Problems and Solutions   Albert Natian, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email. *To best serve the readers and contributors of this section, please adopt — as a problem proposer or a solution proposer — the formats, styles and recommendations delineated at the very end of this document.*

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before January 15, 2022.

• **5649** *Proposed by Kenneth Korbin, New York, NY.*

A trapezoid with perimeter $18 + 14\sqrt{2}$ is inscribed in a circle with diameter $7 + 5\sqrt{2}$. Each of the sides of the trapezoid are of the form $a + b\sqrt{2}$, where a and b are positive integers. Find the dimensions of the trapezoid.

• **5650** *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Spain.*

Let L_n be the nth Lucas number defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. Prove

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \ \leqslant\ \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}.$$

• **5651** *Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $f : [0, +\infty) \to (0, +\infty)$ be a concave function. Show that

$$20\int_0^{1/4} f(x)\,dx + 15\int_0^{1/3} f(x)\,dx + 24\int_0^{5/12} f(x)\,dx \geqslant 75\int_0^{1/6} f(x)\,dx + 10\int_0^{1/2} f(x)\,dx.$$

Editor's note: A real-valued function f defined over an interval I is said to be *concave* (*down*) if and only if $\forall \alpha, \beta \in [0,1]$, with $\alpha + \beta = 1$, $\forall a, b \in I :\ \alpha f(a) + \beta f(b) \leqslant f(\alpha a + \beta b)$.

• **5652** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta*

Turnu-Severin, Mehedinti, Romania.

Prove:
$$1 \leqslant a \leqslant b \leqslant c \implies a^a \cdot e^{c-a} \cdot \left(\sqrt{ab}\right)^{b-a} \cdot \left(\sqrt{bc}\right)^{c-b} \leqslant c^c.$$

• **5653** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

Evaluate the following limit:
$$\lim_{n\to\infty} \frac{\sin\left(F_n^{-1}\right)}{\sin\left(L_n^{-1}\right)} \cdot \left(1+\frac{1}{F_n}\right)^{L_n} \cdot \sum_{y=1}^{n}\sum_{x=1}^{\infty} \frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Editor's note: Here, F_n and L_n respectively denote the Fibonacci and Lucas numbers.

• **5654** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{H_{n+m}}{mn^2(n+m)^2}$$
where $H_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$ denotes the nth harmonic number.

Solutions

• **5631** *Proposed by Kenneth Korbin, New York, NY.*

Trapezoid $ABCD$ with integer length sides is inscribed in a circle with diameter P^3 where P is a prime number great than 14. Base $AB = 7P^2$. Express the lengths of the other three sides in terms P.

Solution by Albert Stadler, Herrliberg, Switzerland.

A cyclic quadrilateral with successive sides a, b, c, d and semiperimeter s has the circumradius (the radius of the circumcircle) given by
$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

(Formula of the Indian mathematician Vatasseri Parameshvara, 15^{th} century, see https://en.wikipedia.org/wiki/Cyclic_quadrilateral).

$a = 7P^2$, $d = b$, $R = P^3/2$, and the formula collapses to

$$P^3 = 2b\sqrt{\frac{b^2 + 7P^2c}{4b^2 - (7P^2 - c)^2}}$$

which is equivalent to each of the following lines

$(i) P^6(4b^2 - (7P^2 - c)^2) = 4b^2(b^2 + 7P^2c)$,

$(ii) 4b^4 + 28b^2cP^2 - 4b^2P^6 + c^2P^6 - 14cP^8 + 49P^{10} = 0$,

$(iii) (2b^2 + 7P^2c - P^6)^2 + c^2P^4(P^2 - 49) = P^{10}(P^2 - 49)$,

$(iv) (cP^4 - 7P^6 + 14b^2)^2 + (P^2 - 49)(2b^2 - P^6)^2 = P^{12}(P^2 - 49)$.

(iii) and (iv) show that given $P > 8$ (ii) has only finitely many solutions in integral tuples (b, c). An exhaustive search gives the following solutions for positive b and c:

P	b	c
17	2023	4697
17	3247	4697
19	2527	6209
19	4997	6209
23	3703	9737
23	9913	9737
29	5887	16289
29	21547	16289
31	6727	18809
31	26753	18809
37	9583	27377
37	47027	27377
41	11767	33929
41	64903	33929

The numerical evidence suggests that c is a quadratic function of P, namely

$$c = 21P^2 - 1372.$$

(Note that $c > 0$ for $P > 8$.) We insert this value into (ii) and solve for b. We find that

$$b \in \{\pm 7P^2, \pm P(P^2 - 98)\}.$$

So $(b,c) = (7P^2, 21P^2 - 1372)$ and $(b,c) = (P|P^2 - 98|, 21P^2 - 1372)$ are positive integral solutions of (ii). In the first case the trapezoid has three sides of equal length.

We have nowhere used that P must be a prime, only that P is an integer > 8. We have not answered the question if there are more solutions in positive integral b and c for P other than the ones in the table above. Likely the answer is no.

