Problems and Solutions

Albert Natian, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email. *To best serve the readers and contributors of this section, please adopt — as a problem proposer or a solution proposer — the formats, styles and recommendations delineated at the very end of this document.* Please keep in mind that the examples given are your best guide!

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before February 15, 2022.

• **5655** *Proposed by Kenneth Korbin, New York, NY.*

Given a Heronian triangle $\triangle ABC$ with altitude $\overline{CD} = y - 1$, sides $\overline{AC} = y$ and $\overline{BC} = y + 1$, find four possible values of y.

Editor's note: A trianlge with integer sides and integer area is called *Heronian.*

• **5656** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Find $\lim_{n\to\infty} n\left(\dfrac{\pi^2}{4} - a_n^2\right)$ where $a_n = \sum_{k=1}^{n} \arctan\left(\dfrac{1}{k^2-k+1}\right)$.

• **5657** *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy.*

Let $\alpha, \beta, \gamma > 0$ and define

$$S_n = \sum_{m=2}^{n} (\ln m)^\gamma \prod_{k=2}^{m} \frac{\alpha + k\ln k}{\beta + (k+1)\ln(k+1)}.$$

Determine sufficient and necessary condition(s) governing the parameters α, β and γ so that $\lim_{n\to\infty} S_n$ exists.

• **5658** *Proposed by Titu Zvonaru, Comănești, Romania.*

Let a, b and c be positive with $a+b+c=3$. Prove $\dfrac{1}{5+a^3}+\dfrac{1}{5+b^3}+\dfrac{1}{5+c^3}\leqslant\dfrac{1}{2}$.

• **5659** *Proposed by Narendra Bhandari, National Academy of Science and Technology, Pokhara University, Nepal.*

Prove that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\left(H_{\frac{n}{2}}-H_{\frac{n-1}{2}}\right)=\frac{\pi^4}{32}-2G^2$$

where G is Catalan's constant defined by

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\quad\text{and}\quad H_{\frac{n}{2}}-H_{\frac{n-1}{2}}=2\sum_{k=0}^{n}\frac{(-1)^k}{k+n+1}$$

• **5660** *Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Suppose the sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ of real numbers satisfy the equation

$$x_n^2+x_{n-1}^2+y_n^2+y_{n-1}^2=(y_nx_{n-1}-x_ny_{n-1})+\sqrt{3}\,(x_nx_{n-1}+y_ny_{n-1})\,.$$

Show that the sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ are periodic and find their period.

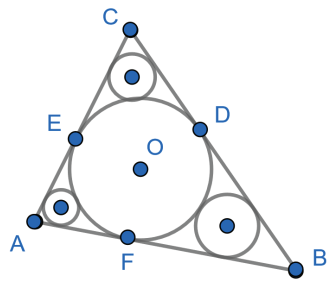
Solutions

• **5637** *Proposed by Kenneth Korbin, New York, NY.*

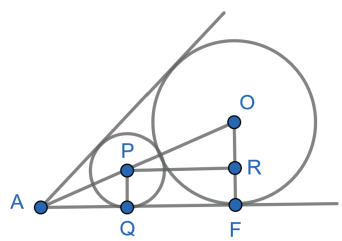
In traingle ABC three circles are tangent to the incircle, and all of these circles are tangent to two sides of the triangle. The radii of these three circles are $r_a=1/4$, $r_b=4/9$, and $r_c=16/49$. Find the sides of $\triangle ABC$.

Solution 1 by David A. Huckaby, Angelo State University, San Angelo, TX.

Let the incircle intersect side BC at point D, side AC at point E, and side AB at point F, as in the figure below.



Let r be the inradius. In the figure below, which includes the incircle and the circle tangent to sides AB and AC, $OP = r + \frac{1}{4}$ and $OR = r - \frac{1}{4}$. Letting A denote the measure of the angle at vertex A, we have $\sin \frac{A}{2} = \sin \angle PAQ = \sin \angle OPR = \frac{OR}{OP} = \frac{r - \frac{1}{4}}{r + \frac{1}{4}}$, so that $r = \frac{1}{4} \cdot \frac{1 + \sin \frac{A}{2}}{1 - \sin \frac{A}{2}}$.



So $4r = \frac{1 + \sin \frac{A}{2}}{1 - \sin \frac{A}{2}} = \frac{1 + \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)}{1 - \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{1 + \cos\left(\frac{\pi - A}{2}\right)}{1 - \cos\left(\frac{\pi - A}{2}\right)} = \frac{2\cos^2\left(\frac{\pi - A}{4}\right)}{2\sin^2\left(\frac{\pi - A}{4}\right)} = \cot^2\left(\frac{\pi - A}{4}\right)$, so that $2\sqrt{r} = \cot\left(\frac{\pi - A}{4}\right)$.

Similar calculations for vertices B and C give $\frac{3}{2}\sqrt{r} = \cot\left(\frac{\pi - B}{4}\right)$ and $\frac{7}{4}\sqrt{r} = \cot\left(\frac{\pi - C}{4}\right)$.

Letting $\alpha = \frac{\pi - A}{2}$, $\beta = \frac{\pi - B}{2}$, and $\gamma = \frac{\pi - C}{2}$, note that $\alpha + \beta + \gamma = \frac{3\pi - (A + B + C)}{2} = \frac{3\pi - \pi}{2} = \pi$. Therefore the following indentity holds (`https://en.wikipedia.org/wiki/List_of_trigonometric_identities`):

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\alpha}{2}\tan\frac{\gamma}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = 1$$

So $\frac{1}{2\sqrt{r}} \cdot \frac{2}{3\sqrt{r}} + \frac{1}{2\sqrt{r}} \cdot \frac{4}{7\sqrt{r}} + \frac{2}{3\sqrt{r}} \cdot \frac{4}{7\sqrt{r}} = 1$, whence $r = \frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{4}{7} + \frac{2}{3} \cdot \frac{4}{7} = 1$.

