Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email. To propose problems, email them to: problems4ssma@gmail.com. To propose solutions, email them to: solutions4ssma@gmail.com. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before September15, 2021

- **5643:** *Proposed by Kenneth Korbin, New York, NY*

  Let K be the product of three different prime numbers each congruent to 1(mod4). How many different Pythagorean triangles have hypotenuse K^2?

- **5644:** *Proposed by Albert Natian, Los Angles Valley College, Valley Glen, CA*

  Pete, Paul and Tom are three sides of a triangle. Tom is originally a perfect 10. Tom becomes 8 when 1 is borrowed from Pete and paid Paul. However Tom becomes 4 when 1 is borrowed from Paul and paid Pete, instead. It turns out none of this borrowing and paying makes any difference on the area of the triangle. What is the area? And who is Pete and Paul?

  (The creation of this problem was inspired by Problem 5553 proposed by Kenneth Korbin in the 2019 October issue of this Journal. My thanks to Kenneth Korbin!)

- **5645:** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turna-Severin, Romania*

  Prove: If $a, b, c > 1$ and for all values of $x > 1$,

$$\log_a x^2 + \log_b x^2 + \log_c x^2 \geq \log_a 2^x + \log_b 2^x + \log_c 2^x, \quad \text{then :}$$

$$\log_a \left(\frac{2}{e}\right) + \log_b \left(\frac{2}{e}\right) + \log_c \left(\frac{2}{e}\right) = 0.$$

- **5646:** *Proposed by Michel Bataille, Rouen, France*

  If a, b are real numbers such that $b > a > 0$. Find the minimal value of MN if M is a point of the parabola $y = ax^2$ and N is a point of the parabola $y = bx^2 + 1$.

- **5647:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

  Find all real solutions to the following system of equations

$$\begin{cases} 3\left(\frac{1}{16}\right)^x + 2\left(\frac{1}{27}\right)^y &= \frac{13}{6}, \\ 3\log_{\frac{1}{16}} x - 2\log_{\frac{1}{27}} y &= \frac{5}{6}. \end{cases}$$

- **5648:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

  Let $a, b > 0$ and $\alpha \in (01]$ be real numbers. Calculate:

$$I(\alpha) = \int_0^1 \frac{x^a - x^b}{(-\ln x)^\alpha} dx.$$

Solutions

- **5625:** *Proposed by Kenneth Korbin, New York, NY*

  Trapezoid $ABCD$ with integer length sides is inscribed in a circle with diameter y^4.

$$\begin{aligned} \text{Side } \overline{AB} &= \overline{CD} = 3xy^3 - 4x^3y \\ \text{Side } \overline{BC} &= 2x^2y^2 - y^4. \end{aligned}$$

  Express the length of side $\overline{AD}$ in terms of positive integers x and y.

Solution 1 by Albert Stadler Herrliberg, Switzerland

A cyclic quadrilateral with successive sides a, b, c, d and semiperimeter s has the circumradius (the radius of the circumcircle) given by

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

(Formula of the Indian mathematician Vatasseri Parameshvara, 15th century, see https://en.wikipedia.org/wiki/Cyclicquadrilateral).

In our case we have $a = c = \overline{AB} = \overline{CD} = 3xy^3 - 4x^3y,\ b = BC = 2x^2y^2 - y^4$, and the formula collapses to

$$R = a\sqrt{\frac{(a^2+bd)}{(2a-b+d)(2a+b-d)}}.$$

Using $a, b > 0$ combined with $x, y > 0$ implies

$$0.707 \approx \frac{1}{\sqrt{2}} < \frac{x}{y} < \frac{\sqrt{3}}{2} \approx 0.866. \tag{*}$$

We solve for d and find

$$d = \frac{b(2R^2 - a^2) \pm a\sqrt{(2R-a)(2R+a)(2R-b)(2R+b)}}{2R^2} =$$

$$= \frac{-(-2x^2+y^2)^2(16x^4 - 16x^2y^2 + y^4) \pm 4(-4x^3 + 3xy^2)x(2x-y)(y-x)(x+y)(2x+y)}{y^4}.$$

The + sign gives

$$d = -8x^4 + 8x^2y^2 - y^4 =$$

$$= -8\left(x - \frac{\sqrt{2-\sqrt{2}}}{2}y\right)\left(x + \frac{\sqrt{2-\sqrt{2}}}{2}y\right)\left(x - \frac{\sqrt{2+\sqrt{2}}}{2}y\right)\left(x + \frac{\sqrt{2+\sqrt{2}}}{2}y\right).$$

We note that $d > 0$ if x/y satisfies $(*)$.

The $-$ sign gives $d = -\dfrac{128x^8 - 256x^6y^2 + 160x^4y^4 - 32x^2y^6 + y^8}{y^4}$.

We see that $128x^8 - 256x^6y^2 + 160x^4y^4 - 32x^2y^6 + y^8 = 128\prod_{j=1}^{8}(x - r_jy)$ where each r_j is one of the eight different numbers $\pm\frac{1}{2}\sqrt{2 \pm \sqrt{2 \pm \sqrt{2}}}$ obtained by taking either the + sign or the $-$ sign. The value of d is positive if

$$0.831 \approx \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}} < \frac{x}{y} < \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} \approx 0.98$$

The value of d is integral if for instance y divides x^2.

To sum up: we have either $d = -8x^4 + 8x^2y^2 - y^4$ with $\dfrac{1}{\sqrt{2}} < \dfrac{x}{y} < \dfrac{\sqrt{3}}{2}$ or

$$d = -\frac{12x^8 - 256x^6y^2 + 160x^4y^4 - 32x^2y^6 + y^8}{y^4} \text{ with } \frac{1}{\sqrt{2}} < \frac{x}{y} < \frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}}$$

and y^4 divides $128x^6(x^2 - 2y^2)$.

Here are a few examples for the two cases:

x	y	a	b	c	d	R
3	4	144	32	144	248	128
15	18	19440	40824	19440	73224	52448
15	18	19440	40824	19440	2824	52448
170	200	149600000	712000000	149600000	966320000	800000000
170	200	149600000	712000000	149600000	432782072	800000000

Solution 2 by Michel Bataille, Rouen, France

The answer, $AD = 8x^2y^2 - y^4 - 8x^4$, will be obtained *via* the following ten steps:
(1) since $AB, BC > 0$, we have $\frac{4x^2}{3} < y^2 < 2x^2$.
(2) $y^8 - BC^2 = y^8 - 4x^4y^4 - y^8 + 4x^2y^6 = 4x^2y^4(y^2 - x^2)$.
(3) Similarly, a simple calculation gives $y^4 - AB = y(y+x)(2x-y)^2$, $y^4 + AB = y(y-x)(2x+y)^2$ so that $y^8 - AB^2 = y^2(y^2-x^2)(4x^2-y^2)^2$.
(4) Ptolemy's Theorem yields $AC^2 = AC \cdot BD = AB \cdot CD + BC \cdot AD = AB^2 + BC \cdot AD$.
(5) Let $\alpha = \angle CAD = \angle BDA$ and $\beta = \angle BDC = \angle BAC$. Since $AC^2 > AB^2$ (from (4)) we have

$$\cos\alpha = \frac{AC^2 + AD^2 - AB^2}{2AC \cdot AD} > 0.$$

(6) The calculation of $2AB^2 - BC^2$ easily gives $2AB^2 - BC^2 = -x^6y^2P\left(\frac{y^2}{x^2}\right)$ where $P(t) = t^3 - 22t^2 + 28t - 32$. Since $P'(t) = 3t^2 - 44t + 28$ is negative when $t \in (1,2)$, P is decreasing on $[1,2]$ and since $P(1) < 0$, we have $P\left(\frac{y^2}{x^2}\right) < 0$ (since $1 < \frac{y^2}{x^2} < 2$ by (1)). It follows that $2AB^2 > BC^2$ so that

$$\cos\beta = \frac{AC^2 + AB^2 - BC^2}{2AB \cdot AC} > \frac{2AB^2 - BC^2}{2AB \cdot AC} > 0.$$

