

$$M = a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right).$$

Since $\sqrt{ab}$ and $\frac{a+b}{2}$ are in $(0,1]$, the functions $x \mapsto x^{\sqrt{ab}}$ and $x \mapsto x^{\frac{a+b}{2}}$ are concave on $(0,\infty)$. It follows that

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \le (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\sqrt{ab}} = a+b$$

and

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \le (a+b)\left(\frac{a}{a+b}\cdot\frac{b}{a} + \frac{b}{a+b}\cdot\frac{a}{b}\right)^{\frac{a+b}{2}} = a+b.$$

By addition, $M \le 2(a+b)$.

If m is a positive real number, the function $x \mapsto m^x$ is convex on R. Taking successively $m = \frac{b}{a}$ and $m = \frac{a}{b}$ and setting $k = \frac{1}{2}\left(\sqrt{ab} + \frac{a+b}{2}\right)$, it follows that

$$\left(\frac{b}{a}\right)^{\sqrt{ab}} + \left(\frac{b}{a}\right)^{\frac{a+b}{2}} \ge 2\left(\frac{b}{a}\right)^{k}$$

and

$$\left(\frac{a}{b}\right)^{\sqrt{ab}} + \left(\frac{a}{b}\right)^{\frac{a+b}{2}} \ge 2\left(\frac{a}{b}\right)^{k}.$$

Using $x + y \ge 2\sqrt{xy}$ for positive x, y, we deduce that

$$M \ge 2\left(a\left(\frac{b}{a}\right)^{k} + b\left(\frac{a}{b}\right)^{k}\right) \ge 2\cdot 2\left(a\left(\frac{b}{a}\right)^{k}\cdot b\left(\frac{a}{b}\right)^{k}\right)^{1/2}$$

and $M \ge 4\sqrt{ab}$ follows.

Solution 3 by Arkady Alt, San Jose, California

Applying inequality $x + y \ge 2\sqrt{xy}, x, y > 0$ to $(x,y) = \left(a\left(\frac{b}{a}\right)^{\sqrt{ab}}, b\left(\frac{a}{b}\right)^{\sqrt{ab}}\right)$ and to $(x,y) = \left(a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}, b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\right)$ we obtain

$$a\left(\frac{b}{a}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} \ge 2\sqrt{a\left(\frac{b}{a}\right)^{\sqrt{ab}}\cdot b\left(\frac{a}{b}\right)^{\sqrt{ab}}} = 2\sqrt{ab} \text{ and}$$

$$a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}} \ge 2\sqrt{a\left(\frac{b}{a}\right)^{\frac{a+b}{2}}\cdot b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}} = 2\sqrt{ab}.$$

Thus, $a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{b}{a}\right)^{a+b}}\right) + b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}} + \sqrt{\left(\frac{a}{b}\right)^{a+b}}\right) \ge 4\sqrt{ab}.$

For function $f(t) = t^p$, which for $p \in [0,1]$ is concave down on $(0,\infty)$, holds inequality $\dfrac{ax^p + by^p}{a+b} \le \left(\dfrac{ax+by}{a+b}\right)^p$ for any $x, y > 0$.

Since $\sqrt{ab}, \dfrac{a+b}{2} \in [0,1]$ then applying this inequality to $(x,y,p) = \left(\frac{b}{a}, \frac{a}{b}, \sqrt{ab}\right)$ and $(x,y,p) = \left(\frac{b}{a}, \frac{a}{b}, \frac{a+b}{2}\right)$

we obtain $\dfrac{a\left(\dfrac{b}{a}\right)^{\sqrt{ab}}+b\left(\dfrac{a}{b}\right)^{\sqrt{ab}}}{a+b} \leq \left(\dfrac{a\cdot\dfrac{b}{a}+b\cdot\dfrac{a}{b}}{a+b}\right)^{\sqrt{ab}}=1 \iff$

$a\left(\dfrac{b}{a}\right)^{\sqrt{ab}}+b\left(\dfrac{a}{b}\right)^{\sqrt{ab}} \leq a+b$ and

$\dfrac{a\left(\dfrac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\dfrac{a}{b}\right)^{\frac{a+b}{2}}}{a+b} \leq \left(\dfrac{a\cdot\dfrac{b}{a}+b\cdot\dfrac{a}{b}}{a+b}\right)^{\frac{a+b}{2}}=1 \iff$

$a\left(\dfrac{b}{a}\right)^{\frac{a+b}{2}}+b\left(\dfrac{a}{b}\right)^{\frac{a+b}{2}} \leq a+b.$

Therefore, $a\left(\left(\dfrac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\dfrac{b}{a}\right)^{a+b}}\right)+b\left(\left(\dfrac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\dfrac{a}{b}\right)^{a+b}}\right) \leq 2\left(a+b\right).$

Solution 4 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY

In the following proof, we use Heinz's inequality:

$$\sqrt{ab} \;\leq\; \frac{a^{1-\alpha}b^{\alpha}+a^{\alpha}b^{1-\alpha}}{2} \;\leq\; \frac{a+b}{2} \quad \text{for } a,b>0,\ \alpha\in[0,1],$$

first with $\alpha=\sqrt{ab}$ and then with $\alpha=(a+b)/2$.
First we rearrange the central term in the proposed inequality:

$$\begin{aligned}
&a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\\
&= a\left(\frac{b}{a}\right)^{\sqrt{ab}} + a\left(\frac{b}{a}\right)^{\frac{a+b}{2}} + b\left(\frac{a}{b}\right)^{\sqrt{ab}} + b\left(\frac{a}{b}\right)^{\frac{a+b}{2}}\\
&= \left(a\cdot\frac{b^{\sqrt{ab}}}{a^{\sqrt{ab}}}+b\cdot\frac{a^{\sqrt{ab}}}{b^{\sqrt{ab}}}\right) + \left(a\cdot\frac{b^{\frac{a+b}{2}}}{a^{\frac{a+b}{2}}}+b\cdot\frac{a^{\frac{a+b}{2}}}{b^{\frac{a+b}{2}}}\right)\\
&= 2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right) + 2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right).
\end{aligned}$$

Now the Heinz inequality yields

$$\begin{aligned}
4\sqrt{ab} &\leq 2\left(\frac{a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}}{2}\right) + 2\left(\frac{a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}}{2}\right)\\
&\leq 2\left(\frac{a+b}{2}\right) + 2\left(\frac{a+b}{2}\right) = 2(a+b).
\end{aligned}$$

Solution 5 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

First we prove that

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\geq 4\sqrt{ab}. \tag{1}$$

To prove this, we will use the well-known inequality that for all $p>0$ and any real number r

$$p^r+\frac{1}{p^r}\geq 2. \tag{2}$$

For all $a>0$ and $b>0$, using (2) in the above step, we can write

$$\begin{aligned}
& a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\\
= & \sqrt{ab}\left[\sqrt{\frac{a}{b}}\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+\sqrt{\frac{b}{a}}\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\right]\\
= & \sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{b}{a}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)+\left(\left(\frac{a}{b}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]\\
& \sqrt{ab}\left[\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)+\left(\left(\frac{b}{a}\right)^{-\frac{1}{2}+\sqrt{ab}}+\left(\frac{a}{b}\right)^{-\frac{1}{2}+\frac{a+b}{2}}\right)\right]\\
\geq & \sqrt{ab}(2+2)=4\sqrt{ab}.
\end{aligned}$$

This completes the proof of (1).

Now, we prove that

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\leq 2(a+b). \tag{3}$$

To prove (3), we notice that, for $a\in(0,1]$ and $b(0,1]$.

$$\text{With } a\leq b\text{, we have}\begin{cases} b^{\sqrt{ab}}-a^{\sqrt{ab}}\geq 0\\ b^{1-\sqrt{ab}}-a^{1-\sqrt{ab}}\geq 0,\end{cases}\quad\text{and}\quad\begin{cases} b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\geq 0\\ b^{1-\frac{a+b}{2}}-a^{1-\frac{a+b}{2}}\geq 0\end{cases}. \tag{4}$$

Also,

$$-2a-2b=-a^{\sqrt{ab}}a^{1-\sqrt{ab}}-a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}-b^{\sqrt{ab}}b^{1-\sqrt{ab}}-b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}} \tag{5}$$

Now, using (4) and (5), we have

$$a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)-2a-2b$$

$$= a^{1-\sqrt{ab}}b^{\sqrt{ab}}+a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}+a^{\sqrt{ab}}b^{1-\sqrt{ab}}+a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$
$$-a^{\sqrt{ab}}a^{1-\sqrt{ab}}-a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}-b^{\sqrt{ab}}b^{1-\sqrt{ab}}-b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}$$

$$=\left(a^{1-\sqrt{ab}}b^{\sqrt{ab}}-a^{\sqrt{ab}}a^{1-\sqrt{ab}}\right)+\left(a^{1-\frac{a+b}{2}}b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}a^{1-\frac{a+b}{2}}\right)$$
$$+\left(a^{\sqrt{ab}}b^{1-\sqrt{ab}}-b^{\sqrt{ab}}b^{1-\sqrt{ab}}\right)+\left(a^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}-b^{\frac{a+b}{2}}b^{1-\frac{a+b}{2}}\right)$$

$$=a^{1-\sqrt{ab}}\left(b^{\sqrt{ab}}-a^{\sqrt{ab}}\right)+a^{1-\frac{a+b}{2}}\left(b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\right)$$
$$-b^{1-\sqrt{ab}}\left(b^{\sqrt{ab}}-a^{\sqrt{ab}}\right)-b^{1-\frac{a+b}{2}}\left(b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\right)$$

$$=-\left[\left(b^{\sqrt{ab}}-a^{\sqrt{ab}}\right)\left(b^{1-\sqrt{ab}}-a^{1-\sqrt{ab}}\right)\right.$$
$$\left.+\left(b^{\frac{a+b}{2}}-a^{\frac{a+b}{2}}\right)\left(b^{1-\frac{a+b}{2}}-a^{1-\frac{a+b}{2}}\right)\right]\le 0.$$

This completes the proof of (3).