Now from the second figure above, $\frac{OF}{PQ} = \frac{AO}{AP}$, that is, $\frac{1}{\frac{1}{4}} = \frac{AO}{AO - 1 - \frac{1}{4}}$, whence $AO = \frac{5}{3}$.

Now since $AF^2 + OF^2 = AO^2$, $AF = \sqrt{\left(\frac{5}{3}\right)^2 - 1^2} = \frac{4}{3}$. Note from the first figure that

$AE = AF$, so that $AE = \dfrac{4}{3}$ as well. Similar calculations for vertices B and C give $BO = \dfrac{13}{5}$, so that $BF = BD = \dfrac{12}{5}$, and $CO = \dfrac{65}{33}$ so that $CD = CE = \dfrac{56}{33}$. So the sides of triangle ABC are $AB = AF + BF = \dfrac{4}{3} + \dfrac{12}{5} = 56/15$, $AC = AE + CE = \dfrac{4}{3} + \dfrac{56}{33} = 100/33$, and $BC = BD + CD = \dfrac{12}{5} + \dfrac{56}{33} = 676/165$.

Solution 2 by Daniel Văcaru, Pitești, Romania.

In article **Chains of Tangent Circles Inscribed in a Triangle**, by Giovanni Lucca, *Forum Geometricorum* 17(2017), 41-44 it is proved that $\dfrac{r}{r_a} = \dfrac{1+\sin\frac{A}{2}}{1-\sin\frac{A}{2}}$, $\dfrac{r}{r_b} = \dfrac{1+\sin\frac{B}{2}}{1-\sin\frac{B}{2}}$ and $\dfrac{r}{r_c} = \dfrac{1+\sin\frac{C}{2}}{1-\sin\frac{C}{2}}$.Then we obtain $\dfrac{\frac{r}{r_a}-1}{\frac{r}{r_a}+1} = \dfrac{4\frac{r}{\sqrt{r_b r_c}} - \left(\frac{r}{r_b}-1\right)\left(\frac{r}{r_c}-1\right)}{\left(\frac{r}{r_b}+1\right)\left(\frac{r}{r_c}+1\right)} \Leftrightarrow \dfrac{r-r_a}{r+r_a} = \dfrac{4r\sqrt{r_b r_c} - r^2 + r r_b + r r_c - r_b r_c}{r^2 + r r_b + r r_c + r_b r_c}$. Solving with Symbolab, we find $r = 1$. We obtain $\sin\dfrac{A}{2} = \dfrac{r-r_a}{r+r_a} = \dfrac{1-\frac{1}{4}}{1+\frac{1}{4}} = \dfrac{3}{5}$, $\sin\dfrac{B}{2} = \dfrac{r-r_b}{r+r_b} = \dfrac{1-\frac{4}{9}}{1+\frac{4}{9}} = \dfrac{5}{13}$ and $\sin\dfrac{C}{2} = \dfrac{1-\frac{16}{49}}{1+\frac{16}{49}} = \dfrac{33}{65}$. We have $\cos A = 1 - 2\sin^2\dfrac{A}{2}$ and analogs relationships, and $\sum\cos A = 1 + \dfrac{r}{R} \Rightarrow 3 - 2\sum\sin^2\dfrac{A}{2} = 1 + \dfrac{r}{R} \Rightarrow \dfrac{r}{R} = 2 - 2\sum\sin^2\dfrac{A}{2} \Rightarrow R = \dfrac{r}{2-2\sum\sin^2\frac{A}{2}} = \dfrac{1}{2-2\left(\left(\frac{3}{5}\right)^2 + \left(\frac{5}{13}\right)^2 + \left(\frac{33}{65}\right)^2\right)} = \dfrac{1}{2\left(1-\frac{647}{845}\right)} = \dfrac{845}{2\cdot 198} = \dfrac{845}{396}$. We have $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2} = 2\cdot\dfrac{3}{5}\cdot\dfrac{4}{5} = \dfrac{24}{25}$ and $a = 2R\sin A = \dfrac{845}{198}\cdot\dfrac{24}{25} = \dfrac{5\cdot 169\cdot 2^3\cdot 3}{2\cdot 3^2\cdot 11\cdot 5^2} = \dfrac{2^2\cdot 13^2}{3\cdot 5\cdot 11} = \dfrac{676}{165}$.Using $\sin B = 2\sin\dfrac{B}{2}\cos\dfrac{B}{2}$, we obtain $\sin B = 2\dfrac{5}{13}\dfrac{12}{13} = \dfrac{120}{169}$. We find $b = 2R\sin B = \dfrac{5\cdot 13^2}{2\cdot 3^2\cdot 11}\cdot\dfrac{2^3\cdot 3\cdot 5}{13^2} = \dfrac{25\cdot 4}{3\cdot 11} = \dfrac{100}{33}$. Again, using $\sin C = 2\sin\dfrac{C}{2}\cos\dfrac{C}{2} = 2\cdot\dfrac{33}{65}\cdot\dfrac{56}{65} = \dfrac{2^4\cdot 3\cdot 7\cdot 11}{5^2\cdot 13^2}$, we find $c = 2R\sin C = \dfrac{5\cdot 13^2}{2\cdot 3^2\cdot 11}\cdot\dfrac{2^4\cdot 3\cdot 7\cdot 11}{5^2\cdot 13^2} = \dfrac{2^3\cdot 7}{3\cdot 5^2} = \dfrac{56}{15}$.