(7) From (4) and $AC^2 = AB^2 + BC^2 - 2AB \cdot BC\cos(\pi - (\alpha+\beta))$, we obtain

$$AD = BC + 2AB\cos(\alpha+\beta).$$

(8) From the Law of Sines, $AB = y^4\sin\alpha$, $BC = y^4\sin\beta$.
(9) From (8), taking (5) and (6) into account and using (2), (3) and (1), we have

$$\cos\alpha = \sqrt{1 - \frac{AB^2}{y^8}} = \frac{\sqrt{y^8 - AB^2}}{y^4} = \frac{(4x^2-y^2)\sqrt{y^2-x^2}}{y^3}$$

and similarly, $\cos\beta = \frac{2x\sqrt{y^2-x^2}}{y^2}$. It follows that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{2x(4x^2-y^2)(y^2-x^2)}{y^5} - \frac{AB \cdot BC}{y^8} = \frac{x}{y}$$

(after a short calculation).
(10) Finally, from (7),

$$AD = 2x^2y^2 - y^4 + \frac{2x}{y}(3xy^3 - 4x^3y) = 8x^2y^2 - y^4 - 8x^4.$$

Solution 3 by Daniel Văcaru, Pitesti, Romania

We have two situations, when $BC > AD$ and $BC < AD$.

We obtain $\overline{AB} = \overline{CD} = y^4\left[3 \cdot \frac{x}{y} - 4\left(\frac{x}{y}\right)^3\right]$. Consider $d \perp AD, BC$ and d is containing the middle of AD and BC and let $\frac{x}{y} = \sin\theta$. Then we have $\overline{AB} = \overline{CD} = y^4\sin 3\theta$, $\overline{BC} = y^4\left[2\left(\frac{x}{y}\right)^2 - 1\right] = y^4(-\cos 2\theta) = y^4\sin\left(2\theta - \frac{\pi}{2}\right)$. We obtain $\overline{AD} =$

$y^4\sin\left[\left(2\theta-\frac{\pi}{2}\right)+6\theta\right]=y^4\sin\left(-\frac{\pi}{2}-8\theta\right)=-y^4\cos 8\theta=-y^4\left(8\cos^4 2\theta-8\cos^2 2\theta+1\right)=$ $-y^4\left[8\left(2\left(\frac{x}{y}\right)^2-1\right)^4-8\left(2\left(\frac{x}{y}\right)^2-1\right)^2+1\right]$ and $\overline{AD}=y^4\sin\left[\left(2\theta-\frac{\pi}{2}\right)-6\theta\right]=$ $y^4\sin\left(-\frac{\pi}{2}-4\theta\right)=-y^4\cos 4\theta=-y^4\left(2\cos^2 2\theta-1\right)=-y^4\left[2\left(2\left(\frac{x}{y}\right)^2-1\right)^2-1\right]=$ $-y^4\left(8\frac{x^4}{y^4}-8\frac{x^2}{y^2}+1\right)=-8x^4+8x^2y^2-y^4.$

Solution 4 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA

Parameshvara's formula (c.f. 1) for the circumradius R of an inscribed quadrilateral is given by

$$4R=\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}},$$

where $a=AB$, $b=BC$, $c=CD$, $d=AD$, and $s=\frac{a+b+c+d}{2}$ is the semiperimeter of the quadrilateral. Since $a=c$, we have $s=a+\frac{b+d}{2}$ and

$$(s-a)(s-b)(s-c)(s-d)=\left(\frac{b+d}{2}\right)^2\left(a^2-\left(\frac{d-b}{2}\right)^2\right)=\frac{1}{16}(b+d)^2(4a^2-(d-b)^2).$$

In addition, $(ab+cd)(ac+bd)(ad+bc)=a^2(b+d)^2(a^2+bd)$, so

$$4R=\frac{4a(b+d)\sqrt{a^2+bd}}{(b+d)\sqrt{4a^2-(d-b)^2}}=\frac{4a\sqrt{a^2+bd}}{\sqrt{4a^2-(d-b)^2}},$$

and $a\sqrt{a^2+bd}=R\sqrt{4a^2-(d-b)^2}$. Squaring both sides and substituting the diameter $D=2R$ gives

$$\begin{aligned} a^4+a^2bd&=a^2D^2-D^2(d-b)^2/4\\ D^2(d-b)^2+4a^2bd&=4a^2D^2-a^4\\ D^2d^2+2b(2a^2-D^2)d+4a^4-4a^2D^2+b^2D^2&=0. \end{aligned}$$

Using the quadratic formula to solve this equation for d, the discriminant Δ is given by

$$\begin{aligned} \Delta&=4b^2(2a^2-D^2)^2-4D^2(4a^4-4a^2D^2+b^2D^2)\\ &=16a^4b^2-16a^2b^2D^2-16a^4D^2+16a^2D^4\\ &=16a^2(D^2-a^2)(D^2-b^2). \end{aligned}$$

Substituting the given values $a=xy(3y^2-4x^2)$, $b=y^2(2x^2-y^2)$, and $D=y^4$ gives $D^2-a^2=y^2(4x^2-y^2)^2(y^2-x^2)$, $D^2-b^2=4x^2y^4(y^2-x^2)$, and

$$\sqrt{\Delta}=8x^2y^4(3y^2-4x^2)(4x^2-y^2)(y^2-x^2).$$

Thus, the two solutions for d are

$$d_1=-8x^4+8x^2y^2-y^4$$

and

$$d_2 = \left(-128x^8 + 256x^6y^2 - 160x^4y^4 + 32x^2y^6 - y^8\right)/y^4.$$

Since $a = xy(3y^2 - 4x^2)$ and $b = y^2(2x^2 - y^2)$ are positive integers, then

$$4 - 2\sqrt{2} < 4/3 < y^2/x^2 < 2 < 4 + 2\sqrt{2},$$

and hence

$$-2\sqrt{2}x^2 < y^2 - 4x^2 < 2\sqrt{2}x^2,$$

which means that

$$\begin{aligned}(y^2 - 4x^2)^2 &< \left(2\sqrt{2}x^2\right)^2 \\ y^4 - 8x^2y^2 + 16x^4 &< 8x^4 \\ 0 &< -8x^4 + 8x^2y^2 - y^4.\end{aligned}$$