Now, combining the inequalities from (1) an (3), we conclude that

$$4\sqrt{ab}\le a\left(\left(\frac{b}{a}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{b}{a}\right)^{a+b}}\right)+b\left(\left(\frac{a}{b}\right)^{\sqrt{ab}}+\sqrt{\left(\frac{a}{b}\right)^{a+b}}\right)\le 2(a+b)$$

Also solved by Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5621: *Proposed by Stanley Rabinowitz, Brooklyn, NY*

Given; non-negative integer n, real numbers a and c with $ac \neq 0$, and the expression $a+cx^2 \geq 0$.

Express: $\displaystyle\int \left(a+bc^2\right)^{\frac{2n+1}{2}}\,dx$ as the sum of elementary functions.

Solution 1 by Michel Bataille, Rouen, France

We will make use of the following lemma: if z is a nonzero complex number, then

(i) $(z+z^{-1})^{2n+2} = \binom{2n+2}{n+1} + \sum_{j=0}^{n} \binom{2n+2}{n-j} (z^{2j+2} + z^{-(2j+2)})$

(ii) $(z-z^{-1})^{2n+2} = (-1)^{n+1}\binom{2n+2}{n+1} + (-1)^n \sum_{j=0}^{n} (-1)^j \binom{2n+2}{n-j} (z^{2j+2} + z^{-(2j+2)})$

Proof: (i) Applying the binomial theorem, we obtain

$$\begin{aligned}(z+z^{-1})^{2n+2} &= \sum_{k=0}^{n} \binom{2n+2}{k} z^{2n+2-2k} + \binom{2n+2}{n+1} + \sum_{k=n+2}^{2n+2} \binom{2n+2}{k} z^{2n+2-2k} \\ &= \binom{2n+2}{n+1} + \sum_{j=0}^{n} \binom{2n+2}{n-j} z^{2j+2} + \sum_{j=0}^{n} \binom{2n+2}{n+2+j} z^{-(2j+2)} \\ &= \binom{2n+2}{n+1} + \sum_{j=0}^{n} \binom{2n+2}{n-j} (z^{2j+2} + z^{-(2j+2)})\end{aligned}$$

(the latter because $\binom{2n+2}{n+2+j} = \binom{2n+2}{2n+2-(n+2+j)}$). The proof of (ii) is similar.

Turning to the problem, three cases are to be considered, depending on the signs of a and c.
Case 1: if $a > 0, c > 0$. The substitution $x = u\sqrt{\frac{a}{c}}$ gives

$$\int (a+cx^2)^{\frac{2n+1}{2}}\, dx = \frac{a^{n+1}}{\sqrt{c}} \int \left(\sqrt{1+u^2}\right)^{2n+1} du. \tag{1}$$

To calculate $\int \left(\sqrt{1+u^2}\right)^{2n+1} du$, we use the substitution $u = \sinh t,\ du = \cosh t$ and obtain

$$\int \left(\sqrt{1+u^2}\right)^{2n+1} du = \int (\cosh t)^{2n+2}\, dt = 2^{-(2n+2)} \int (e^t + e^{-t})^{2n+2}\, dt. \tag{2}$$

From the part (i) of the lemma with $z = e^t$,

$$\begin{aligned}\int (e^t + e^{-t})^{2n+2}\, dt &= \binom{2n+2}{n+1} \cdot t + 2\sum_{j=0}^{n} \binom{2n+2}{n-j} \int (\cosh[(2j+2)t])\, dt \\ &= \binom{2n+2}{n+1} \cdot t + 2\sum_{j=0}^{n} \binom{2n+2}{n-j} \frac{\sinh[(2j+2)t]}{2j+2}.\end{aligned}$$

The answer follows from (1), (2) and $t = \sinh^{-1}\left(x\sqrt{\frac{c}{a}}\right)$: up to an additive constant

$$\int (a+cx^2)^{\frac{2n+1}{2}}\, dx =$$

$$\frac{a^{n+1}}{\sqrt{c}} \left(\frac{1}{2^{2n+2}} \binom{2n+2}{n+1} \sinh^{-1}\left(x\sqrt{\frac{c}{a}}\right) + \frac{1}{2^{2n+1}} \sum_{j=0}^{n} \binom{2n+2}{n-j} \frac{\sinh\left[(2j+2)\sinh^{-1}\left(x\sqrt{\frac{c}{a}}\right)\right]}{2j+2} \right)$$

Case 2: $a > 0,\ c < 0$. Similarly, the successive substitutions $x = u\sqrt{\frac{a}{|c|}}$ ($|u| \leq 1$ and $u = \sin t,\ |t| \leq \pi/2$, leads to

$$\int (a+cx^2)^{\frac{2n+1}{2}}\, dx = \frac{a^{n+1}}{\sqrt{|c|}} \int (\cos t)^{2n+2}\, dt = \frac{1}{2^{2n+2}} \int (e^{it} + e^{-it})^{2n+2}\, dt.$$

Then, the part (i) of the lemma with $z = e^{it}$ yields

$$\int (e^{it} + e^{-it})^{2n+2}\, dt = \binom{2n+2}{n+1} \cdot t + 2\sum_{j=0}^{n} \binom{2n+2}{n-j} \int (\cos(2j+2)t)^{2n+2}\, dt$$
$$= \binom{2n+2}{n+1} \cdot t + 2\sum_{j=0}^{n} \binom{2n+2}{n-j} \frac{\sin(2j+2)t}{2j+2}.$$

The answer is deduced in the same way as above: up to an additive constant, $\int (a+cx^2)^{\frac{2n+1}{2}}\, dx =$

$$\frac{a^{n+1}}{\sqrt{|c|}} \left(\frac{1}{2^{2n+2}} \binom{2n+2}{n+1} \sin^{-1}\left(x\sqrt{\frac{|c|}{a}}\right) + \frac{1}{2^{2n+1}} \sum_{j=0}^{n} \binom{2n+2}{n-j} \frac{\sin\left[(2j+2)\sin^{-1}\left(x\sqrt{\frac{|c|}{a}}\right)\right]}{2j+2} \right)$$

Case 3: $a < 0,\ c > 0$. In that case either $x \geq \sqrt{\frac{|a|}{c}}$ or $x \leq -\sqrt{\frac{|a|}{c}}$. We use the substitution $x = u\sqrt{\frac{|a|}{c}}$ and $u = \cosh t$ (if $u \geq 1$) or $u = -\cosh t$ (if $u \leq -1$) for $t \geq 0$. Setting $\epsilon = \frac{u}{\cosh t}$ we obtain that $\int (a + cx^2)^{\frac{2n+1}{2}}\, dx = \int |a|^{\frac{2n+1}{2}} \left(\frac{cx^2}{|a|} - 1\right)^{\frac{2n+1}{2}}\, dx =$

$$\frac{|a|^{n+1}}{\sqrt{c}} \int \left(\sqrt{u^2-1}\right)^{2n+1} du = \frac{\epsilon |a|^{n+1}}{\sqrt{c}} \int (\sinh t)^{2n+2}\, dt = \frac{\epsilon |a|^{n+1}}{\sqrt{c}} \cdot \frac{1}{2^{2n+2}} \int (e^t - e^{-t})^{2n+2}\, dt$$

Using the part (ii) of the lemma, we obtain (up to an additive constant)

$$\int (e^t - e^{-t})^{2n+2}\, dt = (-1)^{n+1} \binom{2n+2}{n+1} t + (-1)^n \sum_{j=0}^{n} \binom{2n+2}{n-j} \frac{\sinh[(2j+2)t]}{2j+2}.$$

We come back to the variable x as above.

Solution 2 by Moti Levy, Rehovot, Israel

Let

$$I_n(t) := \int \left(1+t^2\right)^{n+\frac{1}{2}} dt. \tag{1}$$

Observing the first three definite integrals,

$$I_0(t) = t\sqrt{t^2+1}\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(t + \sqrt{t^2+1}\right),$$
$$I_1(t) = t\sqrt{t^2+1}\left(\frac{5}{8} + \frac{1}{4}t^2\right) + \frac{3}{8}\ln\left(t + \sqrt{t^2+1}\right),$$
$$I_2(t) = t\sqrt{t^2+1}\left(\frac{11}{16} + \frac{13}{24}t^2 + \frac{1}{6}t^4\right) + \frac{5}{16}\ln\left(t + \sqrt{t^2+1}\right),$$

we claim that

$$I_n(t) = \sqrt{1+t^2} \sum_{k=0}^{n} a_{k,n} t^{2k+1} + (1 - a_{0,n}) \ln\left(t + \sqrt{1+t^2}\right), \tag{2}$$

where the polynomial coefficients satisfies the recurrence

$$2ka_{k-1,n} + (2k+1)\, a_{k,n} = \binom{n+1}{k} \tag{3}$$

with initial condition

$$a_{n,n} = \frac{1}{2\,(n+1)}. \tag{4}$$

Change of variable $t = \sqrt{\frac{c}{a}}x$ gives the required indefinite integral:

$$\int \left(a + cx^2\right)^{\frac{2n+1}{2}} dx = \frac{a^{n+1}}{\sqrt{c}} I_n \left(\sqrt{\frac{c}{a}}x\right).$$

Proof of claim:
To show that $I_n\,(t)$ is the antiderivative of $\left(1+t^2\right)^{n+\frac{1}{2}}$, we differentiate $I_n\,(t)$ defined in (Eq. 2) and use the recurrence defined by (Eq. 3) and initial condition (Eq 4).