Also solved by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

Editor's note: The solutions (i.e., answers) provided by the proposer are

Solution 1: $AB = 7P^2,\ \ BC = 7P^2,\ \ CD = 21P^2 - 1372,\ \ DA = 7P^2.$

Solution 2: $AB = 7P^2,\ \ BC = P^3 - 98P,\ \ CD = 21P^2 - 1372,\ \ DA = P^3 - 98P.$

• **5632** *Proposed by Toyesh Prakash Sharma (Student), St. C.F. Andrews School, Agra, India.*

Show that

$$\frac{1-\sqrt{2}\sin\left(89/2\right)^\circ}{1+\sqrt{2}\sin\left(89/2\right)^\circ} < \tan^2\left(\frac{1}{2}\right)^\circ \cdot \tan^2\left(\frac{2}{2}\right)^\circ \cdot \tan^2\left(\frac{3}{2}\right)^\circ \cdots \tan^2\left(\frac{89}{2}\right)^\circ.$$

Editor's comments: No solution was received for this problem. Albert Stadler has noticed that the claim in the problem is incorrect, which has also been comfirmed by the proposer of the problem. Motivated by the problem nonetheless, Albert Stadler has created an alternative inequality which will be presented as a new problem in a future issue of this section. An interesting problem has value even if it is affected by an error. My thanks go to both Toyesh Prakash Sharma for originating the problem, and to Albert Stadler for putting a longer life into it by an astute alteration.

• **5633** *Proposed by Goran Conar, Varaždin, Croatia.*

Calculate:

$$\lim_{n\to+\infty} \sqrt[n]{\sinh n + \tanh n}.$$

Solution 1 by Daniel Vacaru, Pitesti, Romania.

By d'Alembert, we calculate

$$\lim_{n\to\infty} \frac{\sinh\left(n+1\right) + \tanh\left(n+1\right)}{\sinh n + \tanh n} = \lim_{n\to\infty} \frac{\sinh\left(n+1\right)}{\sinh n} \lim_{n\to\infty} \frac{1+\frac{1}{\cosh(n+1)}}{1+\frac{1}{\cosh n}}.$$

We know that $\lim_{n\to\infty} \dfrac{1}{\cosh n} = 0$. For $\lim_{n\to\infty} \dfrac{\sinh(n+1)}{\sinh n}$, we have

$$\lim_{n\to\infty} \frac{\sinh(n+1)}{\sinh n} = \lim_{n\to\infty} \frac{\frac{e^{2(n+1)}-1}{e^{n+1}}}{\frac{e^{2n}-1}{e^n}} = \frac{1}{e}\lim_{n\to\infty} \frac{e^{2(n+1)-1}}{e^{2n}-2} = \frac{1}{e}\lim_{n\to\infty} \frac{e^2 - \frac{1}{e^{2n}}}{1-\frac{1}{e^{2n}}} = e.$$

We obtain the limit as e.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let

$$y = \lim_{n\to+\infty} \sqrt[n]{\sinh n + \tanh n}.$$

Then,

$$\begin{aligned} \ln y &= \ln\left(\lim_{n\to+\infty} \sqrt[n]{\sinh n + \tanh n}\right) \\ &= \lim_{n\to+\infty} \ln \sqrt[n]{\sinh n + \tanh n} \\ &= \lim_{n\to+\infty} \frac{\ln(\sinh n + \tanh n)}{n}. \end{aligned}$$

By, L'Hôpital's Rule,

$$\begin{aligned} \lim_{n\to+\infty} \frac{\ln(\sinh n + \tanh n)}{n} &= \lim_{n\to+\infty} \frac{\cosh n + \operatorname{sech}^2 n}{\sinh n + \tanh n} \\ &= \lim_{n\to+\infty} \coth n \left(\frac{1+\operatorname{sech}^3 n}{1+\operatorname{sech} n}\right). \end{aligned}$$

Now,

$$\lim_{n\to+\infty} \coth n = 1 \quad \text{and} \quad \lim_{n\to+\infty} \operatorname{sech} n = 0,$$

so

$$\lim_{n\to+\infty} \frac{\ln(\sinh n + \tanh n)}{n} = 1.$$

Finally,

$$y = \lim_{n\to+\infty} \sqrt[n]{\sinh n + \tanh n} = e^1 = e.$$

Solution 3 by Brian D. Beasley, Department of Mathematics, Presbyterian College, Clinton, SC.

We show that the limit is e.

Let $f(n) = \sqrt[n]{\sinh n + \tanh n}$. Then $\ln(f(n)) = (1/n)\ln(\sinh n + \tanh n)$. Using L'Hôpital's Rule, we have

$$\lim_{n\to\infty} \ln(f(n)) = \lim_{n\to\infty} \frac{\ln(\sinh n + \tanh n)}{n} = \lim_{n\to\infty} \frac{\cosh n + \text{sech}^2 n}{\sinh n + \tanh n}.$$

This limit in turn equals

$$\lim_{n\to\infty} \frac{\coth n + \text{sech}^2 n \ \text{csch}\ n}{1 + \text{sech}\ n} = 1.$$

Hence we conclude $\lim_{n\to\infty} f(n) = e^1 = e$.