Thus, $d_1 = -8x^4 + 8x^2y^2 - y^4$ is a positive integer solution for each feasible pair of positive integers x and y. The first few values for x, y, and the side lengths of the trapezoid are given below.

x	y	a	b	d_1
3	4	144	32	248
4	5	220	175	527
5	6	240	504	904
5	7	1645	49	2399
6	7	126	1127	1343

In order for d_2 to be positive, we additionally require that $y < 1.20269x$; in order for d_2 to be an integer, we require that $y^4 \mid 128x^6(2y^2 - x^2)$. Some pairs of positive integers (x, y) that give positive integer values for d_2 are given in the following table. Notice that positive integer scalar multiples of the listed pairs (x, y) will also give solutions (such as $(30, 36)$, $(45, 54)$, etc.)

x	y	a	b	d_2
15	18	19,440	40,824	2824
42	49	302,526	2,705,927	2,157,503
143	169	93,937,129	352,357,057	174,695,903
170	200	149,600,000	712,000,000	432,782,072

References

1 **Larry Hoehn, Circumradius of a cyclic quadrilateral,** *Math. Gazette*, **Mar. 2000, 84 (499), 69-70.**

Also solved by the proposer

- **5626:** *Roger Izard, Dallas, TX*

In triangle ABC, cevians AF, BE, and CD are drawn so that they intersect at point O. Prove or disprove that

$$\overline{AC} \cdot \overline{EO} \cdot \overline{OC} \cdot \overline{DB} \cdot \overline{FB} = \overline{AB} \cdot \overline{OD} \cdot \overline{OB} \cdot \overline{CE} \cdot \overline{CF}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland

There are various properties of the ratios of lengths formed by three cevians all passing through the same arbitrary interior point (see for instance https://en.wikipedia.org/wiki/Cevian), a few of which are

$$\frac{\overline{AD}}{\overline{BD}} \cdot \frac{\overline{BF}}{\overline{CF}} \cdot \frac{\overline{CE}}{\overline{AE}} = 1 \qquad (Ceva'sTheorem)$$

$$\frac{\overline{BO}}{\overline{EO}} = \frac{\overline{BD}}{\overline{AD}} + \frac{\overline{BF}}{\overline{CF}}, \qquad \frac{\overline{CO}}{\overline{DO}} = \frac{\overline{CE}}{\overline{AE}} + \frac{\overline{CF}}{\overline{BF}}.$$

Thus

$$\frac{\overline{AD} \cdot \overline{EO} \cdot \overline{CO} \cdot \overline{BD} \cdot \overline{BF}}{\overline{AB} \cdot \overline{DO} \cdot \overline{BO} \cdot \overline{CE} \cdot \overline{CF}} = \left(\frac{\overline{AC}}{\overline{AB}}\right)\left(\frac{1}{\frac{\overline{BO}}{\overline{EO}}}\right)\left(\frac{\overline{CO}}{\overline{DO}}\right)\left(\frac{\overline{BD}}{\overline{CE}}\right)\left(\frac{\overline{BF}}{\overline{CF}}\right) =$$

$$= \frac{(\overline{AE} + \overline{CE})}{(\overline{AD} + \overline{BD})}\left(\frac{1}{\frac{\overline{BD}}{\overline{AD}} + \frac{\overline{BF}}{\overline{CF}}}\right)\left(\frac{\overline{CE}}{\overline{AE}} + \frac{\overline{CF}}{\overline{BF}}\right)\frac{\overline{BD}}{\overline{CE}}\frac{\overline{BF}}{\overline{CF}} =$$

$$= \frac{\left(\frac{\overline{AE}}{\overline{CE}} + 1\right)}{\left(\frac{\overline{AD}}{\overline{BD}} + 1\right)}\left(\frac{1}{\frac{\overline{BD}}{\overline{AD}} + \frac{\overline{BF}}{\overline{CF}}}\right)\left(\frac{\overline{CE}}{\overline{AE}} + \frac{\overline{CF}}{\overline{BF}}\right)\frac{\overline{BF}}{\overline{CF}} =$$

$$= \frac{\left(\frac{\overline{AD}}{\overline{BD}} \cdot \frac{\overline{BF}}{\overline{CF}} + 1\right)}{\left(\frac{\overline{AD}}{\overline{BD}} + 1\right)}\left(\frac{1}{\frac{\overline{BD}}{\overline{AD}} + \frac{\overline{BF}}{\overline{CF}}}\right)\left(\frac{1}{\frac{\overline{AD}}{\overline{BD}} \cdot \frac{\overline{BF}}{\overline{CF}}} + \frac{\overline{CF}}{\overline{BF}}\right)\frac{\overline{BF}}{\overline{CF}} =$$

$$\frac{(\overline{AD} \cdot \overline{BF} + \overline{BD} \cdot \overline{CF})}{(\overline{AD} \cdot \overline{CF} + \overline{BD} \cdot \overline{CF})}\left(\frac{\overline{AD} \cdot \overline{CF}}{\overline{BD} \cdot \overline{CF} + \overline{BF} \cdot \overline{AD}}\right)\left(\frac{\overline{BD} \cdot \overline{CF} + \overline{AD} \cdot \overline{CF}}{\overline{AD} \cdot \overline{BF}}\right)\frac{\overline{BF}}{\overline{CF}} = 1$$

Solution 2 by Michel Bataille, Rouen, France

We prove the given equality.
Let α, β, γ be the real numbers with $\alpha, \beta, \gamma \neq 0, 1$ such that $\alpha + \beta + \gamma = 1$ and $O = \alpha A + \beta B + \gamma C$. [$(\alpha : \beta : \gamma)$ are the areal coordinates of O relatively to (A, B, C)]. Then, we have

$$(\beta + \gamma)F = \beta B + \gamma C, \quad (\gamma + \alpha)E = \alpha A + \gamma C, \quad (\alpha + \beta)D = \alpha A + \beta B,$$

from which we deduce

$$\beta\overline{FB} + \gamma\overline{FC} = 0, \quad (\alpha + \beta)\overline{BD} = \alpha\overline{BA}, \quad (\gamma + \alpha)\overline{CE} = \alpha\overline{CA}.$$

On the other hand, from $O=(\alpha+\beta)D+\gamma C$, we obtain $(\alpha+\beta)\overline{OD}+\gamma\overline{OC}=0$. Similarly, $(\gamma+\alpha)\overline{OE}+\beta\overline{OB}=0$ and therefore $\overline{AC}\cdot\overline{EO}\cdot\overline{OC}\cdot\overline{DB}\cdot\overline{FB}$ is equal to

$$\left(-\frac{\gamma+\alpha}{\alpha}\overline{CE}\right)\left(\frac{\beta}{\gamma+\alpha}\overline{OB}\right)\left(-\frac{\alpha+\beta}{\gamma}\overline{OD}\right)\left(\frac{\alpha}{\alpha+\beta}\overline{AB}\right)\cdot\frac{\gamma}{\beta}\overline{CF},$$

hence to $\overline{AB}\cdot\overline{OD}\cdot\overline{OB}\cdot\overline{CE}\cdot\overline{CF}$.