$$\begin{aligned}
\frac{dI_n\,(t)}{dt} &= \frac{t}{\sqrt{1+t^2}} \sum_{k=0}^{n} a_{k,n} t^{2k+1} + \sqrt{1+t^2} \sum_{k=0}^{n} a_{k,n}\,(2k+1)\, t^{2k} + (1-a_{0,n})\, \frac{1}{\sqrt{1+t^2}} \\
&= \frac{1}{\sqrt{1+t^2}} \left(\sum_{k=0}^{n} a_{k,n} t^{2k+2} + \left(1+t^2\right) \sum_{k=0}^{n} a_{k,n}\,(2k+1)\, t^{2k} + (1-a_{0,n}) \right) \\
&= \frac{1}{\sqrt{1+t^2}} \left(\sum_{k=0}^{n} a_{k,n} t^{2k+2} + \sum_{k=0}^{n} a_{k,n}\,(2k+1)\, t^{2k} + \sum_{k=0}^{n} a_{k,n}\,(2k+1)\, t^{2k+2} + (1-a_{0,n}) \right) \\
&= \frac{1}{\sqrt{1+t^2}} \left(\sum_{k=1}^{n+1} a_{k-1,n} t^{2k} + \sum_{k=0}^{n} a_{k,n}\,(2k+1)\, t^{2k} + \sum_{k=1}^{n+1} a_{k-1,n}\,(2k-1)\, t^{2k} + (1-a_{0,n}) \right) \\
&= \frac{1}{\sqrt{1+t^2}} \left(\sum_{k=1}^{n} \left(2ka_{k-1,n} + a_{k,n}\,(2k+1)\right) t^{2k} + a_{n,n} t^{2n+2} + a_{n,n}\,(2n+1)\, t^{2n+2} + a_{0,n} + (1-a_{0,n}) \right) \\
&= \frac{1}{\sqrt{1+t^2}} \left(\sum_{k=1}^{n} \left(2ka_{k-1,n} + a_{k,n}\,(2k+1)\right) t^{2k} + (2n+2)\, a_{n,n} t^{2n+2} + a_{0,n} + (1-a_{0,n}) \right) \\
&= \frac{1}{\sqrt{1+t^2}} \left(1 + \sum_{k=1}^{n} \left(2ka_{k-1,n} + (2k+1)\, a_{k,n}\right) t^{2k} + t^{2n+2} \right) = \frac{1}{\sqrt{1+t^2}} \sum_{k=0}^{n+1} \binom{n+1}{k} t^{2k} \\
&= \frac{1}{\sqrt{1+t^2}} \left(1+t^2\right)^{n+1} = \left(1+t^2\right)^{n+\frac{1}{2}}.
\end{aligned}$$

To complete this solution, we find closed form expression for the coefficients $\left(a_{k,n}\right)_{k=0}^{n}$.

The recurrence relation defined in (Eq. 3) is a first order linear recurrence relation with variable coefficients.
We reformulate (Eq. 3) by setting

$$y_{k,n} := a_{n-k,n}.$$

The recurrence relation for $y_{k,n}$ is

$$2\,(n-k)\, y_{k+1,n} = -\,(2n-2k+1)\, y_{k,n} + \binom{n+1}{n-k}, \qquad y_{0,n} = \frac{1}{2\,(n+1)}. \tag{5}$$

or

$$2\left(n-k+1\right)y_{k,n}=-\left(2n-2k+3\right)y_{k-1,n}+\binom{n+1}{k},\qquad y_{0,n}=\frac{1}{2\left(n+1\right)}.\tag{6}$$

One of the methods for solving recurrence relation of this type is to use summation factor $F\left(k,n\right)$ defined as follows:

$$\begin{aligned}F\left(k,n\right)&:=\frac{\prod_{i=1}^{k-1}2\left(n-i+1\right)}{\prod_{j=1}^{k}\left(-\left(2n-2j+3\right)\right)}=\left(-1\right)^{k}\frac{n!}{\left(n-k+1\right)!}\frac{\Gamma\left(n-k+\frac{3}{2}\right)}{2\Gamma\left(n+\frac{3}{2}\right)}\\&=\left(-1\right)^{k}4^{k}\frac{\binom{2n-2k+1}{n-k}}{\binom{2n}{n}}.\end{aligned}\tag{7}$$

If we multiply both sides of (6) by $F\left(k,n\right)$ then the recurrence becomes

$$d_{k,n}=d_{n,k-1}+F\left(k\right)\binom{n+1}{k},\tag{8}$$

where

$$d_{k,n}:=-\left(2n-2k+1\right)F\left(k+1\right)y_{k,n}.$$

Recurrence (8) is linear recurrence with constant coefficients. Hence we can readily write the solution of (6),

$$y_{k,n}=\frac{y_{0,n}+\sum_{m=1}^{k}F\left(m\right)\binom{n+1}{m}}{-\left(2n-2k+1\right)F\left(k+1\right)}.$$

After some simplification we get,

$$y_{k,n}=\frac{\left(-1\right)^{k}}{2\left(2n-2k+1\right)\binom{2n-2k}{n-k}4^{k}}\sum_{m=0}^{k}\left(-1\right)^{m}4^{m}\binom{2n-2m+1}{n-m}\binom{n+1}{m},$$

which implies that

$$a_{k,n}=y_{n-k,n}=\frac{\left(-1\right)^{n-k}}{2\left(k+1\right)\binom{2k}{k}4^{n-k}}\sum_{m=0}^{n-k}\left(-1\right)^{m}4^{m}\binom{2n-2m+1}{n-m}\binom{n+1}{m}.\tag{9}$$

Equivalent expression for the coefficients $\left(a_{k,n}\right)_{k=0}^{n}$ (by reversing the summation direction in (9)) is

$$a_{k,n}=\frac{1}{2\left(2k+1\right)\binom{2k}{k}}\sum_{m=0}^{n-k}\frac{\left(-1\right)^{m}}{4^{m}}\binom{2k+2m+1}{k+m}\binom{n+1}{k+m+1}.\tag{10}$$

Remark: Another equivalent closed form of the coefficients $\left(a_{k,n}\right)_{k=0}^{n}$ is

$$a_{k,n}=\frac{1}{2\left(k+1\right)}\binom{n+1}{k+1}3\mathrm{F}2\left(1,k+\frac{3}{2},k-n;k+2,k+2;1\right)$$

where $3\mathrm{F}2\left(a,b,c;d,e;z\right)$ is the generalized hypergeometric function.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We refer to formulas 2.260.3 and 2.261 of [1] which state:

Let $R(x) = a + bx + cx^2, \Delta = 4ac - b^2$. Then

$$(i) \qquad \int \sqrt{R^{2n+1}(x)}dx = \frac{(2cx+b)\sqrt{R(x)}}{4(n+1)c}\left(R^n(x)+\right.$$

$$\left.\sum_{k=0}^{n-1}\frac{(2n+1)(2n-1)c\ldots(2n-2k+1)}{8^{k+1}n(n-1)\cdots(n-k)}\left(\frac{\Delta}{c}\right)^{k+1}R^{n-k-1}(x)\right)+\frac{(2n+1)!!}{8^{n+1}(n+1)!}\left(\frac{\Delta}{c}\right)^{n+1}\int\frac{dx}{\sqrt{R(x)}}.$$

$$(ii)\int\frac{dx}{\sqrt{R(x)}} = \frac{1}{\sqrt{c}}\ln\left(2\sqrt{R}(x)+2cx+b\right),\ c>0.$$

Reference

[1] I.S. Gradshteyn/ I.M. Ryzhik, Table of Integrals, Series, and Products, corrected and enlarged edition, Academic Press, 1980.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Denote $\int(a+cx^2)^{\frac{2n+1}{2}}dx$ by $I = I(a,c,n,x)$, and let K be a constant, not necessarily the same in each occurrence. We show that

$$I = \frac{\binom{2n+1}{n}}{2}\left(\sum_{k=0}^{n}\frac{((n-k)!)^2}{2^{2k}(2n-2k+1)!}a^k(a+cx^2)^{n-k+\frac{1}{2}}\right)x+$$
$$\frac{\binom{2n+1}{n}a^{n+1}}{2^{2n+1}\sqrt{c}}\ln\left(\frac{\sqrt{c}x+\sqrt{a+cx^2}-\sqrt{a}}{\sqrt{c}x-\sqrt{a+cx^2}+\sqrt{a}}\right)+K, \qquad (1)$$

$$I = \frac{(a+cx^2)^{n+\frac{3}{2}}}{2(n+1)!}\left(\sum_{k=0}^{n}\frac{(2k)!(n-k)!a^k}{(2^{2k})(k!)c^{k+1}x^{2k+1}}\right)+$$
$$\frac{\binom{2n+1}{n}a^{n+1}}{(2^{2n+1})\sqrt{c}}\left(\ln\left(\frac{\sqrt{a+cx^2}+\sqrt{cx}-\sqrt{-a}}{\sqrt{a+cx^2}-\sqrt{cx}+\sqrt{-a}}\right)-\sum_{k=1}^{n+1}\frac{1}{2k-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{cx}}\right)^{2k-1}\right)+K, \qquad (2)$$

and

$$I = \binom{2n+1}{n}\left(\left(\sum_{k=0}^{n}\frac{a^k(a+cx^2)^{n-k+\frac{1}{2}}}{2^{2k+1}(n-k+1)\binom{2n-2k+1}{n-k}}\right)x+\frac{\sin^{-1}\left(\sqrt{\frac{-c}{a}}x\right)}{2^{2n+1}\sqrt{-c}}a^{n+1}\right)+K \qquad (3)$$

according as $a>0$ and $c>0, a<0$ and $c>0, a>0$ and $c<0$. We suppose that $x>0$ in the proofs. If $x<0$, we put $x=-y$ and work with y.

To prove (1) we substitute $x=\sqrt{\frac{a}{c}}\tan\theta$ to get $I=\dfrac{a^{n+1}}{\sqrt{c}}\int \sec^{2n+3}\theta d\theta$.

According to entry, section 2.519 on p. 156 of [1], we have $\int \sec^{2n+3}\theta d\theta =$

$$\frac{\sin\theta}{2(n!)((n+1)!)}\left(\sum_{k=0}^{n}\frac{(2n+1)!((n-k)!)^2}{2^{2k}(2n-2k+1)!}\sec^{2(n-k+1)}\theta\right)+\frac{\binom{2n+1}{n}}{2^{2n+1}}\ln\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+K.$$

Since $\sin\theta=\sqrt{\dfrac{\theta}{a+cx^2}}x, \sec^2\theta=\dfrac{a+cx^2}{a}$, and $\tan\dfrac{\theta}{2}=\dfrac{\sqrt{a+cx^2}-\sqrt{a}}{\sqrt{c}x}$, we obtain (1) readily.