Solution 4 by Bruno Salgueiro Fanego, Viveiro, Spain.

$$\lim_{N\to\infty} \sqrt[n]{\sinh n + \tanh n} = \lim_{n\to\infty} \sqrt[n]{\frac{e^n - e^{-n}}{2} + \frac{e^n - e^{-n}}{e^n + e^{-n}}} =$$

$$= \lim_{n\to\infty} \sqrt[n]{e^n \frac{1-e^{-2n}}{2} + \frac{1-e^{-2n}}{1+e^{-2n}}} = \lim_{n\to\infty} e \sqrt[n]{\frac{1-e^{-2n}}{2} + e^{-n}\frac{1-e^{-2n}}{1+e^{-2n}}} =$$

$$= e \lim_{n\to\infty} \left(\left(1 - e^{-2n}\right) \left(\frac{1}{2} + \frac{e^{-n}}{1+e^{-2n}} \right) \right)^{\frac{1}{n}} = e \left((1-0) \left(\frac{1}{2} + \frac{0}{1+0} \right) \right)^0 = e \left(\frac{1}{2} \right)^0 = e.$$

Solution 5 by David E. Manes, Oneonta, NY.

The value of the limit is the real number e.

We will use the following form of l'Hôpital's rule: If f and g are functions that are defined in an open interval $(a, +\infty)$ and $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$ and if $\lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to+\infty} \frac{f(x)}{g(x)}$ exists and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$.

Let $y = (\sinh n + \tanh n)^{1/n}$. Then $\ln y = \dfrac{\ln(\sinh n + \tanh n)}{n}$. Let $f(n) = \ln(\sinh n + \tanh n)$ and $g(n) = n$. Then $f'(n) = \dfrac{\cosh n + \text{sech}^2\, n}{\sinh n + \tanh n}$ and $g'(n) = 1$. Note that

$$\lim_{n\to+\infty} \sinh n = \lim_{n\to+\infty} \cosh n = +\infty \ \text{ and } \ \lim_{n\to+\infty} \tanh n = 1.$$

Therefore,

$$\lim_{n\to+\infty} f(n) = \lim_{n\to+\infty} \ln(\sinh n + \tanh n) = +\infty = \lim_{n\to+\infty} g(n) = \lim_{n\to+\infty} n$$

and

$$\lim_{n\to+\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to+\infty} \frac{\cosh n + \text{sech}^2\, n}{\sinh n + \tanh n} = \lim_{n\to+\infty} \frac{\left(1 + \dfrac{1}{\cosh^3 n}\right)}{\left(\tanh n + \dfrac{\tanh n}{\cosh n}\right)} = 1.$$

Therefore, by l'Hôpital's rule,

$$\lim_{n\to+\infty} \ln y = \lim_{n\to+\infty} \frac{f(n)}{g(n)} = \lim_{n\to+\infty} \left(\frac{\ln(\sinh n + \tanh n)}{n} \right) = \lim_{n\to+\infty} \frac{f'(n)}{g'(n)} = 1.$$

Since the natural logarithm is a continuous function it follows that

$$\lim_{n\to+\infty} \ln y = \ln(\lim_{n\to+\infty} y) = 1.$$

Taking the exponential of both sides implies $\lim_{n\to+\infty} y = e = \lim_{n\to+\infty} \sqrt[n]{\sinh n + \tanh n}$. This completes the solution.

Solution 6 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

The limit is e.

Let $f(x) = \sinh x + \tanh x$. Notice that for positive values of x,

$$0 < f(x) = \sinh x(1 + \operatorname{sech} x) \leqslant 2 \sinh x = e^x - e^{-x} < e^x$$

and $f'(x) = \cosh x + \operatorname{sech}^2 x > 0$, so that f is increasing on $(0, \infty)$. Thus, for positive integers n, $\sqrt[n]{f(n)} < \sqrt[n]{e^n} = e$ and $\sqrt[n]{f(n)}$ is an increasing sequence. By the Monotone Convergence Theorem, the sequence $\sqrt[n]{f(n)}$ converges; call the limit L. Since $e^{-n} = 1/e^n < e^n/2$, then $2 \sinh n = e^n - e^{-n} > e^n/2$ and for positive integers n,

$$\ln(\sinh n + \tanh n) > \ln \sinh n > \ln \frac{e^n}{4} = n - \ln 4,$$

so that $\ln(\sinh n + \tanh n)$ approaches infinity as n approaches infinity. Thus, we may use l'Hospital's Rule to compute

$$\begin{aligned}
\ln L &= \lim_{n\to\infty} \frac{\ln(\sinh n + \tanh n)}{n} \\
&= \lim_{n\to\infty} \frac{\cosh n + \operatorname{sech}^2 n}{\sinh n + \tanh n} \cdot \frac{\operatorname{csch} n}{\operatorname{csch} n} \\
&= \lim_{n\to\infty} \frac{\coth n + \operatorname{sech}^2 n \operatorname{csch} n}{1 + \operatorname{sech} n} \\
&= \lim_{n\to\infty} \coth n \\
&= \lim_{n\to\infty} \frac{e^n + e^{-n}}{e^n - e^{-n}} \cdot \frac{e^{-n}}{e^{-n}} \\
&= \lim_{n\to\infty} \frac{1 + e^{-2n}}{1 - e^{-2n}} \\
&= 1.
\end{aligned}$$