Solution 3 by Daniel Văcaru, Pitesti, Romania

Let $D\in d, d\|BC,\ d\cap AO=\{D'\}$ and $E\in d', d'\|BC, d'\cap AO=\{E'\}$.

With $\Delta ADD'\simeq\Delta ABF$, we obtain

$$\frac{DD'}{BF}=\frac{AD}{AB}\Leftrightarrow DD'\cdot AB=AD\cdot BF. \qquad (1)$$

With $\Delta AEE'\simeq ACF$, we obtain

$$\frac{EE'}{CF}=\frac{AE}{AC}\Rightarrow EE'\cdot AC=CF\cdot AE. \qquad (2)$$

In the same manner, we have $\dfrac{OD}{OC}=\dfrac{DD'}{FC}$ and $\dfrac{EO}{OB}=\dfrac{EE'}{BF}$. Therefore, $OD\cdot FC=OC\cdot DD'$ and $EO\cdot BF=EE'\cdot OB$.

So,

$$\overline{AC}\cdot\overline{EO}\cdot\overline{OC}\cdot\overline{DB}\cdot\overline{FB}=\overline{AC}\cdot\overline{OC}\cdot\overline{DB}\cdot\overline{EE'}\cdot\overline{OB}, \qquad (3)$$

and

$$\overline{AB}\cdot\overline{OD}\cdot\overline{OB}\cdot\overline{CE}\cdot\overline{CF}=\overline{AB}\cdot\overline{OB}\cdot\overline{CE}\cdot\overline{OC}\cdot\overline{DD'}. \qquad (4)$$

Substitution into the statement of the problem, we have

$$\overline{AC}\cdot\cancel{\overline{OC}}\cdot\overline{DB}\cdot\overline{EE'}\cdot\cancel{\overline{OB}}=\overline{AB}\cdot\cancel{\overline{OB}}\cdot\overline{CE}\cdot\cancel{\overline{OC}}\cdot\overline{DD'}$$
$$\Leftrightarrow\overline{AC}\cdot\overline{DB}\cdot\overline{EE'}=\overline{AB}\cdot\overline{CE}\cdot\overline{DD'}. \qquad (5)$$

Using relationships (1) and (2), we have $\overline{AB}\cdot\overline{CE}\cdot\overline{DD'}=\overline{AD}\cdot\overline{CE}\cdot\overline{BF}$ and $\overline{AC}\cdot\overline{DB}\cdot\overline{EE'}=\overline{AE}\cdot\overline{DB}\cdot\overline{CF}$. But $\overline{AD}\cdot\overline{CE}\cdot\overline{BF}=\overline{AE}\cdot\overline{DB}\cdot\overline{CF}$ by Ceva's Theorem for ΔABC with cevians AF, BE and CD.

Also solved by the proposer.

- **5627:** *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA*

Solve $ax+7=bx+a$, given that $a,b\in\Re$ and

$$a=\begin{cases}2 & \text{if } b \text{ is less than } 3\\ x-6 & \text{if } 3\le b\end{cases}\quad\text{and}\quad b=\begin{cases}3x & \text{if } x \text{ is less than } a\\ 7 & \text{if } a\le x\end{cases}$$

Solution 1 by Bruno Salgueiro Fanego, Viveiro, Spain

There are initially four cases:

$$\{b < 3, x < a\}, \quad \{b < 3, a \leq x\}, \ \{3 \leq b, x < a\}, \ \text{or} \ \ \{3 \leq b, a \leq x\}.$$

The two intermediate cases are impossible because $\{b < 3, a \leq x\}$ would imply $\{a = 2, b = 7\}$, but $b = 7$ contradicts $b < 3$, and $\{3 \leq b, x < a\}$ would imply $a = x - 6$ which contradicts $x < a$.

The first case $\{b < 3, x < a\}$, implies $\{a = 3, b = -3x\}$ and $ax + 7 = bx + a$ is equivalent to $3x + 7 = bx + 3$, that is, to $3x + 7 = 3x^2 + 3$ or $x = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \cdot 3(-5)}}{2 \cdot 3}$ or, $x \in \left\{-1, \dfrac{5}{3}\right\}$. Since $3x = b < 3m$, the only possible solution in this case is $x = -1$, in which case $a = 2$ and $b = -3$. This solution works, and it is the only solution for the first case.

The last case $\{3 \leq b, a \leq x\}$, implies $\{a = x - 6, b = 7\}$ and $ax + 7 = bx + a$ is equivalent to $ax + 7 = 7x + a$, that is, to $(a - 7)x = a - 7$. If $a \neq 7$, this last equation lead us to $x = 1$ and if $a = 7$ the equation is verified for any $x \in \Re$, but the restriction $a = x - 6$ implies $x = 13$. If $x = 1$, then $a = -5$ and $b = 7$, and when $a = 7, b = 7$ and $x = 13$. In both of these two cases the values of x work, so they are the only solutions in this last case.

Hence, we have shown that the solution to the first degree equation in the statement of the problem only has solutions when $(a, b) \in \{(2, -3), (-5, 7), (7, 7)\}$ and that its only solution is $x = -1$ when $(a, b) = (2, -3)$, $x = 1$ when $(a, b) = (-5, 7)$, and $x = 13$ when $(a, b) = (7, 7)$.

Solution 2 by Michael Brozinsky, Central Islip, NY

Assume first that $b < 3$ so that $a = 2$ and $b = 3x$ so that the given equation become $2x + 7 = 3x^2 + 2$ giving $x = \dfrac{5}{3}$ **or** $x = -1$ implying $b = 5$ or $b = -3$ respectively. Since we assumed $b < 3$ the only solution thus far is x=1, a=-2, **and** b=-3.

Assume next that $b \geq 3$ so that $a = x - 6$ and thus $x - 6 + a \geq a$ which means $b = 7$ and the given equation becomes $(x - 6) \cdot x + 7 = 7x + x - 6$ or equivalently $x^2 - 14x + 13 = 0$ giving $x = 13$ **or** x = 1 and so either $x = 13, a = 7b = 7$ **or** $x = 1, a = -5, b = 7$. In summary there are three solutions.

x	a	b
-1	2	-3
13	7	7
1	-5	7

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC

We show that the three solutions for (a, b, x) are $(-5, 7, 1)$, $(7, 7, 13)$, and $(2, -3, -1)$.

If $x \geq a$, then $b = 7$, so $a(x - 1) = 7(x - 1)$ with $a = x - 6$. This implies $x = 1$ or $x = 13$, producing the solutions $(a, b, x) = (-5, 7, 1)$ or $(a, b, x) = (7, 7, 13)$.

If $x < a$, then $b = 3x$; also, a cannot equal $x - 6$, so $b < 3$ and hence $a = 2$. Thus $2x + 7 =$

$3x^2+2$, so $x=-1$ or $x=5/3$. While $x=-1$ yields the solution $(a,b,x)=(2,-3,-1)$, $x=5/3$ implies the contradiction $b=5>3$.