To prove (2), we substitute $x=\sqrt{\dfrac{-a}{c}}\sec\theta$, to get
$I=\dfrac{(-a)^{n+1}}{\sqrt{c}}\int \tan^{2n+2}\theta\sec\theta d\theta$. According to entries 1 and 3 , section 2.516 on pp.155-156 of [1], we have

$$\int \tan^{2n+2}\theta\sec\theta d\theta=\frac{\sin^{2n+3}\theta}{2(n+1)!}\left(\sum_{k=0}^{n}\frac{(-1)^k(2k)!(n-k)!}{2^{2k})(k!)}\sec^{2(n-k+1)}\theta\right)$$
$$+\frac{(-1)^{n+1}\binom{2n+1}{n}}{2^2n+1}\left(\ln\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)-\sum_{k=1}^{n+1}\frac{\sin^{2k-1}\theta}{2k-1}\right)+K.$$

Since $\sin\theta=\dfrac{a+cx^2}{\sqrt{c}x},\ \sec^2\theta=\dfrac{cx^2}{-a}$, and $\tan\dfrac{\theta}{2}=\dfrac{\sqrt{c}x-\sqrt{-a}}{\sqrt{a+cx^2}}$, we obtain (2) readily.

To prove(3), we substitute $x=\sqrt{\dfrac{a}{-c}}\sin\theta$ to get $I=\dfrac{a^{n+1}}{\sqrt{-c}}\int\cos^{2n+2}\theta d\theta$.

According to entry 2, section 2.512 on p. 152 of [1], we have

$$\int \cos^{2n+2}\theta d\theta=\binom{2n+1}{n}\left(\left(\sum_{k=0}^{n}\frac{\cos^{2n-2k+1}\theta}{2^{2k+1}(n-k+1)\binom{2n-2k+1}{n-k}}\right)\sin\theta+\frac{\theta}{2n+1}\right)+K.$$

Since $\sin\theta=\sqrt{\dfrac{-c}{a}}x$ and $\cos\theta=\sqrt{\dfrac{a+cx^2}{a}}$, we obtain (3) readily.

Reference: 1. I.S. Gradshteyn and I.M. Ryzhik: Table of Integrals, Series, and Products, Seventh Edition,Elsevier,Inc. 2007.

Solution 5 by David E. Manes, Oneonta, NY

Note that $ac \neq 0$ implies $a \neq 0$ and $c \neq 0$. Then $a + cx^2 \geq 0$ and $x = 0$ imply $a + cx^2 = a \geq 0$; hence, $a > 0$. We will use the following two results: (**1**) the reduction formula for integrating powers of secants. If $n > 1$, then

$$\int \sec^n x \,\mathrm{d}x = \frac{\tan x \cdot \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \,\mathrm{d}x.$$

(**2**) Using induction and the reduction formula, one obtains that if $m \geq 1$, then

$$\int \sec^{2m+1} x \ \mathrm{d}x = \frac{\binom{2m}{m}}{4^m} \left[\sum_{i=1}^{m} \frac{1}{2i} \cdot \frac{4^i}{\binom{2i}{i}} \cdot \sec^{2i-1} x \cdot \tan x + \ln|\sec x + \tan x| \right] + C.$$

If we rewrite the formula with $m = n + 1$, then

$$\int \sec^{2n+3} x \ \mathrm{d}x = \frac{\binom{2n+2}{n+1}}{4^{n+1}} \left[\sum_{i=1}^{n+1} \frac{1}{2i} \cdot \frac{4^i}{\binom{2i}{i}} \cdot \sec^{2i-1} x \cdot \tan x + \ln|\sec x + \tan x| \right] + C.$$

Observe that the second formula is defined for the value $n = 0$ and yields the correct answer for this case.

Assume $n \geq 0$. Then

$$\int (a + cx^2)^{\frac{2n+1}{2}} \,\mathrm{d}x = a^{\frac{2n+1}{2}} \int \left(1 + \frac{c}{a}x^2\right)^n \sqrt{1 + \frac{c}{a}x^2} \,\mathrm{d}x.$$

Let $\sqrt{\frac{c}{a}}x = \tan\theta$. Then $\sqrt{\frac{c}{a}} \,\mathrm{d}x = \sec^2\theta \,\mathrm{d}\theta$ so that $\mathrm{d}x = \sqrt{\frac{a}{c}} \sec^2\theta \,\mathrm{d}\theta$. Furthermore,

$\frac{c}{a}x^2 = \tan^2\theta$ so that $1 + \frac{c}{a}x^2 = 1 + \tan^2\theta = \sec^2\theta$. Therefore,

$$\left(1 + \frac{c}{a}x^2\right)^n = \frac{(a + cx^2)^n}{a^n} = \sec^{2n}\theta \quad \text{and} \quad \sqrt{1 + \frac{c}{a}x^2} = \frac{\sqrt{a + cx^2}}{\sqrt{a}} = \sec\theta.$$

Let $J = \int (a + cx^2)^{\frac{2n+1}{2}} \,\mathrm{d}x$. Rewriting the integral in terms of θ, one obtains

$$\begin{aligned}
J &= a^{\frac{2n+1}{2}} \int \sec^{2n}\theta \cdot \sec\theta \cdot \sqrt{\frac{a}{c}} \cdot \sec^2\theta \,\mathrm{d}\theta \\
&= \frac{a^{n+1}}{\sqrt{c}} \int (\sec\theta)^{2n+3} \,\mathrm{d}\theta \\
&= \frac{a^{n+1}}{\sqrt{c}} \cdot \frac{\binom{2n+2}{n+1}}{4^{n+1}} \left[\ln|\sec\theta + \tan\theta| + \sum_{i=1}^{n+1} \frac{1}{2i} \cdot \frac{4^i}{\binom{2i}{i}} \cdot \sec^{2i-1}\theta \cdot \tan\theta \right] + C \\
&= \frac{a^{n+1}}{\sqrt{c}} \cdot \frac{\binom{2n+2}{n+1}}{4^{n+1}} \left[\ln(\sqrt{a + cx^2} + \sqrt{c}x) + \sum_{i=1}^{n+1} \frac{1}{2i} \cdot \frac{4^i}{\binom{2i}{i}} \cdot \frac{(a + cx^2)^i}{a^i} \cdot \frac{\sqrt{a}}{\sqrt{a + cx^2}} \cdot \frac{\sqrt{c}}{\sqrt{a}}x \right] + C \\
&= \frac{a^{n+1}}{\sqrt{c}} \cdot \frac{\binom{2n+2}{n+1}}{4^{n+1}} \left[\ln(\sqrt{a + cx^2} + \sqrt{c}x) + \sum_{i=1}^{n+1} \frac{1}{2i} \cdot \frac{4^i}{\binom{2i}{i}} \cdot \frac{(a + cx^2)^{\frac{2i-1}{2}}}{a^i} \cdot \sqrt{c}x \right] + C.
\end{aligned}$$

If $n=0$, then

$$J=\int\sqrt{a+cx^2}\ \mathrm{d}x=\frac{1}{2}\cdot\frac{a}{\sqrt{c}}\left[\ln(\sqrt{a+cx^2}+\sqrt{c}x)+\frac{1}{a}\sqrt{c}x\cdot\sqrt{a+cx^2}\right]+C$$
$$=\frac{1}{2}\cdot\frac{a}{\sqrt{c}}\cdot\ln\left(\sqrt{a+cx^2}+\sqrt{c}x\right)+\frac{1}{2}x\sqrt{a+cx^2}+C.$$

If $n=1$, then

$$J=\int\left(a+cx^2\right)^{3/2}\ \mathrm{d}x=\frac{1}{4}x\left(a+cx^2\right)^{3/2}+\frac{3a}{8}x\left(a+cx^2\right)^{1/2}+\frac{3a^2}{8\sqrt{c}}\ln\left(\sqrt{a+cx^2}\right)+C.$$

On the other hand, if $n=5$, then

$$\begin{aligned}\int\left(a+cx^2\right)^{11/2}\ \mathrm{d}x \quad = \quad & \frac{1}{12}x\left(a+cx^2\right)^{11/2}+\frac{11a}{120}x\left(a+cx^2\right)^{9/2}+\frac{33a^2}{320}x\left(a+cx^2\right)^{7/2}\\ &+\frac{77a^3}{640}x\left(a+cx^2\right)^{5/2}+\frac{77a^4}{572}x\left(a+cx^2\right)^{3/2}+\frac{231a^5}{1024}x\left(a+cx^2\right)^{1/2}\\ &+\frac{231}{1024}\cdot\frac{a^6}{\sqrt{c}}\ln\left(\sqrt{a+cx^2}+\sqrt{c}x\right).\end{aligned}$$

Also solved by the proposer.

5622: *Albert Natian Los Angelos Valley College, Valley Glen, CA*

Suppose f is a real-valued function such that for all real numbers x;

$$[f(x-8/15)]^2+[f(x+47/30)]^2+[f(x+2/75)]^2=$$
$$=f(x-8/15)f(x+47/30)+f(x+47/30)f(x+2/75)+f(x+2/75)f(x-8/15).$$

If $f\left(\frac{49}{5}\right)=\frac{11}{3}$, then find $f\left(\frac{1}{2}f\left(\frac{28}{50}\right)-\frac{2}{25}f\left(-42\right)\right)$.

Solution 1 by Michel Bataille, Rouen, France

Suppose f is a real-valued function such that for all real numbers x:

$$[f(x-8/15)]^2+[f(x+47/30)]^2+[f(x+2/75)]^2=$$
$$f(x-8/15)f(x+47/30)+f(x+47/30)f(x+2/75)+f(x+2/75)f(x-8/15).$$

If $f\left(\frac{49}{5}\right)=\frac{11}{3}$, then find $f\left(\frac{1}{2}f\left(\frac{28}{50}\right)-\frac{2}{25}f(-42)\right)$.