Solution 3 by the *Eagle Problem Solvers*, Georgia Southern University, Statesboro, GA and Savannah, GA.

The side lengths of ΔABC are $a = \dfrac{676}{165}$, $b = \dfrac{100}{33}$, and $c = \dfrac{56}{15}$. If we drop a perpendicular line from the center of the circle with radius r_a to the radius of the incircle perpendicular to AC, then from the right triangle formed with hypotenuse along the angle bisector from vertex A to the incenter, we see that

$$\sin\frac{A}{2} = \frac{r - r_a}{r + r_a} = \frac{4r-1}{4r+1},$$

where r is the inradius and A is the angle at vertex A. Similarly, $\sin\frac{B}{2} = \frac{9r-4}{9r+4}$ and $\sin\frac{C}{2} = \frac{49r-16}{49r+16}$. Notice that

$$\cos\frac{B}{2} = \sqrt{1-\sin^2\frac{B}{2}} = \frac{\sqrt{(9r+4)^2-(9r-4)^2}}{9r+4} = \frac{12\sqrt{r}}{9r+4}$$

and

$$\cos\frac{C}{2} = \sqrt{1-\sin^2\frac{C}{2}} = \frac{\sqrt{(49r+16)^2-(49r-16)^2}}{49r+16} = \frac{56\sqrt{r}}{49r+16}.$$

Since $\frac{A}{2}+\frac{B}{2}+\frac{C}{2} = \frac{\pi}{2}$, then

$$\begin{aligned}
\sin\frac{A}{2} &= \cos\left(\frac{B}{2}+\frac{C}{2}\right) \\
\frac{4r-1}{4r+1} &= \cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2} \\
&= \frac{12\sqrt{r}}{9r+4}\cdot\frac{56\sqrt{r}}{49r+16} - \frac{9r-4}{9r+4}\cdot\frac{49r-16}{49r+16} \\
&= \frac{672r - \left(441r^2 - 340r + 64\right)}{441r^2+340r+64} \\
\frac{4r-1}{4r+1} &= \frac{-441r^2+1012r-64}{441r^2+340r+64} \\
(4r-1)\left(441r^2+340r+64\right) &= (4r+1)\left(-441r^2+1012r-64\right) \\
3528r^3 - 2688r^2 - 840r &= 0 \\
168r(21r+5)(r-1) &= 0.
\end{aligned}$$

Since the inradius $r > 0$, then $r = 1$, so $\sin\frac{A}{2} = \frac{3}{5}$, $\sin\frac{B}{2} = \frac{5}{13}$, and $\sin\frac{C}{2} = \frac{33}{65}$. Using the double angle formula for sine,

$$\begin{aligned}
\sin A &= 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25} \\
\sin B &= 2\cdot\frac{5}{13}\cdot\frac{12}{13} = \frac{120}{169} \\
\sin C &= 2\cdot\frac{33}{65}\cdot\frac{56}{65} = \frac{3696}{4225}.
\end{aligned}$$

If R is the circumradius of ΔABC, then from the Law of Sines

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

so

$$2R = \frac{25a}{24} = \frac{169b}{120} = \frac{4225c}{3696}.$$

Thus, $b = \dfrac{125a}{169}$, $c = \dfrac{154a}{169}$, and the semiperimeter is $s = \dfrac{a+b+c}{2} = \dfrac{224a}{169}$. The circumradius $R = \dfrac{25a}{48}$ is also given by $R = \dfrac{abc}{4\Delta}$, where $\Delta = rs = s$ is the area of ΔABC. Thus,

$$\begin{aligned}
\frac{25a}{48} &= \frac{abc}{4s} \\
\frac{25}{12} &= \frac{bc}{s} \\
&= \frac{125 \cdot 154a^2}{169^2} \cdot \frac{169}{224a} \\
\frac{25}{12} &= \frac{125 \cdot 154a}{169 \cdot 224} \\
a &= \frac{25 \cdot 169 \cdot 224}{12 \cdot 125 \cdot 154} = \frac{676}{165},
\end{aligned}$$

$b = \dfrac{125}{169} \cdot \dfrac{676}{165} = \dfrac{100}{33}$, and $c = \dfrac{154}{169} \cdot \dfrac{676}{165} = \dfrac{56}{15}$.

Solutions were also received from David E. Manes, Oneonta, NY; Arkady Alt, San Jose, CA; Albert Stadler, Herrliberg, Switzerland.

• **5638** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania.*

Let a, b, c be real numbers such that $a, b, c \geqslant -1$, and $a + b + c = 3$. Then:

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{3}{4}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle.

First we prove that $\big((x+1)/(x+3)\big)^2 \leqslant \frac{1}{8}(x+1)$ for $x \geqslant -1$:

$$\left(\frac{x+1}{x+3}\right)^2 \leqslant \frac{1}{8}(x+1) \iff \frac{x+1}{(x+3)^2} \leqslant \frac{1}{8} \iff 8x + 8 \leqslant x^2 + 6x + 9 \iff 0 \leqslant (x-1)^2.$$

(This says that the line tangent to the curve $y = \big((x+1)/(x+3)\big)^2$ at $x = 1$ lies on or above the curve for $x \geqslant -1$.) Now we see that

$$\sum_{cyclic} \left(\frac{a+1}{a+3}\right)^2 \leqslant \sum_{cyclic} \frac{a+1}{8} = \frac{a+b+c+3}{8} = \frac{3}{4}.$$

Equality holds if and only if $a = b = c = 1$.

Solution 2 by Kee Wai Lau, Hong Kong, China.