Solution 4 by Kee-Wai Lau, Hong Kong, China

We show that the solutions are $(a,b,x)=(2,-3,-1),(-5,7,1)$, and $(7,7,13)$. There are three cases to consider.

Case I: $x<a$ and $b<3$.

We have $a=2, c=\dfrac{b}{3}<1$ and the given equation becomes

$2x+7=3x^2+2$. This gives $(a,b,x)=(2,-3,-1)$

Case II: $x<a$ and $b\geq 3$.

We have $a>x=a+6>a$, which is impossible.

Case III: $x\geq a$.

We have $b=7, a=x-6$ and the given equation becomes

$(x-6)x+7=7x+x=6$. This gives $(a,b,x)=(-5,7,1)$ and $(7,7,13)$.

Also solved by Michel Bataille, Rouen, France; Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA; Albert Stadler Herrliberg, Switzerlland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

- **5628:** *Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania*

Prove that if $x,y,z,u,v,w\in(0,\infty)$, then

$$\frac{x^2}{u}e^{\frac{u}{x}}+\frac{y^2}{v}e^{\frac{v}{y}}+\frac{z^2}{w}e^{\frac{w}{z}}\geq\frac{(x+y+z)^2}{u+v+w}e^{\frac{u+v+w}{x+y+z}}.$$

Solution 1 by Ángel Plaza
Universidad de Las Palmas de Gran Canaria, Spain

The solution follows by Jensen's inequality, considering function $f(t)=\dfrac{1}{t}e^t$, which is convex because $f''(t)=\frac{e^t\left(t^2-2t+2\right)}{t^3}=\frac{e^t\left((t-1)^2+1\right)}{t^3}>0$ for $t>0$. By doing $a=\frac{u}{x}$, $b=\frac{v}{y}$, $c=\frac{w}{z}$, the given inequality may be written as

$$xf(a)+yf(b)+zf(c)\geq(x+y+z)f\left(\frac{xa+yb+zc}{x+y+z}\right)$$

because $xa+yb+zc=u+v+w$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY

The function $f(x)=\frac{1}{x}e^x$ is convex on $(0,\infty)$: $f''(x)=e^x\left((x-1)^2+1\right)/x^3>0$. Therefore we can apply the weighted form of Jensen's inequality

$$\frac{w_1f(a)+w_2f(b)+w_3f(c)}{w_1+w_2+w_3}\ \geq\ f\left(\frac{w_1a+w_2b+w_3c}{w_1+w_2+w_3}\right)$$

with weights x, y, z and variables $u/x, v/y, w/z$:

$$\begin{aligned} x\cdot\left(\frac{x}{u}e^{\frac{u}{x}}\right)+y\cdot\left(\frac{y}{v}e^{\frac{v}{y}}\right)+z\cdot\left(\frac{z}{w}e^{\frac{w}{z}}\right) &\geq (x+y+z)\,\frac{x+y+z}{u+v+w}\,e^{\frac{u+v+w}{x+y+z}} \\ &= \frac{(x+y+z)^2}{u+v+w}\,e^{\frac{u+v+x}{x+y+z}}. \end{aligned}$$

Solution 3 by Kee-Wai Lau, Hong Kong, China

Since $e^t = +\sum_{k=12}^{\infty}\frac{t^k}{k!}$ we need only prove that

$$\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^2}{w}\geq\frac{(x+y+x)^2}{u++w}. \tag{1}$$

and

$$x\left(\frac{u}{x}\right)^k+y\left(\frac{v}{y}\right)^k+z\left(\frac{w}{z}\right)^k\geq(x+y+z)\left(\frac{u+v+w}{x+y+z}\right)^k. \tag{2}$$

for any nonnegative integer k

By the inequality of Cauchy-Schwarz, we have

$$(x+y+z)^2=\left(\frac{x}{\sqrt{u}}\sqrt{u}+\frac{y}{\sqrt{v}}\sqrt{v}+\frac{x}{\sqrt{w}}\sqrt{w}\right)^2\leq\left(\frac{x^2}{u}+\frac{y^2}{v}+\frac{z^{‘2}}{w}\right)(u+v+w)$$

and (1) holds. For $k = 0, 1$, (2) is trivial. For $k \geq 2$ and $t > 0$, let $f(t) = t^k$, which is convex. Hence by Jensen's inequality, we have

$$\frac{x}{x+y+z}\left(\frac{u}{x}\right)^k+\frac{y}{x+y+z}\left(\frac{v}{y}\right)^k\frac{z}{x+y+z}\left(\frac{w}{z}\right)^k.$$

Also solved by Michel Bataille, Rouen, France; Albert Stadler Herrliberg, Switzerland; Daniel Văcaru, Pitesti, Romania, and the proposer.

- **5629:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

A square is divided into a finite number of blue and red rectangles, each with sides parallel to the sides of the square. Within each blue rectangle the ratio between its width and its height is written. Within each red rectangle the ratio between its height and its width is written. Finally, the sum S of these numbers is computed. If the total area of the blue rectangles equals twice the total area of the red rectangles, what is the smallest possible value of S?

Solution by proposer

Suppose that after dividing the square we have n blue rectangles and m red rectangles, respectively. Let $a_i b_i$ denote the width and height of each blue rectangle, and let $c_i d_j$

denote the width and height of each red rectangle. Also denote by L the length of the initial square. We claim that either holds:

$$\sum_{i=1}^{n} a_i \geq L \text{ or } \sum_{j=1}^{m} d_j > L.$$

Indeed, suppose that there exists a horizontal line across the square that is covered entirely with blue rectangles. Then, the total width of these rectangles is at least L, and the claim is proven. Otherwise, there is a red rectangle intersecting every horizontal one, and hence the total height of these rectangles is at least L.

Now, WLOG we can assume that $\sum_{i=1}^{n} a_i \geq L$. Applying Cauchy's inequality to vectors

$$\vec{u} = \left(\sqrt{\frac{a_1}{b_1}}, \sqrt{\frac{a_2}{b_2}}, \cdots \sqrt{\frac{a_n}{b_n}}\right)$$

and

$$\vec{v} = \left(\sqrt{(a_1 b_1}, \sqrt{a_2 b_2}, \cdots \sqrt{a_n b_n}\right),$$

$$\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right)\left(\sum_{i=1}^{n} a_i b_i\right) \geq \left(\sum_{i=1}^{n} a_i\right)^2 \geq L^2.$$

Since we know that$=\displaystyle\sum_{i=1}^{n} a_i b_i = \frac{2L^2}{3}$, then $\displaystyle\sum_{i=1}^{n} \frac{a_i}{b_i} \geq \frac{3}{2}$. Moreover, each $c_j \leq L$, so

$$\sum_{j=1}^{m} \frac{d_j}{c_j} = \sum_{j=1}^{m} \frac{c_j d_j}{c_j^2} \geq \frac{1}{L^2}\sum_{j=1}^{m} c_j d_j = \frac{1}{3}.$$

Adding up the preceding, yields

$$S \geq \frac{3}{2} + \frac{1}{2} = \frac{11}{6}.$$

Equality holds when $S = \dfrac{11}{6}$. It can be achieved by making the top $\dfrac{2}{3}$ of the square a blue rectangle, and the remaining $\dfrac{1}{3}$ bottom rectangle red.