For real numbers a,b,c, we have $2(a^2+b^2+c^2-(ab+bc+ca))=(a-b)^2+(b-c)^2+(c-a)^2$ so that the condition $a^2+b^2+c^2=ab+bc+ca$ is equivalent to $a=b=c$. It follows that the hypothesis means that for all real x

$$f(x-8/15)=f(x+47/30)=f(x+2/75),$$

and in consequence, $f(x) = f((x + 8/15) - 8/15) = f(x + 8/15 + 47/30) = f(x + 21/10)$ and $f(x) = f(x + 8/15 + 2/75) = f(x + 14/25)$.

Thus, the numbers $\frac{14}{25}$ and $\frac{21}{10}$ (and $m \cdot \frac{14}{25} + n \cdot \frac{21}{10}$ for $m, n \in Z$) are periods of f. We deduce that

$$f(-42) = f\left((-20) \cdot \frac{21}{10}\right) = f(0), \qquad f\left(\frac{28}{50}\right) = f\left(\frac{14}{25}\right) = f(0).$$

Also, we obtain

$$\frac{11}{3} = f\left(\frac{49}{5}\right) = f\left(2 \cdot \frac{21}{10} + 10 \cdot \frac{14}{25}\right) = f(0).$$

Thus,

$$f\left(\frac{1}{2} f\left(\frac{28}{50}\right) - \frac{2}{25} f(-42)\right) = f\left(\frac{21}{50} f(0)\right) = f\left(\frac{21}{50} \cdot \frac{11}{3}\right) = f\left(\frac{77}{50}\right) = f\left(\frac{21}{10} - \frac{14}{25}\right) = f(0) = \frac{11}{3}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland

Suppose that a, b, c are real numbers satisfying $a^2 + b^2 + c^2 = ab + bc + ca$. Then

$$0 = a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-a)^2$$

implying that $a = b = c$. Therefore, for all real numbers x

$$f\left(x - \frac{8}{15}\right) = f\left(x + \frac{47}{30}\right) = f\left(x + \frac{2}{75}\right),$$

or equivalently replacing x by $x + \frac{8}{15}$

$$f(x) = f\left(x + \frac{21}{10}\right) = f\left(x + \frac{14}{25}\right).$$

This means that

$$f(x) = f\left(x + \frac{21}{10} m + \frac{14}{25} n\right) = f\left(x + \frac{7(15m + 4n)}{50}\right)$$

for all integers m and n. 15 and 4 are relatively prime. Thus the set $\{15m + 4n \big| m, n \in Z\}$ equals the set of all integers and therefore

$$f(x) = f\left(x + \frac{7n}{50}\right)$$

for all integers n which means that f is periodic with period $\dfrac{7}{50}$.

We reduce all arguments modulo $\dfrac{7}{50}$ and get one by one

$$\frac{11}{3} = f\left(\frac{49}{5}\right) = f\left(\frac{49}{5} - 70 \cdot \frac{7}{50}\right) = f(0),$$

$$f(-42) = f\left(-42 + 300 \cdot \frac{7}{50}\right) = f(0) = \frac{11}{3},$$

$$f\left(\frac{28}{50}\right) = f\left(\frac{28}{50} - 4 \cdot \frac{7}{50}\right) = f(0) = \frac{11}{3},$$

and finally

$$f\left(\frac{1}{2} f\left(\frac{28}{50}\right) - \frac{2}{25} f(-42)\right) = f\left(\left(\frac{1}{2} - \frac{2}{25}\right)\frac{11}{3}\right) = f\left(\frac{21}{50} \cdot \frac{11}{3}\right) = \\ = f\left(\frac{77}{50}\right) = f\left(\frac{77}{50} - 11 \cdot \frac{7}{50}\right) = f(0) = \frac{11}{3}$$

.

Solution 3 by Kee-Wai Lau, Hong Kong, China

We show that $f\left(\frac{1}{2} f\left(\frac{28}{50}\right) - \frac{2}{25} f(-42)\right) = \frac{11}{3}$.

From the given functional equation we deduce that

$$\left(f\left(x - \frac{8}{15}\right) - f\left(x + \frac{47}{30}\right)\right)^2 + \left(f\left(x + \frac{47}{30}\right) - f\left(x + \frac{2}{75}\right)\right)^2 + \left(f\left(x + \frac{2}{75}\right) - f\left(x - \frac{8}{15}\right)\right)^2 = 0$$

or

$$f\left(x - \frac{8}{15}\right) = f\left(x + \frac{47}{30}\right) = f\left(x + \frac{2}{75}\right).$$

Let $x = y + \frac{8}{15}$, so that $f(y) = f\left(y + \frac{21}{10}\right) = f\left(y + \frac{14}{25}\right)$. Hence by induction we have

$f(y) = f\left(y + \frac{21m}{10}\right)$ for any integer m, and

$f(y) = f\left(y + \frac{14n}{25}\right)$ for any integer n. Thus for any real number y and integers m and n, we have

$$f(y) = f\left(y + \frac{21m}{10} + \frac{14n}{25}\right) \qquad (1)$$

Putting $y = \frac{28}{50}, m = 2, n = 9$, into (1), we obtain $f\left(\frac{28}{50}\right) = f\left(\frac{49}{5}\right)$.

Putting $y = -42, m = 2, n = 85$, into (1), we obtain $f(-42) = f\left(\frac{49}{5}\right)$, and

Putting $y = \frac{77}{50}, m = 1, n = 11$, into (1), we obtain $f\left(\frac{77}{50}\right) = f\left(\frac{49}{5}\right)$. Hence,

$$f\left(\frac{1}{2}f\left(\frac{28}{50}\right)-\frac{2}{25}f\left(-42\right)\right)=f\left(\left(\frac{1}{2}-\frac{2}{25}\right)f\left(\frac{49}{5}\right)\right)=f\left(\frac{22}{50}\right)\left(\frac{11}{3}\right)=f\left(\frac{77}{50}\right)=f\left(\frac{49}{5}\right)=\frac{11}{3}.$$

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the desired value is 11/3.

If a, b, and c are real numbers with $a^2+b^2+c^2=ab+bc+ca$, then

$$(a-b)^2+(b-c)^2+(c-a)^2=2(a^2+b^2+c^2)-2(ab+bc+ca)=0,$$

so $a=b=c$. Thus for all real numbers x, $f(x-8/15)=f(x+47/30)=f(x+2/75)$. By applying horizontal translations, we obtain the following identities for all real numbers x:

$$f(x)=f(x+21/10)=f(x+14/25);$$

$$f(x-21/10)=f(x)=f(x-77/50);$$

$$f(x-14/25)=f(x+77/50)=f(x).$$

Since $14/25=49/5-(77/50)(6)$, applying the second identity repeatedly yields $f(14/25)=f(49/5)=11/3$. Then the first identity implies $f(0)=f(14/25)=11/3$, so the third identity produces $f(77/50)=f(0)=11/3$. Finally, since $-42=0-(21/10)(20)$, applying the second identity repeatedly yields $f(-42)=f(0)=11/3$. Hence we conclude

$$f\left(\frac{1}{2}f\left(\frac{14}{25}\right)-\frac{2}{25}f\left(-42\right)\right)=f\left(\frac{1}{2}\left(\frac{11}{3}\right)-\frac{2}{25}\left(\frac{11}{3}\right)\right)=f\left(\frac{77}{50}\right)=\frac{11}{3}.$$

Solution 5 by Hyun Bin Yoo, South Korea

Let $X_1=x-\dfrac{8}{15}$, $X_2=x+\dfrac{47}{30}$, $X_3=x+\dfrac{2}{75}$. Then

$$\begin{aligned}&f(X_1)^2+f(X_2)^2+f(X_3)^2=f(X_1)f(X_2)+f(X_2)f(X_3)+f(X_3)f(X_1)\\&\Leftrightarrow (f(X_1)-f(X_2))^2+(f(X_2)-f(X_3))^2+(f(X_3)-f(X_1))^2=0\\&\Leftrightarrow f(X_1)=f(X_2)=f(X_3)\\&\Leftrightarrow f\left(x-\frac{8}{15}\right)=f\left(x+\frac{47}{30}\right)=f\left(x+\frac{2}{75}\right)\end{aligned}$$

Now substitute x with $x+\dfrac{8}{15}$. That gives $f(x)=f\left(x+\dfrac{10}{21}\right)=f\left(x+\dfrac{14}{25}\right)$. In other words, $f(x)=f(x+0.07\cdot 3)=f(x+0.07\cdot 8)$. So $f(x)=f(x+0.07\cdot(3a+8b))$, where a and b are integers. Since $3\cdot 3+8\cdot(-1)=1$, $3a+8b$ can represent any integer k. ($a=3k$, $b=-k$) Then the equation can be further reduced into $f(x)=f(x+0.07\cdot k)$ where k is an integer.

Notice that $\dfrac{49}{5}=0.07\cdot 140$, $\dfrac{28}{50}=0.07\cdot 8$ and $-42=0.07\cdot 600$. Since all three numbers are a multiple of 0.07, there differences are also a multiple of 0.07. So $f\left(\dfrac{49}{5}\right) = f\left(\dfrac{28}{50}\right) = f(-42)$.

$$f\left(\frac{1}{2}f\left(\frac{28}{50}\right) - \frac{2}{25}f(-42)\right) = f\left(\left(\frac{1}{2} - \frac{2}{25}\right)\frac{11}{3}\right) = f(0.07 \cdot 22) = f\left(\frac{49}{5}\right) = \frac{3}{11}$$

Also solved by the proposer.