Since $\ln L = 1$, then $L = e^1 = e$.

Solution 7 by Peter Fulop, Gyomro, Hungary.

Expressing $\sinh(n)$ and $\tanh(n)$ by exponential function and performing several cancellations we get:

$$\sqrt[n]{\sinh(n)+\tanh(n)} = \sqrt[n]{\frac{(e^n+1)^2(e^{2n}-1)}{2(e^{2n}+1)e^n}} \tag{1}$$

Taking the limit of (1).

$$\lim_{n\to\infty} \underbrace{\sqrt[n]{\frac{(e^{2n}-1)}{(e^{2n}+1)}}}_{\to 1} \sqrt[n]{\frac{(e^n+1)^2}{2e^n}} \tag{2}$$

Obviously the first term of (2) goes to 1.

Determine the value of the limit of second term

$$\lim_{n\to\infty} \frac{\sqrt[n]{(e^n+1)^2}}{\sqrt[n]{2}e} \tag{3}$$

where the $\lim\limits_{n\to\infty} \sqrt[n]{2} \to 1$ and for the numerator is valid the following inequality:

$$\underbrace{\lim_{n\to\infty} \sqrt[n]{(e^n)^2}}_{\to e^2} < \lim_{n\to\infty} \sqrt[n]{(e^n+1)^2} < \lim_{n\to\infty} \sqrt[n]{(2e^n)^2} \tag{4}$$

The limit of LHS equals to e^2 and the RHS of (4) also goes to e^2 because $\lim\limits_{n\to\infty} \sqrt[n]{4}$ goes to 1. It follows that $\lim\limits_{n\to\infty} \sqrt[n]{(e^n+1)^2} = e^2$. So the limit of (3) goes to e.

Turning back to (2), the limit equals e: $\lim\limits_{n\to\infty} \sqrt[n]{\sinh(n)+\tanh(n)} = e$.

Solution 8 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the nth root quotient criterion

$$\begin{aligned}
\lim_{n\to+\infty} \sqrt[n]{\sinh n+\tanh n} &= \lim_{n\to+\infty} \frac{\sinh n+\tanh n}{\sinh(n-1)+\tanh(n-1)} \\
&= \lim_{n\to+\infty} \frac{\sinh n}{\sinh(n-1)} \\
&= e.
\end{aligned}$$

Solution 9 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\sqrt[n]{\sinh n+\tanh n}=\sqrt[n]{\frac{e^n-e^{-n}}{2}+\frac{e^n-e^{-n}}{e^n+e^{-n}}}=e\sqrt[n]{\frac{1-e^{-2n}}{2}+e^{-n}\frac{e^n-e^{-n}}{e^n+e^{-n}}}\to e$$

as n tends to infinity, since $\dfrac{e^n-e^{-n}}{e^n+e^{-n}}\to 1$ and $\sqrt[n]{\dfrac{1}{2}}\to 1$.

Solution 10 by Kee-Wai Lau, Hong Kong, China.

Since $\sinh n=\dfrac{e^n-e^{-n}}{2}$ and $\tanh n=\dfrac{e^n-e^{-n}}{e^n+e^{-n}}$, so for $n\geqslant 1$, we have

$$\frac{e^n}{3}<\sinh n<\frac{e^n}{2}\quad\text{and}\quad 0<\tanh n<\frac{e^n}{2}.$$

Hence

$$\frac{e}{\sqrt[n]{3}}<\sqrt[n]{\sinh n+\tanh n}<e.$$

It follows that $\lim\limits_{n\to+\infty}\sqrt[n]{\sinh n+\tanh n}=e.$

Also solved by Team Just ∫du It, Newark Academy; Livingston, NJ.