- **5630:** *Proposed by Arkady Alt, San Jose, CA*

Find the integer part of the minimal value of $k + \dfrac{n}{k}, k \in N$.

Solution 1 by Michel Bataille, Rouen, France

Let μ denote the minimal value of $k + \dfrac{n}{k}$ when $k \in N$.
If $n = 0$, we clearly have $\lfloor \mu \rfloor = \mu = 1$.

Let f_n be the function defined on $(0, \infty)$ by $f_n(x) = x + \dfrac{n}{x}$.

If $n < 0$, f_n is increasing on $(0, \infty)$, hence $\mu = 1 + \dfrac{n}{1} = n + 1$ and $\lfloor \mu \rfloor = \lfloor n + 1 \rfloor$.

From now on, we suppose that $n > 0$ and for simplicity, we set $m = \lfloor \sqrt{n} \rfloor$. Note that $m^2 \leq n < (m+1)^2 = m^2 + 2m + 1$.

We prove that $\lfloor\mu\rfloor = 2m$ if $n < m^2 + m$ and $\lfloor\mu\rfloor = 2m+1$ if $n \geq m^2 + m$.

The function f_n is decreasing on $(0, \sqrt{n})$ and increasing on $[\sqrt{n}, \infty)$, hence the minimum of f_n on $(0, \infty)$ is $f(\sqrt{n}) = 2\sqrt{n}$. It immediately follows that $\mu = \min\{f(m), f(m+1)\}$. Now, $f(m+1) - f(m) = m + 1 + \dfrac{n}{m+1} - m - \dfrac{n}{m} = 1 - \dfrac{n}{m(m+1)}$ and therefore $\mu = f(m)$ if $n < m(m+1)$ and $\mu = f(m+1)$ if $n \geq m(m+1)$. In the former case, we have $m \leq \dfrac{n}{m} < m+1$, hence $2m \leq \mu = m + \dfrac{n}{m} < 2m+1$ and so $\lfloor\mu\rfloor = 2m$. In the latter case, $(m+1)^2 > n \geq m(m+1)$ and $\mu = f(m+1) = m + 1 + \dfrac{n}{m+1}$ satisfies $2m+1 \leq \mu < 2m+2$ so that $\lfloor\mu\rfloor = 2m+1$.

Solution 2 by Albert Stadler, Herrlierg, Switzerland

We denote by $[x]$ the integer part of x and claim that the integer part of the minimal value of $k + n/k, k \in N$, equals either $[2\sqrt{n}]$ or $[2\sqrt{n}] + 1$, and it equals $[2\sqrt{n}] + 1$ if and only if there is a natural number m such that $n = m(m+1)$.

The function $x \to x + n/x$ is decreasing for $x < \sqrt{n}$ and increasing for $x > \sqrt{n}$. The minimum at $\sqrt{n}$ equals $2\sqrt{n}$.

Therefore

$$2\sqrt{n} \leq \min_{k\in N}\left(k + \frac{n}{k}\right) = \min\left([\sqrt{n}] + \frac{n}{[\sqrt{n}]},\ [\sqrt{n}] + 1 + \frac{n}{[\sqrt{n}] + 1}\right).$$

The inequality

$$[\sqrt{n}] + 1 + \frac{n}{[\sqrt{n}] + 1} < 2[\sqrt{n}] + 2$$

is equivalent to $\sqrt{n} < [\sqrt{n}] + 1$ which is true. So

$$[2\sqrt{n}] \leq \left[\min_{k\in N}\left(k + \frac{n}{k}\right)\right] \leq 2\left[\sqrt{n}\right] + 1.$$

Clearly, $2[\sqrt{n}] + 1 - [2\sqrt{n}] \in \{0, 1\}$. So the integer part of $\min\limits_{k\in N}\left(k + \dfrac{n}{k}\right)$ equals either $[2\sqrt{n}]$ or $[2\sqrt{n}] + 1$. It remains to investigate for which n we have

$$[2\sqrt{n}] + 1 = \min\left([\sqrt{n} + \left[\frac{n}{[\sqrt{n}]}\right],\ [\sqrt{n}] + 1 + \left[\frac{n}{[\sqrt{n}] + 1}\right]\right). \tag{*}$$

Clearly, if $n = m(m+1)$ above equation holds true, since $[\sqrt{n}] = m$, the right-hand side equals $2m + 1$ and the left-hand side equals

$$[2\sqrt{n}] + 1 = 2m + 1 + \left[2\sqrt{m(m+1)} - 2m\right] = 2m + 1 + \Bigg[\underbrace{\frac{2m}{\sqrt{m(m+1)} + m}}_{<1}\Bigg] = 2m + 1$$

as well. It remains to prove that if (*) holds true then $n = m(m+1)$ for some natural number m.

Let $m = [\sqrt{n}],\ r = n - m^2$. Then $0 \leq r \leq 2m$, and

$$[2\sqrt{n}]+1=[2\sqrt{m^2+r}]+1=2m+1+[2\left(\sqrt{m^2+r}-m\right)]=\begin{cases}2m+1, & 0\le r\le m\\ 2m+2, & m+1\le r\le 2m\end{cases}$$

$$[\sqrt{n}]+\left[\frac{n}{[\sqrt{n}]}\right]=m+\left[\frac{m^2+r}{m}\right]=2m+\left[\frac{r}{m}\right]=\begin{cases}2m & 0\le r\le m-1\\ 2m+1, & m\le r\le 2m-1\\ 2m+2 & r=2m\end{cases}$$

$$[\sqrt{n}]+1\left[\frac{n}{[\sqrt{n}]+1}\right]=m+1+\left[\frac{m^2+r}{m+1}\right]=2m+\left[\frac{r+1}{m+1}\right]=\begin{cases}2m & 0\le r\le m-1\\ 2m+1, & m\le r\le 2m\end{cases}$$

This shows that (*) can only hold true for $r=m$ which implies that $n=m^2+m=m(m+1)$, as claimed.

Solution 3 by Kee-Wai Lau, Hong-Kong, China

We show that the integer part of the minimal value of $k+\dfrac{n}{k},\ k\in N$ equals

$$\begin{cases}2\lfloor\sqrt{n}\rfloor, & \lfloor\sqrt{n}\rfloor^2\le n<\lfloor\sqrt{n}\rfloor\left(\lfloor\sqrt{n}\rfloor+1\right)\\ 2\lfloor\sqrt{n}\rfloor+1, & \lfloor\sqrt{n}\rfloor\left(\lfloor\sqrt{n}\rfloor+1\right)\le n<\left(\lfloor\sqrt{n}\rfloor+1\right)^2\end{cases}$$

where $\lfloor t\rfloor$ is the greatest integer not exceeding t.

Suppose that $m^2\le n<(m+1)^2$, where m is any positive integer.

For real x, the convex function $x+\dfrac{n}{x}$ attains its minimal value when $x=\sqrt{n}$.