Since

$$\left(\frac{a+1}{a+3}\right)^2 = \frac{1}{8}(a+1)\left(1-\left(\frac{a-1}{a+3}\right)^2\right) \leqslant \frac{a+1}{8}$$

and similarly $\left(\frac{b+1}{b+3}\right)^2 \leqslant \frac{b+1}{8}$ and $\left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{c+1}{8}$, then

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{(a+1)+(b+1)+(c+1)}{8} = \frac{3}{4}.$$

Solution 3 by Peter Fulop, Gyomro, Hungary.

Let a,b,c be real numbers such that a,b,c $\geqslant -1$, and $a + b + c = 3$. Then:

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{3}{4} \tag{1}$$

Let's applying the AM-GM unequality for the denominators of (1)

$$\sqrt{2(a+1)} \leqslant \frac{2+a+1}{2} = \frac{a+3}{2} \tag{2}$$

$$2\sqrt{2}\sqrt{(a+1)} \leqslant a+3 \tag{3}$$

The unequality of (3) is valid for c and b, we could increase the LHS of (1) by using LHS of (3):

$$\frac{(a+1)^2}{8(a+1)} + \frac{(b+1)^2}{8(b+1)} + \frac{(c+1)^2}{8(c+1)} = \frac{a+b+c+3}{8} = \frac{6}{8} \tag{4}$$

Used that $a + b + c = 3$, the staement is prooved.

Solution 4 by Daniel Sitaru, National Economic College ¨Theodor Costescu¨ Drobeta Turnu-Severin, Romania .

Let a, b, c be real numbers such that $a, b, c \geqslant -1$ and $a + b + c = 3$.Then

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{3}{4}.$$

Solution proposed by Daniel Văcaru and Marin Chirciu, Pitești, Romania
We prove that

$$\left(\frac{x+1}{x+3}\right)^2 \leqslant \frac{x+1}{8}, , \forall x \geqslant -1$$

It is equivalent to $8(x+1) \leqslant x^2 + 6x + 9 \Leftrightarrow x^2 - 2x + 1 \geqslant 0$.
Then

$$\sum \left(\frac{a+1}{a+3}\right)^2 \leqslant \frac{\sum x + 3}{8} = \frac{3}{4}.$$

Solution 5 by David E. Manes, Oneonta, NY.

Assume $-1 \leqslant a \leqslant b \leqslant c$ and $-1 \leqslant a \leqslant 0$. Then the function $f(x) = \left(\dfrac{x+1}{x+3}\right)^2$ is increasing on the closed interval $[-1, 0]$ with maximum value $f(0) = 1/9$. The inequality then reads if $0 \leqslant b \leqslant c$ and $b + c = 3$, then

$$\left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{3}{4} - \frac{1}{9} = \frac{23}{36}.$$

Let $b = 3 - c$. Then

$$\frac{c^2 - 8c + 16}{c^2 - 12c + 36} + \frac{c^2 + 2c + 1}{c^2 + 6c + 9} \leqslant \frac{23}{36}.$$

Equivalently, one can show

$$36\left[(c^2 - 8c + 16)(c^2 + 6c + 9) + (c^2 + 2c + 1)(c^2 - 12c + 36)\right] - \\ 23(c^2 + 6c + 9)(c^2 - 12c + 36) \leqslant 0$$

Doing the arithmetic, one obtains for the left side of the inequality

$$49c^4 - 294c^3 + 261c^2 + 540c - 972$$

which is $\leqslant 0$ on the closed interval $\left[(3/14)(7 - \sqrt{89 - 16\sqrt{43}}, (3/14)(7 + \sqrt{89 - 16\sqrt{43}}\right]$ (approximate values $[-1.484, 4.484]$). This interval includes the admissible values of b and c. Therefore, if $-1 \leqslant a \leqslant 0$ and $a + b + c = 3$, then

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 < \frac{3}{4}.$$

Assume $-1 \leqslant a \leqslant b \leqslant 0$. Then the maximum value of $f(x) = \left(\frac{x+1}{x+3}\right)^2$ for $f(a) + f(b)$ occurs when $a = b = 0$ with value $f(0) + f(0) = 2/9$. Moreover, for this case $c = 3$ and $f(c) = 4/9$ so that $f(a) + f(b) + f(c) = 2/3 < 3/4$.

For the last case assume that $0 < a \leqslant b \leqslant c$. We will use the following form of Chebyshev's inequality: let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n$ be real numbers. Then

$$\sum_{i=1}^{n} a_i b_i \leqslant \frac{1}{n} \left(\sum_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} b_i\right).$$

Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$. Consider

$$\sum_{cyclic} \left[\left(\frac{a+1}{a+3}\right)^2 - \frac{1}{4}\right].$$

Then

$$\begin{aligned}
\left(\frac{a+1}{a+3}\right)^2 - \frac{1}{4} &= \frac{a^2+2a+1}{a^2+6a+9} - \frac{1}{4} \\
&= \frac{4a^2+8a+4-a^2-6a-9}{4(a+3)^2} = \frac{3a^2+2a-5}{4(a+3)^2} \\
&= (a-1)\left(\frac{3a+5}{4(a+3)^2}\right),
\end{aligned}$$

with similar results for the b- and c-terms. Note that $a-1 \leqslant b-1 \leqslant c-1$ is increasing and $(3x+5)/(4(x+3)^2$ is a decreasing function for positive values of x. Therefore,

$$\frac{3a+5}{4(a+3)^2} \geqslant \frac{3b+5}{4(b+3)^2} \geqslant \frac{3c+5}{4(c+3)^2}$$

so that by Chebyshev's inequality

$$\begin{aligned}
\sum_{cyclic} \left[\left(\frac{a+1}{a+3}\right)^2 - \frac{1}{4}\right] &= \sum_{cyclic} (a-1)\left(\frac{3a+5}{4(a+3)^2}\right) \\
&\leqslant \frac{1}{3} \sum_{cyclic} (a-1) \sum_{cyclic} \left(\frac{3a+5}{4(a+3)^2}\right) \\
&= 0,
\end{aligned}$$

since $\sum_{cyclic} (a-1) = a-1+b-1+c-1 = 0$. Hence,

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Equality occurs if and only if $a = b = c = 1$.