5623: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let P be an interior point to an equilateral triangle of altitude one. If x, y, z are the distances from P to the sides of the triangle, then prove that

$$x^2 + y^2 + z^2 \geq x^3 + y^3 + z^3 + 6xyz.$$

Solutions 1,2 and 3 by Bruno Salgueiro Fanego, Viveiro, Spain

Solution 1:

Since

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx)$$

and

$$x^3 + y^3 + z^3 = (x + y + z)^3 - 3(x + y + z)(xy + yz + zx) + 3xyz,$$

the inequality to prove is equivalent to

$$(x + y + z)^2 - 2(xy + yz + zx)^3 \geq (x + y + z)^3 - 3(x + y + z)(xy + yz + zx) + 9xyz.$$

From Viviani's theorem, $x + y + z = 1$, so this last inequality is the same as

$$xy + yz + zx \geq 9xyz,$$

that is (since x, y, z are strictly positive because P is interior to the equilateral triangle)

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 9.$$

or, since $x + y + z = 1$, to

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9.$$

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 3 + \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z} = 3 + \frac{x}{y} + \frac{1}{\frac{x}{y}} + \frac{y}{z} + \frac{1}{\frac{y}{z}} + \frac{z}{x} + \frac{1}{\frac{z}{x}} \geq$$

$$\geq 3 + 2 + 2 + 2 = 9,$$

where we have used that $t + \dfrac{1}{t} \geq 2$ for any $t > 0$, with equality if and only if $t = 1$ (which is equivalent to $t^2 - 2t + 1 \geq 0$, that is, to $(t-1)^2 \geq 0$) with $t = \dfrac{x}{y}, t = \dfrac{y}{z}$ and $t = \dfrac{z}{x}$.

Thus, the inequality is proven and also that equality occurs if and only if, $\frac{x}{y} = \frac{y}{z} = \frac{z}{x} = 1$, that is, $x = y = z = \frac{1}{3}$. In plane geometry the barycenter of a triangleis is also called the centroid of the triangle. Is the point in the triangle where the three medians meet. In equilateral triangles, the incenter, circumcenter and orthocenter also meet at the barycenter.

Solution 2:

From Viviani's theorem, $x+y+z=1$, so the equality in the problem is successively equivalent to

$$x^2 - x^3 + y^2 - y^3 + z^2 - z^3 \geq 6xyz,$$

$$x^(1-x) + y^2(1-y) + z^2(1-z) \geq 6xyz,$$

$$x^2(y+z) + y^2(z+x) + z^2(x+y) \geq 6xyz,$$

$$xy(x+y) + yz(y+z) + zx(z+x) \geq 6xyz.$$

Since P is interior to the triangle, $x > 0, y > 0$ and $z > 0$, so this last inequality (and, hence, the inequality in the problem) is equivalent to

$$\frac{z}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z} \geq 6,$$

which is true because, for any $t > 0$, we have $t + \frac{1}{t} \geq 2$ with equality if and only if $t = 1$, so taking $t = x/y$, $t = y/z$, and $t = z/x$ and adding the obtained inequalities we obtain the required result, and note that equality occurs on it if, and only if, $x/y = y/z = z/x = 1$; that is, if and only if $x = y = z = 1/3$. This means that P is the barycenter of the equilateral triangle (see definition above).

Solution 3:

From Viviani's theorem $x + y + z = 1$, and we can think of this as a constraint and apply the Lagrange multiplers method to the funion $L(x, y, z; \lambda) = f(x, y, z) - \lambda g(x, y, z)$ where $f(x, y, z) = x^2 + y^2 + z^2 - x^3 - y^3 - z^3 - 6xyz$ is the objective function that is subjected to the constraint constraint $g(x, y, z) = x + y + z - 1$. The critical points in this expression are points $(x, y, z) \in (0, +\infty)$ for which

$$\frac{\partial L}{\partial x}(x,y,z) = \frac{\partial L}{\partial y}(x,y,z) = \frac{\partial L}{\partial z}(x,y,z) = \frac{\partial L}{\partial \lambda}(x,y,z) = 0.$$

Thus, $2x - 3x^2 - 6yz = 2y - 3y^2 - 6zx = 2z - 3z^2 - 6xy = \lambda$ and $x + y + z = 1$, That is

$0 = 2(x-y) - 3(x^2 - y^2) - 6z(x-y) = (2 - 3x = 3y = 6z)(x-y)$ and

$0 = 2(u-z) - 3(y^2 - z^2) - 6x(y-z) = (2 - 3y - 3z - 6x)(y-z)$ and $x + y + z = 1$. From this it follows that:

$(x,y,z) \in \left\{ \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \ \left(\frac{1}{9},\frac{1}{9},\frac{7}{9}\right) \right\}, \left(\frac{1}{9},\frac{7}{9},\frac{1}{9}\right), \left(\frac{7}{9},\frac{1}{9},\frac{1}{9}\right) \Big\}$ with $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ being the point where f attains its minimum value.

In summary,:

$f(x,y,z) \geq (1/3,1/3,1/3) = 0$ for any $(x,y,z) \in (0,+\infty)$; equality is obtained if, and only if, $(x,y,z) = (1/3,1/3,1/3)$.

Solution 4 by Michael Brozinsky, Central Islip,NY

We first note that if $A+B=C+D$ then $A \geq C$ is equivalent to $B \leq D$. Without loss of generality let $x \leq y \leq z$ and let $k = x+y$ so that $k \leq \dfrac{2}{3}$ since $x+y+z=1$ as the sum of the distances from an interior of point of an equilateral triangle is easily shown (by dissection) to be equal to the altitude. Now since

$$\begin{aligned} x^2+y^2+z^2+2xy+2xz+2yz &= (x+y+z)^2 = 1 = (x+y+z)^3 = \\ &= x^3+y^3+z^3+6xyz+3x^2y+3xy^2+3xz^2+3y^2z+3yz^2 \end{aligned}$$

it suffices to show (by our first note above) that

$$2xy+2xz_+2yz \leq 3x^2y+3xy^2+3x^2+3x^2z+3xz^2+3y^2z+3yz^2$$

or replacing z by$1-k$ and y by $k-x$

$$\begin{aligned} & 2x(k-x)+2x(1-k)+2(k-x)(1-k)-(3x^2(k-x)+3x(k-x)^2+ \\ + \quad & 3x^2(1-k)+3x(1-k)^2+3(k-x)^2(1-k)+3(k-x)(1-k)^2 \leq 0 \end{aligned}$$

which simplifies to

$$(9k-8)x^2+(-3k^2+6k(1-k)+2k)x-3k^2(1-k)-3k(1-k)^2+2k(1-k) \leq 0. \qquad (*)$$

Now the left hand side of $(*)$ viewed as a quadratic in x has a negative leading coefficient since $k \leq \dfrac{2}{3} < \dfrac{8}{9}$, and by completing the square becomes

$$(9k-8)\left(x-\frac{1}{2}k\right)^2 - \frac{1}{4}(-2+3k)\,k$$

which is thus less than or equal to 0 with equality only if

$$k - \frac{2}{3} \textbf{ and } x = \frac{1}{2}k = \frac{1}{3} \textbf{ and thus } y = \frac{1}{3} \textbf{ and } z = \frac{1}{3}.$$

Solution 5 by Titu Zvonaru, Comănesti, Romania

Let a be the side of the equilateral triangle and $h = 1$ its altitude. It is easy to see that $ax + ay + az = ah$, hence $x + y + z = 1$. The desired inequality is equivalent to

$$(x+y+z)(x^2+y^2+z^2) \geq x^3+y^3+z^3+6xyz$$

$$x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2 \geq 6xyz,$$

which is true by the AM-GM inequality for positive numbers $x^2y, xy^2, y^2z, yz^2, x^2z, xz^2$.

Solution 6 by (Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain

By Viviani's theorem, $x + y + z = 1$. By multiplying the left-hand side by $x + y + z$, the inequality becomes homogeneous. Then, after clarifying the inequality becomes

$$x^2y + x^2z + xy^2 + yz^2 + y^2z + xz^2 \geq 6xyz$$

which follows by the AM-GM inequality.

Solution 7 by Arkady Alt, San Jose, California

Let F and a be, respectively, the area and sidelength of the triangle.

Then $F = \dfrac{x \cdot a}{2} + \dfrac{y \cdot a}{2} + \dfrac{z \cdot a}{2} = \dfrac{a \cdot 1}{2}$.Hence, $x+y+z = 1$ and by AM-GM Inequality we have

$x^2 + y^2 + z^2 = \left(x^2 + y^2 + z^2\right)(x + y + z) =$

$x^3 + y^3 + z^3 + \sum\limits_{cyc} x^2 (y + z) \geq x^3 + y^3 + z^3 + \sum\limits_{cyc} x^2 \cdot 2\sqrt{yz} \geq$

$x^3 + y^3 + z^3 + 6\sqrt[3]{x^2y^3z^3} = x^3 + y^3 + z^3 + 6xyz.$

Solution 8 by Albert Natian, Los Angeles Valley College, Valley Glen, California/

Lemma. Suppose $x, y, z \in [0, 1]$ and define

$$s := x + y + z, \quad \sigma := x^2 + y^2 + z^2, \quad w := x^3 + y^3 + z^3, \quad p := xyz.$$

If $s = 1$, then $\sigma \geq w + 6p$.

Proof. Suppose $s = 1$. We will show via the method of Lagrange Multipliers that

$$Q := \sigma - w - 6p \geq 0.$$

But first we make the observation that

$$1 = s^3 = (x + y + z)^3 = -2w + 3\sigma + 6p,$$

$$6p = 1 + 2w - 3\sigma.$$

So

$$Q = \sigma - w - 6p = \sigma - w - (1 + 2w - 3\sigma),$$

$$Q = 4\sigma - 3w - 1.$$

The Lagrangian $\mathcal{L}$, with multiplier λ, is given by

$$\mathcal{L} = 4\sigma - 3w - 1 - \lambda s$$

which yields

$$\frac{\partial \mathcal{L}}{\partial x} = 8x - 9x^2 - \lambda, \quad \frac{\partial \mathcal{L}}{\partial y} = 8y - 9y^2 - \lambda, \quad \frac{\partial \mathcal{L}}{\partial z} = 8z - 9z^2 - \lambda.$$

Setting the latter three partials equal to zero, followed by re-arrangement, we get

$$9x^2 - 8x + \lambda = 0, \quad 9y^2 - 8y + \lambda = 0, \quad 9z^2 - 8z + \lambda = 0,$$

each of which has solution

$$\frac{1}{9}\left(4 \pm \sqrt{16 - 9\lambda}\right)$$

resulting in either $\lambda = 5/3$ or $\lambda = 7/9$. Only for $\lambda = 5/3$ is Q minimized with minimum value 0 at $x = y = z = 1/3$. Thus $\sigma \geq w + 6p$.