• **5634** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turna-Severin, Romania.*

a) Find all real numbers x such that $\tan 3x=\tan 2x+\tan x$.
b) Find:

$$\Omega=\int\tan\left(x+\frac{\pi}{3}\right)\tan 3x\tan\left(2x-\frac{\pi}{3}\right)dx.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.
Problem:

a) Find all real numbers x such that $\tan 3x=\tan 2x+\tan x$.

b) Find

$$\Omega=\int\tan\left(x+\frac{\pi}{3}\right)\tan 3x\tan\left(2x-\frac{\pi}{3}\right)\,dx.$$

Solution:

a) Using the identity

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

it follows that

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \quad \text{and} \quad \tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

The equation $\tan 3x = \tan 2x + \tan x$ then becomes

$$\frac{3\tan x - \tan^3 x}{1-3\tan^2 x} = \frac{2\tan x}{1-\tan^2 x} + \tan x = \frac{3\tan x - \tan^3 x}{1-\tan^2 x},$$

or

$$\frac{2\tan^3 x(3-\tan^2 x)}{(1-3\tan^2 x)(1-\tan^2 x)} = 0.$$

This yields

$$\tan x = 0 \quad \text{or} \quad \tan x = \pm\sqrt{3}.$$

Thus, the solutions to the equation $\tan 3x = \tan 2x + \tan x$ are all real numbers x of the form

$$x = k\frac{\pi}{3},$$

where k is any integer.

b) Note

$$\tan 3x = \tan\left(x + \frac{\pi}{3} + 2x - \frac{\pi}{3}\right) = \frac{\tan\left(x+\frac{\pi}{3}\right) + \tan\left(2x - \frac{\pi}{3}\right)}{1 - \tan\left(x+\frac{\pi}{3}\right)\tan\left(2x-\frac{\pi}{3}\right)},$$

so

$$\tan\left(x+\frac{\pi}{3}\right)\tan 3x \tan\left(2x - \frac{\pi}{3}\right) = \tan 3x - \tan\left(x+\frac{\pi}{3}\right) - \tan\left(2x-\frac{\pi}{3}\right).$$

Thus,

$$\begin{aligned}
&\int \tan\left(x+\frac{\pi}{3}\right)\tan 3x \tan\left(2x-\frac{\pi}{3}\right)\,dx \\
&\qquad = \int \tan 3x\,dx - \int \tan\left(x+\frac{\pi}{3}\right)\,dx - \int \tan\left(2x-\frac{\pi}{3}\right)\,dx \\
&\qquad = \frac{1}{3}\ln|\sec 3x| - \ln\left|\sec\left(x+\frac{\pi}{3}\right)\right| - \frac{1}{2}\ln\left|\sec\left(2x-\frac{\pi}{3}\right)\right| + C.
\end{aligned}$$

Solution 2 by David E. Manes, Oneonta, NY.

The solutions for the equation in a) are $x = \pm n\pi$ or $x = \pm\dfrac{\pi}{3} + n\pi$, where n is any integer. For part b) we will show

$$\Omega = \ln\left|\frac{\sqrt[3]{\sec(3x)}}{\sec\left(\frac{2\pi}{3} - x\right)\sqrt{\sec\left(\frac{\pi}{3} - 2x\right)}}\right| + C,$$

where C is a constant of integration.

To obtain the solutions in a), we will use the following identities:

$$\tan 3x = \tan x\left(\frac{3 - \tan^2 x}{1 - 3\tan^2 x}\right) \quad \text{and} \quad \tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

Substituting these values in the given equation in a), one obtains

$$\tan x\left(\frac{3 - \tan^2 x}{1 - 3\tan^2 x}\right) = \frac{2\tan x}{1 - \tan^2 x} + \tan x.$$

Dividing out $\tan x$ and noting that if $\tan x = 0$, then $x = \pm n\pi$ (n-any integer) and these values do satisfy the given equation. Then

$$\frac{3 - \tan^2 x}{1 - 3\tan^2 x} = \frac{2}{1 - \tan^2 x} + 1.$$

Multiplying this equation by $(1 - 3\tan^2 x)(1 - \tan^2 x)$ and simplifying, we get $6\tan^2 x - 2\tan^4 x = 0$ or $2\tan^2 x(3 - \tan^2 x) = 0$ so that either $\tan x = 0$ or $\tan x = \pm\sqrt{3}$ in which cases $x = \pm n\pi$ or $x = (\pm\pi/3) + n\pi$, where n is any integer. This completes the solution for part a).

To evaluate the integral Ω we will use the triple tangent identity that states if $x + y + z = \pi$, then $\tan x \cdot \tan y \cdot \tan z = \tan x + \tan y + \tan z$. To begin, the tangent function has period π whence $\tan(x + \pi/3) = \tan(x + \pi/3 - \pi) = \tan(x - 2\pi/3)$. Moreover, the tangent is an odd function implies $\tan(x - 2\pi/3) = -\tan((2\pi/3) - x)$ and $\tan(2x - \pi/3) = -\tan((\pi/3) - 2x)$. Therefore,

$$\tan\left(x + \frac{\pi}{3}\right)\tan 3x \tan\left(2x - \frac{\pi}{3}\right) = \tan\left(\frac{2\pi}{3} - x\right)\tan 3x \tan\left(\frac{\pi}{3} - 2x\right).$$