Solution 6 by Arkady Alt, San Jose, CA.

Let $x := a + 1, y := b + 1, z := c + 1$. Then $x, y, z \geqslant 0, x + y + z = 6$ and

$$\sum_{cyc} \left(\frac{a+1}{a+3}\right)^2 \leqslant \frac{3}{4} \iff \sum_{cyc} \left(\frac{x}{x+2}\right)^2 \leqslant \frac{3}{4}.$$

Since $\dfrac{x}{(x+2)^2} \leqslant \dfrac{1}{8} \iff (x-2)^2 \geqslant 0$ then $\displaystyle\sum_{cyc} \left(\frac{x}{x+2}\right)^2 \leqslant \sum_{cyc} \frac{x}{8} = \frac{6}{8} = \frac{3}{4}$.

Solution 7 by Michel Battaille, Rouen, France.

Let $x = \dfrac{a+1}{2}$, $y = \dfrac{b+1}{2}$, $z = \dfrac{c+1}{2}$. The constraints become $x, y, z \geqslant 0$, $x + y + z = 3$ and the inequality to be proved now is

$$\frac{x^2}{(x+1)^2} + \frac{y^2}{(y+1)^2} + \frac{z^2}{(z+1)^2} \leqslant \frac{3}{4}.$$

Now, $(x+1)^2 - 4x = (x-1)^2 \geqslant 0$, hence $(x+1)^2 \geqslant 4x$ and therefore $\dfrac{x^2}{(x+1)^2} \leqslant \dfrac{x^2}{4x} = \dfrac{x}{4}$. A similar result holds for the other term of the left-hand side of the inequality so that

$$\frac{x^2}{(x+1)^2} + \frac{y^2}{(y+1)^2} + \frac{z^2}{(z+1)^2} \leqslant \frac{x+y+z}{4} = \frac{3}{4},$$

as desired.

Solution 8 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Let a, b, c be real numbers such that $a, b, c \geqslant -1$ and $a + b + c = 3$. Then:

$$\left(\frac{a+1}{a+3}\right)^2 + \left(\frac{b+1}{b+3}\right)^2 + \left(\frac{c+1}{c+3}\right)^2 \leqslant \frac{3}{4}.$$

By setting $x = a + 1$, $y = b + 1$, and $z = c + 1$, the problem reads

$$\frac{x^2}{(x+2)^2} + \frac{y^2}{(y+2)^2} + \frac{z^2}{(z+2)^2} \leqslant \frac{3}{4},$$

where $x, y, z \geqslant 0$ and $x + y + z = 6$.

The function $f(x) = \dfrac{x^2}{(x+2)^2}$ is increasing for $x \in (0, 6)$, and so, by the rearrangement inequality

$$3 \sum \cdot (x+2)^2 \frac{x^2}{(x+2)^2} \geqslant \sum (x+2)^2 \cdot \sum \frac{x^2}{(x+2)^2}.$$

Therefore
$$\sum \frac{x^2}{(x+2)^2} \leqslant \frac{3\sum x^2}{\sum (x+2)^2} = \frac{3\sum x^2}{36+\sum x^2},$$
where in the last identity we have used that $x+y+z=6$. So, it is enough to prove that
$$\frac{3\sum x^2}{36+\sum x^2} = 3\left(1-\frac{36}{36+\sum x^2}\right)$$
has a maximum for $x=y=z=2$, for $x,y,z \geqslant 0$ and $x+y+z=6$. This is equivalent to that $\sum x^2$ has a minimum for for $x,y,z \geqslant 0$ and $x+y+z=6$, which follows trivially for example by Lagrange multipliers. □

Solution 9 by Moti Levy, Rehovot, Israel.

Let
$$x=a+1, \quad y=b+1, \quad z=c+1.$$
Then the original inequality becomes
$$\frac{x^2}{(x+2)^2}+\frac{y^2}{(y+2)^2}+\frac{z^2}{(z+2)^2} \leqslant \frac{3}{4}, \qquad x,y,z \geqslant 0, \quad x+y+z=6.$$
Define the function
$$f(t) := \begin{cases} t^2/(t+2)^2 & \text{if} \quad t \geqslant 1, \\ t/9 & \text{if} \quad 0 \leqslant t < 1. \end{cases}$$
The function $f(t)$ is concave for $t \geqslant 0$, since $\frac{d^2 f}{dt^2} \leqslant 0$, for $t \geqslant 0$. By definition of the function $f(t)$,
$$\frac{x^2}{(x+4)^2}+\frac{y^2}{(y+4)^2}+\frac{z^2}{(z+4)^2} \leqslant f(x)+f(y)+f(z), \qquad x,y,z \geqslant 0.$$
By Jensen's inequality
$$f(x)+f(y)+f(z) \leqslant 3f\left(\frac{x+y+z}{3}\right) = 3f(2) = 3\frac{2^2}{(2+2)^2} = \frac{3}{4}.$$

Solutions were also received from Mohammad Bakkar (student), Tishreen University, Latakia, Syria; Albert Stadler, Herrliberg, Switzerland; Bruno Salgueiro Fanego, Viveiro, Spain; Nikos Ntorvas, Athens, Greece; Titu Zvonaru, Comănești, Romania.

• **5639** *Proposed by Dorin Mărghidanu, Corabia, Romania.*

If $n \in \mathbb{N}$, calculate the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)}$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX.