If P were an interior point to any triangle of side lengths a, b and c, with x, y, z being, respectively, the distances from P to the sides with lengths a, b and c, then the area $\mathcal{A}$ of the triangle can be expressed as

$$\mathcal{A} = \frac{1}{2}\left(ax + by + cz\right).$$

For an equilateral triangle with side length a,

$$\mathcal{A} = \frac{1}{2}a\left(x + y + z\right).$$

The side a and area $\mathcal{A}$ of an equilateral triangle whose altitude is one are given by $a = 2/\sqrt{3}$ and $\mathcal{A} = 1/\sqrt{3}$. Thus

$$\frac{1}{\sqrt{3}} = \frac{1}{2} \cdot \frac{2}{\sqrt{3}}\left(x + y + z\right),$$

$$x + y + z = 1$$

which allows us to conclude, by the above Lemma, that

$$\sigma \geq w + 6p,$$

$$x^2 + y^2 + z^2 \geq x^3 + y^3 + z^3 + 6xyz.$$

Solution 9 by Samuel Aguilar (student) and the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA

By Viviani's Theorem, $x + y + z = 1$. (See, e.g., Ken-Ichiroh Kawasaki, "Proof Without Words: Viviani's Theorem," *Math. Mag.* **78**(3), 2005, p. 213.)

By the AM-HM Inequality,

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x + y + z}{3} = \frac{1}{3},$$

and

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 9.$$

Thus, multiplying both sides by xyz and subtracting $3xyz$, we get

$$\begin{aligned} yz+zx+xy-3xyz &\geq 6xyz\\ yz(1-x)+zx(1-y)+xy(1-z) &\geq 6xyz\\ x^3+y^3+z^3+yz(y+z)+zx(z+x)+xy(x+y) &\geq x^3+y^3+z^3+6xyz\\ (x+y+z)(x^2+y^2+z^2) &\geq x^3+y^3+z^3+6xyz, \end{aligned}$$

which gives the desired inequality, since $x+y+z=1$.

Also solved by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Michel Bataille, Rouen, France; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Henry Ricardo, Westchester Area Math Circle Purchase, NY; Albert Stadler, Herrliberg, Switzerland, and the proposer.

5624: *Proposed by Seán M. Stewart, Bomaderry, NSW, Australia*

Evaluate: $\displaystyle\int_0^1\left(\frac{\tan^{-1}x-x}{x^2}\right)^2dx$

Solution 1 by Moti Levy, Rehovot, Israel

$$I:=\int_0^1\left(\frac{\arctan(x)-x}{x^2}\right)^2dx=\int_0^1\frac{1}{x^4}\left(\arctan(x)-x\right)^2dx.$$

By integration by parts,

$$\begin{aligned} I&=-\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2+\frac{2}{3}\int_0^1\frac{1}{x\left(x^2+1\right)}\left(x-\arctan(x)\right)dx\\ &=-\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2+\frac{2}{3}\int_0^1\frac{1}{\left(x^2+1\right)}dx-\frac{2}{3}\int_0^1\frac{\arctan(x)}{x\left(x^2+1\right)}dx\\ &=-\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2+\frac{\pi}{6}-\frac{2}{3}\int_0^1\frac{\arctan(x)}{x\left(x^2+1\right)}dx. \end{aligned}$$

Using the partial fractions of $\frac{1}{x(x^2+1)}=\frac{1}{2x}+\frac{1-x^2}{2x(1+x^2)}$, we get

$$\int_0^1\frac{\arctan x}{x\left(x^2+1\right)}dx=\frac{1}{2}\int_0^1\frac{\arctan x}{x}dx+\int_0^1\arctan(x)\frac{1-x^2}{2x\left(1+x^2\right)}dx.$$

The first integral is related to Catalan constant G

$$G=\int_0^1\frac{\arctan x}{x}dx$$

The second integral is simplified by the change of variable $x = \tan\left(\frac{v}{2}\right)$,

$$\int_0^1 \arctan(x)\,\frac{1-x^2}{2x\,(1+x^2)}dx = \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{v}{\tan(v)}dv,$$

and by integration by parts

$$\int_0^{\frac{\pi}{2}} \frac{v}{\tan(v)}dv = -\int_0^{\frac{\pi}{2}} \ln(\sin(x))\,dx.$$

It is well known that

$$\int_0^{\frac{\pi}{2}} \ln(\sin(x))\,dx = -\frac{\pi}{2}\ln(2)\,,$$

hence

$$\int_0^1 \arctan(x)\,\frac{1-x^2}{2x\,(1+x^2)}dx = \frac{\pi}{8}\ln(2)\,.$$

We conclude that

$$I = \frac{1}{3}\pi - \frac{1}{12}\pi\ln 2 - \frac{1}{48}\pi^2 - \frac{1}{3} - \frac{1}{3}G.$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA

With

$$\tan^{-1} x = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1},$$

it follows that

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = K,$$

where K is Catalan's constant. Next, let

$$I_1 = \int_0^{\pi/4} \ln(\cos\theta)\,d\theta \quad \text{and} \quad I_2 = \int_0^{\pi/4} \ln(\sin\theta)\,d\theta.$$

Then,

$$I_2 - I_1 = \int_0^{\pi/4} \ln(\tan\theta)\,d\theta = \int_0^1 \frac{\ln x}{1+x^2}\,dx = -\int_0^1 \frac{\tan^{-1} x}{x}\,dx = -K,$$

and

$$I_1 + I_2 = \int_0^{\pi/4} \ln\left(\frac{1}{2}\sin 2\theta\right)\,d\theta = -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta)\,d\theta = -\frac{\pi}{2}\ln 2.$$

Consequently,

$$\int_0^{\pi/4} \ln(\cos\theta)\,d\theta = \frac{1}{2}K - \frac{\pi}{4}\ln 2 \quad \text{and} \quad \int_0^{\pi/4} \ln(\sin\theta)\,d\theta = -\frac{1}{2}K - \frac{\pi}{4}\ln 2.$$

Now, by integration by parts and then partial fractions,

$$\begin{aligned}
\int_0^1 \left(\frac{\tan^{-1}x - x}{x^2}\right)^2 dx &= -\frac{(\tan^{-1}x - x)^2}{3x^3}\bigg|_0^1 - \frac{2}{3}\int_0^1 \frac{\tan^{-1} -x}{x(1+x^2)}\,dx \\
&= -\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2 - \frac{2}{3}\int_0^1 \frac{\tan^{-1}x}{x}\,dx + \frac{2}{3}\int_0^1 \frac{x\tan^{-1}x}{1+x^2}\,dx + \frac{2}{3}\int_0^1 \frac{1}{1+x^2}\,dx \\
&= -\frac{\pi^2}{48} + \frac{\pi}{3} - \frac{1}{3} - \frac{2}{3}K + \frac{2}{3}\int_0^1 \frac{x\tan^{-1}x}{1+x^2}\,dx.
\end{aligned}$$

With the substitution $x = \tan\theta$ and then integration by parts,

$$\begin{aligned}
\int_0^1 \frac{x\tan^{-1}x}{1+x^2}\,dx &= \int_0^{\pi/4} \theta\tan\theta\,d\theta \\
&= -\theta\ln(\cos\theta)\bigg|_0^{\pi/4} + \int_0^{\pi/4} \ln(\cos\theta)\,d\theta \\
&= \frac{\pi}{8}\ln 2 + \frac{1}{2}K - \frac{\pi}{4}\ln 2 = \frac{1}{2}K - \frac{\pi}{8}\ln 2.
\end{aligned}$$

Finally,

$$\int_0^1 \left(\frac{\tan^{-1}x - x}{x^2}\right)^2 dx = -\frac{\pi^2}{48} + \frac{\pi}{3} - \frac{1}{3} - \frac{1}{3}K - \frac{\pi}{12}\ln 2.$$

Solution 3 Michel Bataille, Rounen, France

Let I be the integral to be evaluated. We show that $I = \frac{\pi}{3} - \frac{\pi\ln 2}{12} - \frac{\pi^2}{48} - \frac{G+1}{3}$ where G denotes the Catalan number defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \int_0^1 \frac{\tan^{-1}x}{x}.$$

For later use here is a result linked to G and whose proof is postponed to the end:

$$\int_0^1 \frac{x\tan^{-1}x}{1+x^2}\,dx = \frac{G}{2} - \frac{\pi\ln 2}{8}. \tag{1}$$

Now, let a be such that $0 < a < 1$. Then, expanding the integrand and because $\int_a^1 \frac{dx}{x^2} = \frac{1}{a} - 1$, we see that

$$I = \lim_{a\to 0^+} \int_a^1 \left(\frac{\tan^{-1}x - x}{x^2}\right)^2 dx = \lim_{a\to 0^+}\left(J(a) - 2K(a) + \frac{1}{a} - 1\right) \tag{2}$$

where

$$J(a) = \int_a^1 \frac{(\tan^{-1}x)^2}{x^4}\,dx \qquad \text{and} \qquad K(a) = \int_a^1 \frac{\tan^{-1}x}{x^3}\,dx.$$

In the following calculations, since we are interested in the limit as $a \to 0^+$, we gather in $o(1)$ the terms that tend to 0 as $a \to 0^+$.