Summing the three angles, we get $((2\pi/3) - x) + (3x) + ((\pi/3) - 2x) = \pi$. By the triple tangent identity, it follows that

$$\tan\left(\frac{2\pi}{3} - x\right)\tan 3x \tan\left(\frac{\pi}{3} - 2x\right) = \tan\left(\frac{2\pi}{3} - x\right) + \tan 3x + \tan\left(\frac{\pi}{3} - 2x\right).$$

Therefore,

$$\begin{aligned}\Omega &= \int \tan\left(x+\frac{\pi}{3}\right)\tan 3x \tan\left(2x-\frac{\pi}{3}\right)dx\\ &= \int \tan\left(\frac{2\pi}{3}-x\right)dx + \int \tan 3x\,dx + \int \tan\left(\frac{\pi}{3}-2x\right)dx.\end{aligned}$$

Since $\int \tan u\,du = \ln|\sec u| + C$, it follows that using u-substitutions one can show

$$\int \tan\left(\frac{2\pi}{3}-x\right)dx = -\ln\left|\sec\left(\frac{2\pi}{3}-x\right)\right| + C_1,\ \int \tan 3x\ dx = \frac{1}{3}\ln|\sec(3x)| + C_2,$$

$$\int \tan\left(\frac{\pi}{3}-2x\right)dx = -\frac{1}{2}\ln\left|\sec\left(\frac{\pi}{3}-2x\right)\right| + C_3,$$

where C_1, C_2, C_3 are constants of integration. Hence,

$$\begin{aligned}\Omega &= \int \tan\left(x+\frac{\pi}{3}\right)\tan 3x \tan\left(2x-\frac{\pi}{3}\right)dx\\ &= -\ln\left|\sec\left(\frac{2\pi}{3}-x\right)\right| + \frac{1}{3}\ln|\sec(3x)| - \frac{1}{2}\ln\left|\sec\left(\frac{\pi}{3}-2x\right)\right|\\ &= \ln\left|\frac{\sqrt[3]{\sec(3x)}}{\sec\left(\frac{2\pi}{3}-x\right)\sqrt{\sec\left(\frac{\pi}{3}-2x\right)}}\right| + C,\end{aligned}$$

where C is a constant of integration. This completes the solution of part b).

Also solved by Peter Fulop, Gyomro, Hungary; Kee-Wai Lau, Hong Kong, China.

• **5635** *Proposed by Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Assuming that

$$\left(1-3x+3x^2\right)^8\left(1-x-x^2+x^3+3x^4+3x^5\right)^{17} = a_0 + a_1x + a_2x^2 + \cdots + a_{100}x^{100} + a_{101}x^{101},$$

calculate the value of

$$a_1 + a_2 + a_3 + \cdots + a_{49} + a_{50} \quad \text{and} \quad a_0 + a_4 + a_8 + \cdots + a_{100}.$$

Editor's comments: This is a well-stated problem with a definite solution. Except for the proposer, no one else has supplied a solution. The original solution provided by the proposer contained an irremovable error. And a subsequent later solution provided by the proposer is not of the type that is attained via a key feature discovered and utilized within the problem; instead, it is essentially a direct brute-force computation of each term of the said 50 terms, and their subsequent summation. For this reason, the proposer-provided solution is not mathematically informative and therefore not shown here. Nevertheless, this problem poses a new challenge to the extent that there is no known non-brute-force solution that *efficiently* produces the answer via some key feature(s) of the problem. Nor is there yet a known demonstration of the impossibility of a non-brute-force solution. As such, it remains an open problem.

• **5636** *Proposed by Ovidiu Furdui Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Prove that

$$\sum_{n=0}^{\infty} n! \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right)^2 = e \sum_{n=1}^{\infty} \frac{1}{n \cdot n!}.$$

Solution 1 by Seán M. Stewart, Bomaderry, NSW, Australia.

Let

$$S = \sum_{n=0}^{\infty} n! \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right)^2.$$

Observe that

$$\begin{aligned} e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!} &= e - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \\ &= \sum_{k=n+1}^{\infty} \frac{1}{k!} = \sum_{k=0}^{\infty} \frac{1}{(k+n+1)!}, \end{aligned} \tag{5}$$

after a reindexing of $k \mapsto k + n + 1$ has been made.

Now for $n = 0, 1, 2, \ldots$ consider the integral

$$\frac{1}{n!} \int_0^1 (1-t)^n e^t \, dt.$$

Evaluating the integral we find

$$\frac{1}{n!} \int_0^1 (1-t)^n e^t \, dt = \frac{1}{n!} \int_0^1 (1-t)^n \sum_{k=0}^{\infty} \frac{t^k}{k!} \, dt = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 t^k (1-t)^n \, dt.$$

As $t \in (0,1)$, the interchange made here between the summation and integration is permissible due to the positivity of all terms involved. Recalling the integral definition for the beta function of

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad x, y > 0,$$

we have

$$\frac{1}{n!}\int_0^1 (1-t)^n e^t\,dt = \frac{1}{n!}\sum_{k=0}^{\infty}\frac{1}{k!}\,\mathrm{B}(k+1,n+1) = \sum_{k=0}^{\infty}\frac{1}{(k+n+1)!},$$

or

$$\frac{1}{n!}\int_0^1 (1-t)^n e^t\,dt = e - \sum_{k=0}^{n}\frac{1}{k!}. \tag{6}$$

Here the result in (5) has been used together with the relation of the beta function to the gamma function Γ of $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $\Gamma(n+1) = n!$ when n is a non-negative integer.