For $k \geqslant 1$,

$$\begin{aligned}\frac{1}{2^k - 1} - \frac{1}{2^{k+1} - 1} &= \frac{2^{k+1} - 2^k}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} \\ &= \frac{2^k\left(2 - 1\right)}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} \\ &= \frac{2^k}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)}.\end{aligned}$$

It follows that for $k \geqslant 1$,

$$\frac{2^{k-1}}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} = \frac{1}{2}\left(\frac{1}{2^k - 1} - \frac{1}{2^{k+1} - 1}\right)$$

and we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2^k - 1} - \frac{1}{2^{k+1} - 1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2^{n+1} - 1}\right)\end{aligned}$$

for all $n \geqslant 1$. As a result,

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} &= \lim_{n\to\infty}\frac{1}{2}\left(1 - \frac{1}{2^{n+1} - 1}\right) \\ &= \frac{1}{2}.\end{aligned}$$

Solution 2 proposed by Team $i$nsπir$e$d Student Problem Solving Group, Newark Academy, Livingston, NJ.

There are two factors in the denominator, therefore we can assume that the series could be a telescoping series, written:

$$\lim_{n\to\infty}\sum_{k=1}^{n} \frac{A}{2^k - 1} + \frac{B}{2^{k+1} - 1}.$$

To get to this structure, partial fractions can be used to find the coefficients A and B. To set up the partial fractions, we have:

$$\frac{A}{2^k - 1} + \frac{B}{2^{k+1} - 1} = \frac{2^{k-1}}{(2^k - 1)(2^{k+1} - 1)}.$$

Multiply through by the denominator of the right side, cancel out terms, and distribute the variables A and B to get:

$$A(2^{k+1}-1)+B(2^{k}-1)=2^{k-1}.$$

This can be rearranged to give:

$$4A2^{k-1}+2B2^{k-1}-A-B=2^{k-1}+0,$$

or

$$2^{k-1}(4A+2B)-A-B=2^{k-1}+0$$

from which we have the system:

$$\left\{\begin{array}{c} -A-B=0 \\ \\ 4A+2B=1 \end{array}\right\}.$$

Solving the system gives:

$$B=\frac{-1}{2},\quad A=\frac{1}{2}.$$

A and B can be placed back into the original summands to yield:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{A}{2^{k}-1}+\frac{B}{2^{k+1}-1}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2(2^{k}-1)}-\frac{1}{2(2^{k+1}-1)}.$$

The $\frac{1}{2}$ can be pulled out of the entire limit:

$$\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^{k}-1}-\frac{1}{2^{k+1}-1}.$$

Written in expanded form as a telescoping series, we have:

$$\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^{k}-1}-\frac{1}{2^{k+1}-1}$$

$$=\frac{1}{2}\lim_{n\to\infty}[(1-\frac{1}{3})+(\frac{1}{3}-\frac{1}{7})+(\frac{1}{7}-\frac{1}{15})+(\frac{1}{15}-\frac{1}{31})+...+(\frac{1}{2^{n}-1}-\frac{1}{2^{n+1}-1})].$$

This collapses to:

$$\frac{1}{2}\lim_{n\to\infty}(1-\frac{1}{2^{n+1}-1})=\frac{1}{2}.$$

Therefore, for $n\in\mathbb{N}$:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^{k-1}}{(2^{k}-1)(2^{k+1}-1)}=\frac{1}{2}.$$

Solution 3 by Mohammad Bakkar (student), Tishreen University, Latakia, Syria.

$$\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \sum_{k=1}^{n} \frac{2^k}{\left(2^{k+1}-2\right)\left(2^{k+1}-1\right)} = \sum_{k=1}^{n} 2^k \left(\frac{1}{\left(2^{k+1}-2\right)} - \frac{1}{\left(2^{k+1}-1\right)}\right).$$

Notice that $\dfrac{2^{k+1}}{2^{k+1}-2} = \dfrac{2^k}{2^{k+1}-1}$. So it's a telescopic sum:

$$\sum_{k=1}^{n} 2^k \left(\frac{1}{\left(2^{k+1}-2\right)} - \frac{1}{\left(2^{k+1}-1\right)}\right) = 1 - \frac{2^n}{2^{n+1}-1},$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \lim_{n\to\infty} \left(1 - \frac{2^n}{2^{n+1}-1}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

Clearly

$$\frac{1}{2^k-1} - \frac{1}{2^{k+1}-1} = \frac{\left(2^{k+1}-1\right)-\left(2^k-1\right)}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = 2\frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)}.$$

Therefore

$$\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \frac{1}{2}\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2^{n+1}-1} \to \frac{1}{2}$$

as n tends to infinity.

Solution 5 by David E. Manes, Oneonta, NY.

The value of the limit is $\dfrac{1}{2}$.

For each positive integer n, let $s_n = \displaystyle\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)}$. We will show by induction that

$$s_n = \frac{2^n-1}{2^{n+1}-1}.$$

If $n = 1$, then $s_1 = \dfrac{1}{1\cdot 3} = \dfrac{1}{3} = \dfrac{2^1-1}{2^2-1}$. Therefore, the closed form equation for s_n is valid for $n = 1$. Assume the result is true for any integer $n \geqslant 1$. For the integer $n+1$, one finds