First, integrating by parts, we obtain

$$\begin{aligned}
K(a) &= \int_a^1 (\tan^{-1} x)\, d\left(\frac{-1}{2x^2}\right) \\
&= \left[\frac{-\tan^{-1} x}{2x^2}\right]_a^1 + \frac{1}{2}\int_a^1 \frac{dx}{x^2(1+x^2)} \\
&= -\frac{\pi}{8} + \frac{\tan^{-1} a}{2a^2} + \frac{1}{2}\left(\int_a^1 \frac{dx}{x^2} - \int_a^1 \frac{dx}{1+x^2}\right) \\
&= -\frac{\pi}{8} + \frac{\tan^{-1} a}{2a^2} + \frac{1}{2a} - \frac{1}{2} - \frac{\pi}{8} + o(1) = -\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2a} + \frac{\tan^{-1} a}{2a^2} + o(1).
\end{aligned}$$

Similarly $J(a) = \int_a^1 (\tan^{-1} x)^2\, d\left(\frac{-1}{3x^3}\right)$ yields:

$$\begin{aligned}
J(a) &= -\frac{\pi^2}{48} + \frac{(\tan^{-1} a)^2}{3a^3} + \frac{2}{3}\int_a^1 \frac{\tan^{-1} x}{x^3(1+x^2)}\, dx \\
&= -\frac{\pi^2}{48} + \frac{(\tan^{-1} a)^2}{3a^3} + \frac{2}{3}\left(\int_a^1 \frac{\tan^{-1} x}{x^3}\, dx - \int_a^1 \frac{\tan^{-1} x}{x}\, dx + \int_a^1 \frac{x\tan^{-1} x}{1+x^2}\, dx\right) \\
&= -\frac{\pi^2}{48} + \frac{(\tan^{-1} a)^2}{3a^3} + \frac{2}{3}\left(K(a) - G + o(1) + \frac{G}{2} - \frac{\pi\ln 2}{8} + o(1)\right).
\end{aligned}$$

Finally,

$$J(a) = -\frac{\pi^2}{48} + \frac{(\tan^{-1} a)^2}{3a^3} - \frac{\pi}{6} - \frac{G+1}{3} + \frac{1}{3a} + \frac{\tan^{-1} a}{3a^2} - \frac{\pi\ln 2}{12} + o(1)$$

and returning to (2), we obtain

$$J(a) - 2K(a) + \frac{1}{a} - 1 = \frac{\pi}{3} - \frac{\pi^2}{48} - \frac{\pi\ln 2}{12} - \frac{G+1}{3} + \frac{(a-\tan^{-1} a)^2}{3a^3} + o(1).$$

Since $a - \tan^{-1} a \sim \frac{a^3}{3}$ as $a \to 0$, we have $\frac{(a-\tan^{-1} a)^2}{3a^3} = o(1)$ and the announced result follows.

Proof of (1) The substitution $x = \tan t$ and an integration by parts in succession give

$$\int_0^1 \frac{x\tan^{-1} x}{1+x^2}\, dx = \int_0^{\pi/4} t\cdot\tan t\, dt = \int_0^{\pi/4} td(-\ln(\cos t))\, dt = \frac{\pi\ln 2}{8} + \int_0^{\pi/4} \ln(\cos t)\, dt$$

and (1) follows since

$$\int_0^{\frac{\pi}{4}} \ln(\cos x)\, dx = \frac{G}{2} - \frac{\pi\ln(2)}{4},$$

a result that I have proved in *Math. Gazette*, Vol. 86, March 2004, p. 156. [for convenience, the proof is repeated below].
Let

$$U = \int_0^{\frac{\pi}{4}} \ln(\cos t)\, dt \quad\text{and}\quad V = \int_0^{\frac{\pi}{4}} \ln(\sin t)\, dt.$$

Then

$$\begin{aligned}
V + U &= \int_0^{\pi/4} \ln\left(\frac{1}{2}\sin(2t)\right) dt = \frac{\pi}{4}\ln(1/2) + \int_0^{\pi/4} \ln(\sin(2t)) dt \\
&= -\frac{\pi}{4}\ln(2) + \frac{1}{2}\int_0^{\pi/2} \ln(\sin u) du = -\frac{\pi}{2}\ln(2)
\end{aligned}$$

and

$$V-U=\int_0^{\pi/4}\ln(\tan t)dt=[t\ln(\tan t)]_0^{\pi/4}-\int_0^{\pi/4}t\,\frac{1+\tan^2 t}{\tan t}dt$$
$$=-2\int_0^{\pi/4}\frac{t}{\sin(2t)}dt=-\frac{1}{2}\int_0^{\pi/2}\frac{u}{\sin(u)}du=-G$$

where the last equality follows from

$$\int_0^{\pi/2}\frac{u}{\sin(u)}du=2\int_0^1\frac{\tan^{-1}(x)}{x}\,dx$$

[substitution $u=2\tan^{-1}(x)$].
Thus, $2U=-\frac{\pi}{2}\ln(2)+G$ and, as a bonus, $2V=-\frac{\pi}{2}\ln(2)-G$.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the given integral, denote by I equals

$$\frac{(16-4\ln 2-\pi)\pi-16-16G}{48}=0.0214\cdots,$$

where G is Catalan's constant $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$. Integrating by parts, we have

$$I=\frac{1}{3}\int_0^1\left(\tan^{-1}x-x\right)^2 d\left(\frac{1}{x^3}\right)=-\frac{1}{3}\left(\frac{\pi}{4}-1\right)^2-\frac{2}{3}J,$$

where $J=\int_0^1\frac{\tan^{-1}x-x}{x(1+x^2)}dx$. By the substitution $x=\tan\theta$, we obtain $J=K-\frac{\pi}{4}$ where $K=\int_0^{\pi/4}\theta\cot\theta d\theta$. It is known ([1], p.434, section 3.747 entry 8) that $K=\frac{\pi}{8}\ln 2+\frac{1}{2}G$.

Hence our claim for I.

Reference:

1. I.S. Gradshteyn and I.M. Ryzhik: *Table of Integrals,Series and Products,*Seventh Edition, Elsevier, Inc. 2007.

Solution 5 by Albert Stadler, Herrliberg, Switzerland

Let $G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ be Catalan's constant. It is known (see for instance https://en.wikipedia.org/wiki/Catalan%27s_constant) that G can be represented by integrals, specifically,

$$G=\int_0^1\frac{\arctan x}{x}dx=\int_0^{\frac{\pi}{4}}\ln\cot x dx.$$

We note that

$$\int_0^1 \frac{x\arctan x}{1+x^2}dx \underbrace{=}_{x=\tan y} \int_0^{\frac{\pi}{4}} y\tan y dy = -y\ln\cos y\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \ln\cos y dy =$$

$$= -\frac{\pi}{4}\ln\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\cot x dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\sin y dy + \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln\cos y dy =$$

$$= \frac{\pi}{8}\ln 2 + \frac{1}{2}G + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\cos y dy = \frac{1}{2}G - \frac{\pi}{8}\ln 2,$$

because

$$I = \int_0^{\frac{\pi}{2}} \ln\cos y dy = \int_0^{\frac{\pi}{2}} \ln\sin y dy =$$

and thus

$$2I = \int_0^{\frac{\pi}{2}} (\ln\cos y + \ln\sin y)dy = \int_0^{\frac{\pi}{2}} \ln(\sin y\cos y)dy = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin(2y)\right)dy =$$

$$= -\frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{2}} \ln(\sin(2y))dy = -\frac{\pi}{2}\ln 2 + \int_0^{\pi} \ln(\sin(y))dy = -\frac{\pi}{2}\ln 2 + I.$$

So $I = \int_0^{\frac{\pi}{2}} \ln\cos y dy = -\frac{\pi}{2}\ln 2.$

After these preliminary remarks let's turn to the integral of the problem statement. We integrate by parts and get

$$\int_0^1 \left(\frac{\arctan x - x}{x^2}\right)^2 dx = -\frac{1}{3}x^{-3}(\arctan x - x)^2\Big|_0^1 + \int_0^1 \frac{2}{3}x^{-3}(\arctan x - x)\left(\frac{-x^2}{1+x^2}\right)dx =$$

$$= \frac{-1}{3}\left(\frac{\pi}{4}-1\right)^2 - \frac{2}{3}\int_0^1 (\arctan x - x)\frac{1}{x(1+x^2)}dx =$$

$$= \frac{-1}{3}\left(\frac{\pi}{4}-1\right)^2 - \frac{2}{3}\int_0^1 (\arctan x - x)\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx =$$

$$= \frac{-1}{3}\left(\frac{\pi}{4}-1\right)^2 - \frac{2}{3}\int_0^1 \frac{\arctan x}{x}dx + \frac{2}{3}\int_0^1 dx + \frac{2}{3}\int_0^1 \frac{x\arctan x}{1+x^2}dx - \frac{2}{3}\int_0^1 2\left(1 - \frac{1}{1+x^2}\right)dx =$$

$$\frac{-1}{3}\left(\frac{\pi}{4}-1\right)^2 - \frac{2}{3}G + \frac{2}{3} + \frac{1}{3}G - \frac{\pi}{12}\ln 2 - \frac{2}{3}\left(1 - \frac{\pi}{4}\right) =$$

$$= -\frac{1}{3} - \frac{1}{3}G + \frac{\pi}{3} - \frac{\pi^2}{48} - \frac{\pi}{12}\ln 2$$

Also solved by Albert Natian, Los Angeles Valley College, Valley Glen, CA, and by the proposer.

Editor's Note

Time passes quickly, and this is especially true when one is having fun doing what they do. I took over the editorship of this column in 2001 and now 20 years have flown by, and hard as it is for me to admit, it is time for me to step down.

My association with this Column goes back much further than 20 years and I can recall attempting some of the problems in the Column when I was a student in high school, where our librarian had the good sense to have a school subscription to the SSMJ. Over the years I have developed wonderful relationships with so many of you, and I deeply appreciate your collegiality and the help and support you have given to me. Normally I would list the names of those of you who went above and beyond in helping our column become as popular as it is within the SSMA community and also among other problem solvers of columns in other journals. But there are so many of you that have helped me over the years that I would for sure inadvertently omit some of your names, and then feel terribly, terribly embarrassed by the omissions. So let me just say to you one and all, "thank you."

Albert Natian of Los Angeles Valley College in Valley Glen, California will be the next editor. Starting with this issue, please send him your solutions and proposals. His details are listed bellow.

To propose problems, email them to:

problems4ssma@gmail.com

To propose solutions, email them to:

solutions4ssma@gmail.com

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To propose problems and solutions via regular mail, send them to:

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