Hence the minimal value of $k+\dfrac{n}{k}, k\in N$ equals

$$\min\left(m+\frac{n}{m}, m+1+\frac{n}{m+1}\right)=\begin{cases}m+\dfrac{n}{m}, & m^2\le n<m(m+1)\\ m+1+\dfrac{n}{m+1}, & m(m+1)\le n<(m+1)^2.\end{cases}$$

Hence our claim.

Solution 4 by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA

The integer part of the minimal value of $k+\frac{n}{k}, k\in N$ is given by

$$\begin{cases}\lceil n+1\rceil & \text{if } n\le -1\\ \lfloor n+1\rfloor & \text{if } -1<n<1\\ 2\left\lfloor\sqrt{n}\right\rfloor & \text{if } 1\le n<\left\lfloor\sqrt{n}\right\rfloor^2+\left\lfloor\sqrt{n}\right\rfloor\\ 2\left\lfloor\sqrt{n}\right\rfloor+1 & \text{if } n>1 \text{ and } \ge\left\lfloor\sqrt{n}\right\rfloor^2+\left\lfloor\sqrt{n}\right\rfloor\end{cases}$$

Consider the function $f(x)=x+\frac{n}{x}$, where x is a positive real number. Then its derivative $f'(x)=1-\frac{n}{x^2}$ is positive for all $x\in N$ if $n<1$, in which case the minimum value of $k+\frac{n}{k}$ is $1+n$. Recall that if $n+1<0$, then the integer part of $n+1$ is $\lceil n+1\rceil$. On the other hand, if n is a positive real number, then $f'(x)<0$ for $0<x<\sqrt{n}$ and $f'(x)>0$ for $x>\sqrt{n}$, so that $f(\sqrt{n})=2\sqrt{n}$ is the minimum value of $f(x)$ over the continuous interval $(0,\infty)$. If $n=a^2$, where $a\in N$, then the minimum value of $k+\frac{n}{k}$, where $k\in N$, is $f(a)=2\sqrt{n}=2a$, which is an integer. If $n>1$ and n is not a perfect square, then the minimum value of $k+\frac{n}{k}$, where $k\in N$, is either

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}$$

or

$$f(\lceil\sqrt{n}\rceil)=\lceil\sqrt{n}\rceil+\frac{n}{\lceil\sqrt{n}\rceil}=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}.$$

If $n>1$, notice that $\lfloor\sqrt{n}\rfloor\le\sqrt{n}$, so that $\lfloor\sqrt{n}\rfloor^2\le n$ and $\frac{n}{\lfloor\sqrt{n}\rfloor}\ge\lfloor\sqrt{n}\rfloor$, so that

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}\ge 2\lfloor\sqrt{n}\rfloor.$$

In addition, $(\lfloor\sqrt{n}\rfloor+1)(\lfloor\sqrt{n}\rfloor-1)=\lfloor\sqrt{n}\rfloor^2-1\le n-1<n$, so that $\frac{n}{\lfloor\sqrt{n}\rfloor+1}>\lfloor\sqrt{n}\rfloor-1$, and

$$f(\lceil\sqrt{n}\rceil)=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}>\lfloor\sqrt{n}\rfloor+1+\lfloor\sqrt{n}\rfloor-1=2\lfloor\sqrt{n}\rfloor.$$

Since $\sqrt{n}<\lfloor\sqrt{n}\rfloor+1$, then $n<(\lfloor\sqrt{n}\rfloor+1)^2$ and $\frac{n}{\lfloor\sqrt{n}\rfloor+1}<\lfloor\sqrt{n}\rfloor+1$ so

$$f(\lceil\sqrt{n}\rceil)=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}<2\lfloor\sqrt{n}\rfloor+2.$$

Thus, for $n>1$, the integer part of the minimum value of $f(k)=k+\frac{n}{k}$ is either $2\lfloor\sqrt{n}\rfloor$ or $2\lfloor\sqrt{n}\rfloor+1$. We consider two cases, depending on whether $n<\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor$. First, notice that

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}=2\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{\lfloor\sqrt{n}\rfloor}$$

and

$$f(\lceil\sqrt{n}\rceil)=\lfloor\sqrt{n}\rfloor+1+\frac{n}{\lfloor\sqrt{n}\rfloor+1}=2\left(\lfloor\sqrt{n}\rfloor+1\right)-\frac{(\lfloor\sqrt{n}\rfloor+1)^2-n}{\lfloor\sqrt{n}\rfloor+1}.$$

Case 1: $n<\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor$. Then $\frac{n}{\lfloor\sqrt{n}\rfloor}<\lfloor\sqrt{n}\rfloor+1$ and

$$f(\lfloor\sqrt{n}\rfloor)=\lfloor\sqrt{n}\rfloor+\frac{n}{\lfloor\sqrt{n}\rfloor}<2\lfloor\sqrt{n}\rfloor+1.$$

In addition,

$$\left(\lfloor\sqrt{n}\rfloor+1\right)^2-n=\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor-n+\lfloor\sqrt{n}\rfloor+1>\lfloor\sqrt{n}\rfloor+1,$$

so that

$$f(\lceil\sqrt{n}\rceil)=2\left(\lfloor\sqrt{n}\rfloor+1\right)-\frac{(\lfloor\sqrt{n}\rfloor+1)^2-n}{\lfloor\sqrt{n}\rfloor+1}<2\left(\lfloor\sqrt{n}\rfloor+1\right)-1=2\lfloor\sqrt{n}\rfloor+1.$$

Thus, in Case 1, the integer part of the minimal value of $f(k)$ is $2\lfloor\sqrt{n}\rfloor$.
Case 2: $n\ge\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor$. Then

$$f(\lfloor\sqrt{n}\rfloor)=2\lfloor\sqrt{n}\rfloor+\frac{n-\lfloor\sqrt{n}\rfloor^2}{\lfloor\sqrt{n}\rfloor}\ge 2\lfloor\sqrt{n}\rfloor+1.$$

In addition,

$$\left(\lfloor\sqrt{n}\rfloor+1\right)^2-n=\lfloor\sqrt{n}\rfloor^2+\lfloor\sqrt{n}\rfloor-n+\lfloor\sqrt{n}\rfloor+1\le\lfloor\sqrt{n}\rfloor+1,$$

so that

$$f(\lceil\sqrt{n}\rceil)=2\left(\lfloor\sqrt{n}\rfloor+1\right)-\frac{(\lfloor\sqrt{n}\rfloor+1)^2-n}{\lfloor\sqrt{n}\rfloor+1}\ge 2\left(\lfloor\sqrt{n}\rfloor+1\right)-1=2\lfloor\sqrt{n}\rfloor+1.$$

Thus, in Case 2, the integer part of the minimal value of $f(k)$ is $2\lfloor\sqrt{n}\rfloor+1$.