$$\begin{aligned}
s_{n+1} &= \sum_{k=1}^{n+1} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = s_n + \frac{2^n}{(2^{n+1}-1)(2^{n+2}-1)} \\
&= \frac{2n-1}{2^{n+1}-1} + \frac{2^n}{(2^{n+1}-1)(2^{n+2}-1)} = \left(\frac{1}{2^{n+1}-1}\right)\left(2^n - 1 + \frac{2^n}{2^{n+2}-1}\right) \\
&= \left(\frac{1}{2^{n+1}-1}\right)\left(\frac{(2^n-1)(2^{n+2}-1)+2^n}{2^{n+2}-1}\right) = \left(\frac{1}{2^{n+1}-1}\right)\left(\frac{2^{2n+2}-2^{n+2}+1}{2^{n+2}-1}\right) \\
&= \left(\frac{1}{2^{n+1}-1}\right)\left(\frac{2^{n+1}(2^{n+1}-2)+1}{2^{n+2}-1}\right) = \left(\frac{1}{2^{n+1}-1}\right)\left(\frac{2^{n+1}(2^{n+1}-1)-2^{n+1}+1}{2^{n+2}-1}\right) \\
&= \left(\frac{1}{2^{n+1}-1}\right)\left(\frac{2^{n+1}(2^{n+1}-1)-(2^{n+1}-1)}{2^{n+2}-1}\right) = \frac{2^{n+1}-1}{2^{n+2}-1}.
\end{aligned}$$

Therefore, the closed form equation for $s_n$is true for the integer $n+1$. Hence, by induction, $s_n = \dfrac{2^n-1}{2^{n+1}-1}$ for each positive integer n. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} &= \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n-1}{2^{n+1}-1} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1}{2}-\frac{1}{2^{n+1}}\right)}{\left(1-\frac{1}{2^{n+1}}\right)} \\
&= \frac{1}{2},
\end{aligned}$$

as claimed.

Solution 6 by Daniel Văcaru, Pitești, Romania.

We have

$$\frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \frac{1}{2}\cdot\frac{\left(2^{k+1}-1\right)-\left(2^k-1\right)}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \frac{1}{2}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right).$$

It follows

$$\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \frac{1}{2}\left(\frac{1}{2^1-1}-\frac{1}{2^{n+1}-1}\right)$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{2^1-1}-\frac{1}{2^{n+1}-1}\right) = \frac{1}{2}.$$

Solution 7 by Arkady Alt, San Jose, CA.

Since
$$\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}=\frac{2^k}{\left(2^k-1\right)\left(2^{k+1}-1\right)}$$
then
$$\sum_{k=1}^{n}\frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right)$$
$$=\frac{1}{2}\left(\frac{1}{2^1-1}-\frac{1}{2^{n+1}-1}\right)$$
$$=\frac{1}{2}-\frac{1}{2\left(2^{n+1}-1\right)}$$
and, therefore, $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)}=\frac{1}{2}-\lim\limits_{n\to\infty}\frac{1}{2\left(2^{n+1}-1\right)}=\frac{1}{2}.$

Solution 8 by Michel Battaille, Rouen, France.

We observe that
$$\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right).$$
It follows that
$$\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right),$$
a telescopic sum. As a result,
$$\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}\left(\frac{1}{2^1-1}-\frac{1}{2^{n+1}-1}\right)=\frac{1}{2}\left(1-\frac{1}{2^{n+1}-1}\right).$$
Since $\lim\limits_{n\to\infty}\frac{1}{2^{n+1}-1}=0$, we conclude
$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}.$$

Solution 9 by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Spain.

Solution: By induction, we will show that
$$\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{2^n-1}{2^{n+1}-1}.$$

The identity holds for $n = 1$. Assuming it is also true for n, in order to prove that it holds for $n+1$ we have to check that

$$\frac{2^n - 1}{2^{n+1} - 1} + \frac{2^{n+1-1}}{(2^{n+1} - 1)(2^{n+1+1} - 1)} = \frac{2^{n+1} - 1}{2^{n+2} - 1}.$$

Let LHS denote the left-hand side of the last identity. Then,

$$\begin{aligned} LHS &= \frac{2^n - 1}{2^{n+1} - 1} + \frac{2^n}{(2^{n+1} - 1)(2^{n+2} - 1)} \\ &= \frac{(2^n - 1)(2^{n+2} - 1) + 2^n}{(2^{n+1} - 1)(2^{n+2} - 1)} \\ &= \frac{2^{2n+2} - 2^{n+2} + 1}{(2^{n+1} - 1)(2^{n+2} - 1)} \\ &= \frac{(2^{n+1} - 1)^2}{(2^{n+1} - 1)(2^{n+2} - 1)} = \frac{2^{n+1} - 1}{2^{n+2} - 1}. \end{aligned}$$

Since $\displaystyle\sum_{k=1}^{n} \frac{2^{k-1}}{(2^k - 1)(2^{k+1} - 1)} = \frac{2^n - 1}{2^{n+1} - 1}$, then

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{2^{k-1}}{(2^k - 1)(2^{k+1} - 1)} = \frac{1}{2}.$$

Solution 10 by Moti Levy, Rehovot, Israel.

By partial fractions and telescoping,

$$\frac{2^{k-1}}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} = \frac{1}{2}\left(\frac{1}{2^k - 1} - \frac{1}{2^{k+1} - 1}\right),$$

$$\sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2^k - 1} - \frac{1}{2^{k+1} - 1}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2^1 - 1}\right) - \frac{1}{2^{n+1} - 1} = \frac{1}{2} - \frac{1}{2^{n+1} - 1},$$

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{2^{k-1}}{\left(2^k - 1\right)\left(2^{k+1} - 1\right)} = \lim_{n \to \infty}\left(\frac{1}{2} - \frac{1}{2^{n+1} - 1}\right) = \frac{1}{2}.$$

Solution 11 by Peter Fulop, Gyomro, Hungary.