Returning to the sum S, using (6) it may be rewritten as

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} n!\left(\frac{1}{n!}\int_0^1 (1-t)^n e^t\,dt\right)^2 \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 (1-t)^n e^t\,dt \int_0^1 (1-x)^n e^x\,dx \\
&= \int_0^1\int_0^1 e^{x+t}\sum_{n=0}^{\infty}\frac{[(1-t)(1-x)]^n}{n!}\,dx\,dt \\
&= \int_0^1\int_0^1 e^{x+t}e^{(1-t)(1-x)}\,dx\,dt \\
&= \int_0^1\int_0^1 e^{1+xt}\,dx\,dt = e\int_0^1\int_0^1 e^{xt}\,dx\,dt \\
&= e\int_0^1 \frac{e^t-1}{t}\,dt = e\int_0^1\sum_{n=1}^{\infty}\frac{t^{n-1}}{n!}\,dt \\
&= e\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 t^{n-1}\,dt = e\sum_{n=1}^{\infty}\frac{1}{n\cdot n!},
\end{aligned}$$

as required to prove. Any interchanges we have made between summation and integration signs are permissible due to the positivity of all terms involved.

Solution 2 Peter Fulop, Gyomro, Hungary.

Prove that

$$\sum_{n=0}^{\infty} n!\Big(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \ldots\ldots\ldots - \frac{1}{n!}\Big)^2 = e\sum_{n=1}^{\infty}\frac{1}{nn!}$$

1 LHS

It is known that $e = \sum\limits_{k=0}^{\infty}\frac{1}{k!}$ the LHS can be formed in the following way:

$$LHS = \sum_{n=0}^{\infty} n!\Big(\sum_{k=0}^{\infty}\frac{1}{k!} - \sum_{k=0}^{n}\frac{1}{k!}\Big)^2 \tag{7}$$

Let's combine the two inside sums of (7) and perform a re-indexig:

$$LHS = \sum_{n=0}^{\infty} n!\Big(\sum_{k=n+1}^{\infty}\frac{1}{k!}\Big)^2 = \sum_{n=0}^{\infty} n!\Big(\sum_{k=1}^{\infty}\frac{1}{(n+k)!}\Big)^2 \tag{8}$$

Let's transform the $\sum_{k=1}^{\infty}\frac{1}{(n+k)!}$ into integral form using Γ and β functions:

$$\sum_{k=1}^{\infty}\frac{1}{(n+k)!} = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{n!(k-1)!}{(n+k)!(k-1)!} = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{\Gamma(n+1)\Gamma(k)}{\Gamma(n+k+1)(k-1)!} \tag{9}$$

Using the connection between Γ and β and then the integral form of β we get:

$$\sum_{k=1}^{\infty}\frac{1}{(n+k)!} = \sum_{k=1}^{\infty}\frac{\beta(k,n+1)}{n!(k-1)!} = \sum_{k=1}^{\infty}\int_0^1 \frac{x^{k-1}(1-x)^n}{(k-1)!n!}dx \tag{10}$$

By the exchanging the order of summation and integration:

$$\sum_{k=1}^{\infty}\frac{1}{(n+k)!} = \int_0^1 \frac{(1-x)^n}{n!}\sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}dx = \int_0^1 \frac{e^x(1-x)^n}{n!}dx \tag{11}$$

Turning back to (8)

$$LHS = \sum_{n=0}^{\infty} n!\Big(\int_0^1 \frac{e^x(1-x)^n}{n!}dx\Big)^2, \tag{12}$$

$$LHS = \sum_{n=0}^{\infty} n!\Big(\int_0^1 \frac{e^x(1-x)^n}{n!}dx\Big)\Big(\int_0^1 \frac{e^y(1-y)^n}{n!}dy\Big). \tag{13}$$

Let's exchange the order of the summation and the first integral then the second integral:

$$LHS = \int_0^1\int_0^1 \sum_{n=0}^{\infty} e^x(1-x)^n\frac{e^y(1-y)^n}{n!}dxdy \tag{14}$$

Performing the summation in (14) we get:

$$LHS = e\int_0^1\int_0^1 e^{xy}dxdy \tag{15}$$

2 RHS

$$RHS = e\sum_{n=1}^{\infty}\frac{1}{nn!} = \Big(\sum_{n=0}^{\infty}\frac{1}{n!}\Big)\Big(\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+1)!}\Big) \tag{16}$$

Let's form the Cauchy product of the two sums of (16)

$$RHS = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!}\frac{1}{(1+k)(1+k)!} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\frac{1}{(1+k)^2} \tag{17}$$