Solution 5 by Brian D. Beasley, Presbyterian College, Clinton, SC

For each real number n, we define $f(n)=\lfloor\min\{k+n/k : k\in N\}\rfloor$ and note that f is a non-decreasing function on R. If $n\le 1$, then the minimum value of $k+n/k$ for $k\in N$ occurs when $k=1$, so $f(n)=\lfloor 1+n\rfloor$. If $n>1$, then there is a unique positive integer m with either

$$m^2\le n<m(m+1)\quad\text{or}\quad m(m+1)\le n<(m+1)^2.$$

We observe that f increases by 1 only at each $n=m^2$ and at each $n=m(m+1)$: Let $\varepsilon\in(0,1)$. Then $f(m^2)=2m$, while $f(m^2-\varepsilon)=2m-1$ by taking $k=m$. Similarly, $f(m(m+1))=2m+1$, while $f(m(m+1)-\varepsilon)=2m$ by taking $k=m$ or $k=m+1$. Hence we conclude that if $m^2\le n<m(m+1)$, then $f(n)=2m$, while if $m(m+1)\le n<(m+1)^2$, then $f(n)=2m+1$.

Addendum. It is interesting to note that when n is a positive integer, then $f(n)=\lfloor 2\sqrt{n}\rfloor$ unless $n=m(m+1)$, in which case $f(n)=\lfloor 2\sqrt{n}\rfloor+1$.

Solution 6 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA

For an integer n, let $M(n)=\left[\min\left\{k+\frac{n}{k}, k\in N\right\}\right]$,
where [.] is the greatest integer function. We shall see that

$$M(n)=\begin{cases} n+1,\ \text{if } n\le 0;\\ [2\sqrt{n}]+1, \text{if } n\ge 1 \text{ has the form } m(m+1);\\ [2\sqrt{n}],\ \text{otherwise}\end{cases}.$$

Note that in the second case, $M(n)=M\left(m^2+m\right)=2m{+}1$.

By calculus, we know that the function $f_n(k)=k+\frac{n}{k}$, with k considered as a continuous (positive) variable, achieves an absolute minimum of $2\sqrt{n}$ at the sole critical point $\sqrt{n}$. Thus, when we restrict k to integer values, the minimum will be close to $2\sqrt{n}$ and this restricted minimum must occur near $\sqrt{n}$. That is, it must occur at $k=[\sqrt{n}]$ or at $k=[\sqrt{n}]+1$.

Let us validate our claim. For $n\le 0$, $\mathrm{f}_n(1)=1+n$ while $f_n(k)=k+\frac{n}{k}$ for $k>1$.

Thus the minimal value is the integer $n+1$, which therefore is $M(n)$.

Now suppose that n is positive and trapped between given consecutive squares: $m^2\le n<(m+1)^2$.

Thus, $m \le \sqrt{n} < m+1$, so $m = [\sqrt{n}]$.

In the nice case where n is a square, $m^2 = n$, then choosing k to be n yields the calculus-predicted absolute minimum: $f_n(m) = m + \dfrac{m^2}{m} = 2m = 2\sqrt{n} = [2\sqrt{n}] = M(n)$.

In the special case $n = m(m+1) = m^2 + m$, we find more nice behavior. The two candidates for the occurrence of the minimum are at $k = [\sqrt{n}]$ or $k = [\sqrt{n}] + 1$. But these are just m and $m+1$, and
$f_n(m) == m + \dfrac{m(m+1)}{m} = 2m+1$, and $f_n(m+1) = m+1+\dfrac{m(m+1)}{m+1} = 2m+1$.
Therefore $M(n) = 2m+1$, which is $[2\sqrt{n}] + 1$.

Finally, consider the case that $m^2 < n < (m+1)^2$ and $n \neq m(m+1)$.
We still have $m \le \sqrt{n} < m+1$, so $m = [\sqrt{n}]$. The two candidates for the location of our minimum value: at $k = m$ or $k = m+1$. Some algebra shows that

$$f_n(m) = m + \frac{n}{m} < f_n(m+1) = m+1+\frac{n}{m+1} \iff n < m(m+1).$$

So the location of n in the interval $\left(m^2, (m+1)^2\right)$ determines the appropriate choice for k. But for each choice, the minimum value turns out to be $[2\sqrt{n}]$.

We present the (ticky) details verifying that the first choice behaves as claimed: $n < m(m+1)$ using $k = m$, so that $f_n(k) = m + \dfrac{n}{m}$.

Because $m^2 < n < m(m+1)$, we have $m < \dfrac{n}{m} < m+1$, so $\left[\dfrac{n}{m}\right] = m$.

Therefore $[f_n(k)] = \left[n + \dfrac{n}{m}\right] = m + m = 2m = M(n)$.

Moreover, $[2\sqrt{n}] = 2m$ also. This is true because (1) $2m < 2\sqrt{n}$; and if we had $2m+1 < 2\sqrt{n}$, we would conclude by squaring that $4m^2 + 4m + 1 < 4n < 4m^2 + 4m$, which is a contradiction by our choice for n.

Therefore, $M(n) = 2m = [2\sqrt{n}]$.

The argument for $k = m+1$ is similar.

The proof of our formula for $M(n)$ is complete.

Comment (by authors): This interesting problem has connections to three classic problems.

- (1) The ancient Babylonian method for computing the square root of n: make a guess ,k. Compute n/k, then average the result with k, producing a better approximation to the desired root. Repeat as long as you want to the method converges to n.
- (2) But of course, this method turns out to be Newton's method applied to the function $f(x) = x - n^2$.
- (3) The favorite Calculus I example, done in class to help the students see the power of calculus and understand graphs: "What does the graph of $f(x) = x + \dfrac{n}{x}$ look like?

We have two terms competing; for positive x close to zero, the $\dfrac{n}{x}$ term dominates and the graph climbs to infinity; for big positive x, the x term wins and the graph also climbs to infinity. $f(x)$ is continuous and always positive, so the graph must "min out" somewhere.

The derivative tells us where: at $x = \sqrt{n}$. We also learn that the minimum occurs where the two terms balance–each equals $\sqrt{n}$. (There is something to be said for compromise and sharing power.)
Also solved by the proposer.

Mea Culpa

Solutions were received from Paul M. Harms of North Newton, Kansas to problems 5619 and 5623. The solutions were received nearly 5 weeks after they were mailed. The reason for the late delivery was attributed by postal authorities to the COVID virus.

Editor's Note

Time passes quickly, and this is especially true when one is having fun doing what they do. I took over the editorship of this column in 2001 and now 20 years have flown by, and hard as it is for me to admit, it is time for me to step down.

My association with this Column goes back much further than 20 years and I can recall attempting some of the problems in the Column when I was a student in high school, where our librarian had the good sense to have a school subscription to the SSMJ. Over the years I have developed wonderful relationships with so many of you, and I deeply appreciate your collegiality and the help and support you have given to me. Normally I would list the names of those of you who went above and beyond in helping our column become as popular as it is within the SSMA community and also among other problem solvers of columns in other journals. But there are so many of you that have helped me over the years that I would for sure inadvertently omit some of your names, and then feel terribly, terribly embarrassed by the omissions. So let me just say to you one and all, "thank you."
Albert Natian of Los Angeles Valley College in Valley Glen, California will be the next editor. Starting with this issue, please send him your solutions and proposals. His details are listed bellow.

To propose problems, email them to:
problems4ssma@gmail.com
To propose solutions, email them to:
solutions4ssma@gmail.com
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To propose problems and solutions via regular mail, send them to:

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