Taking into account that $\frac{1}{(1+k)^2} = \int_0^1\int_0^1 x^k y^k dxdy$, the (17) equals to the following:

$$RHS = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\int_0^1\int_0^1 x^k y^k dxdy = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1\int_0^1\sum_{k=0}^{n}\binom{n}{k}x^k y^k dxdy \tag{18}$$

Using the binomial theorem $\sum_{k=0}^{n}\binom{n}{k}x^k y^k = (1+xy)^n$

$$RHS = \int_0^1\int_0^1\sum_{n=0}^{\infty}\frac{(1+xy)^n}{n!}dxdy \tag{19}$$

Performing the summation in (19) we get:

$$RHS = e\int_0^1\int_0^1 e^{xy}dxdy \tag{20}$$

The statement is proved because LHS=RHS by (15) and (20).

Solution 3 by Narendra Bhandari, Bajura district, Nepal.

Since

$$e - \sum_{k=0}^{n}\frac{1}{k!} = \sum_{k=n+1}^{\infty}\frac{1}{k!} = \sum_{k=0}^{\infty}\frac{1}{(k+n+1)!} = \sum_{k=0}^{\infty}\frac{n!.k!}{(n+k+1)!(n!.k!)}$$

and by the definition of beta function, $\int_0^1 x^k(1-x)^n dx = \frac{n!.k!}{(n+k+1)}$ the latter summation can be written as

$$\frac{1}{n!}\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^1 x^k(1-x)^n dx = \frac{1}{n!}\int_0^1(1-x)^n\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)dx = \frac{1}{n!}\int_0^1(1-x)^n e^x dx$$

by the use of reflection property of integral and on squaring as

$$\mathcal{A}(n)=\left(e-\sum_{k=0}^{n}\frac{1}{k!}\right)^2=\left(\frac{1}{n!}\int_0^1 x^n e^{1-x}dx\right)^2=\frac{e^2}{(n!)^2}\int_0^1\int_0^1 (xy)^n e^{-(x+y)}dxdy$$

Therefore, $\sum_{n=0}^{\infty} n!\mathcal{A}(n)$ is equal to

$$e^2\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1\int_0^1 (xy)^n e^{-(x+y)}dxdy=e^2\int_0^1\int_0^1 e^{-(x+y)}\left(\sum_{n=0}^{\infty}\frac{(xy)^n}{n!}\right)dxdy$$

$$=e^2\int_0^1\int_0^1 e^{xy}e^{-(x+y)}dxdy=e\int_0^1\int_0^1 e^{(1-x)(1-y)}dxdy=e\int_0^1\frac{1-e^{-y+1}}{y-1}dy$$

In the above steps we use the series of e^{xy} and it can be noted that $-x-y+xy=(1-x)(1-y)-1$. Further, We substitute $1-y=t$ in the last integral we have

$$e\int_0^1\frac{e^t-1}{t}dt=e\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 t^{n-1}dt=e\sum_{n=1}^{\infty}\frac{1}{n.n!}$$

and hence the proposed result is proved. The interchange of integral and summation sign is easily justified by Fubini/Tonelli's theorem.Also it can be noted that the closed form does exist which is $e\left(\mathrm{Ei}(1)-\gamma\right)$ where $\mathrm{Ei}(x)$ and γ are Exponential integral and Euler Mascheroni constant respectively. Using the definition of exponential integral we observe that

$$\lim_{\epsilon\to 0+}\int_\epsilon^1\frac{e^t-1}{t}dt=\mathrm{Ei}(1)-\lim_{\epsilon\to 0+}\left(\mathrm{Ei}(\epsilon)-\log(\epsilon)\right)$$

$$=\mathrm{Ei}(1)-\lim_{\epsilon\to 0+}\Big(\gamma+\log(\epsilon)+\mathcal{O}\left(\epsilon^2\right)-\log(\epsilon)\Big)=\mathrm{Ei}(1)-\gamma$$

we use the Puisex series of $\mathrm{Ei}(x)$.

Reference : https://mathworld.wolfram.com/ExponentialIntergral.html

*********** *********** ***********

*********** *********** ***********

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributers.

We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following recommendations. As you peruse below, you may construe that the recommendations amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Formats, Styles and Recommendations

If possible, when submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s).

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]", followed by your affiliation, city, country. Make sure you do the same for your collaborator.

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "Statement of the Problem".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "Solution of the Problem".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase "Proposed problem to SSMJ".

2. On the second line, write "Problem proposed by [your First Name, your Last Name]" and followed by the name(s) of your collaborator(s).

3. Continuing with the above (on the same line), state your affiliation, city, country (for each collaborator).

4. On a new line state the title of the problem, if any.

5. On a new line below the above, write in bold type: "Statement of the Problem".

6. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

7. Below the statement of the problem, write in bold type: "Solution of the Problem".

8. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Proposed problem to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