The partial fraction decomposition is available for the summand:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}\Big(\frac{1}{(2^k-1)}-\frac{1}{(2^{k+1}-1)}\Big) \tag{5}$$

Let's see the next term of (5) $k \to k+1$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}\Big(\frac{1}{(2^{k+1}-1)}-\frac{1}{(2^{k+2}-1)}\Big) \tag{6}$$

It can be realized that the sum is telescopic:

$$\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}\Big(\frac{1}{(2^1-1)}-\frac{1}{(2^{n+1}-1)}\Big) \tag{7}$$

Taking the limit of (7) we get the result:

$$\lim_{n\to\infty}\frac{1}{2}\Big(\frac{1}{(2^1-1)}-\frac{1}{(2^{n+1}-1)}\Big)=\frac{1}{2} \tag{8}$$

Solution 12 by Henry Ricardo, Westchester Area Math Circle.

We can write the finite sum as a telescoping series:

$$\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)} \;=\; \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right) \;=\; \frac{1}{2}\left(1-\frac{1}{2^{n+1}-1}\right),$$

which tends to $1/2$ as $n\to\infty$.

Solution 13 by Brian Bradie, Christopher Newport University, Newport News, VA.

Because

$$\frac{1}{2}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right)=\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)},$$

it follows that

$$\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right)=\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^{n+1}-1}.$$

Thus,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^{k-1}}{(2^k-1)(2^{k+1}-1)}=\frac{1}{2}.$$

Soluiton 14 by G. C. Greubel, Newport News, VA.

The limit takes the form

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)} &= \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right)\\ &= \frac{1}{2}\left(1-\frac{1}{2^{n+1}-1}\right)\\ &= \frac{1}{2}.\end{aligned}$$

Solution 15 by Bruno Salgueiro Fanego, Viveiro, Spain.

By partial decomposition, we obtain a telescoping sum:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2^{k-1}}{\left(2^k-1\right)\left(2^{k+1}-1\right)}=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1/2}{2^k-1}-\frac{1/2}{2^{k+1}-1}\right)$$

$$=\frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2^k-1}-\frac{1}{2^{k+1}-1}\right)$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2-1}-\frac{1}{2^2-1}+\frac{1}{2^2-1}-\frac{1}{2^3-1}+\frac{1}{2^3-1}-\frac{1}{2^4-1}+\cdots\right.$$

$$\left.\cdots+\frac{1}{2^{n-1}-1}-\frac{1}{2^n-1}+\frac{1}{2^n-1}-\frac{1}{2^{n+1}-1}\right)$$

$$=\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2-1}-\frac{1}{2^{n+1}-1}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{2^\infty-1}\right)=\frac{1}{2}\left(1-\frac{1}{\infty}\right)$$

$$=\frac{1}{2}.$$

Solution 16 by Seán M. Stewart, Bomaderry, NSW, Australia.

Denote the limit to be found by S. We show that $S = \dfrac{1}{2}$. Suppose

$$f(z) = \lim_{n\to\infty} f_n(z),$$

where, for $n \in \mathbb{N}$,

$$f_n(z) = \sum_{k=1}^{n} \frac{z^{k-1}}{(z^k - 1)(z^{k+1} - 1)}, \quad z \in \mathbb{C}, |z| \neq 1.$$

Noting that

$$\frac{z^k - z^{k+1}}{(1 - z^k)(1 - z^{k+1})} = \frac{z^k(1 - z)}{(1 - z^k)(1 - z^{k+1})} = \frac{1}{1 - z^k} - \frac{1}{1 - z^{k+1}},$$

we see that

$$\frac{z^{k-1}}{(1 - z^k)(1 - z^{k+1})} = \frac{1}{z(1 - z)}\left(\frac{1}{1 - z^k} - \frac{1}{1 - z^{k+1}}\right).$$

Thus

$$f_n(z) = \frac{1}{z(1 - z)} \sum_{k=1}^{n} \left(\frac{1}{1 - z^k} - \frac{1}{1 - z^{k+1}}\right) = \frac{1}{z(1 - z)}\left(\frac{1}{1 - z} - \frac{1}{1 - z^{n+1}}\right),$$

as the sum telescopes. So

$$f(z) = \lim_{n\to\infty} f_n(z) = \begin{cases} \dfrac{1}{(1 - z)^2} & \text{if } |z| < 1, \\ \dfrac{1}{z(1 - z)^2} & \text{if } |z| > 1. \end{cases}$$

The limit to be found is given by $f(2)$. From the above result we immediately see that

$$S = f(2) = \frac{1}{2(1 - 2)^2} = \frac{1}{2},$$

as required to show.

Solutions were also received from Brian D. Beasley, Presbyterian College, Clinton, SC; Kee-Wai Lau, Hong Kong, China; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Titu Zvonaru, Comănești, Romania.

• **5640** *Proposed by Titu Zvonaru, Comănești, Romania.*

Let a, b, c be real numbers such that $ab + bc + ca = 0$. Prove that

$$\left(a^2 + b^2 + c^2\right)\left(a^2b^2 + b^2c^2 + c^2a^2\right) \geqslant 54a^2b^2c^2 - 162\max\left(a^4bc, ab^4c, abc^4\right).$$

Solution 1 by Arkady Alt, San Jose, CA.

Noting that inequality is obvious if at least one of a, b, c is equal to zero, we assume that $abc \neq 0$. Since

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)=(a+b+c)^2$$

and

$$a^2b^2+b^2c^2+c^2a^2=(ab+bc+ca)^2-2abc(a+b+c)=-2abc(a+b+c)$$

then

$$\begin{aligned}\left(a^2+b^2+c^2\right)\left(a^2b^2+b^2c^2+c^2a^2\right) &= -2abc(a+b+c)^3 \\ &= -\frac{8a^3b^3c^3(a+b+c)^3}{4a^2b^2c^2} \\ &= \frac{\left(a^2b^2+b^2c^2+c^2a^2\right)^3}{4a^2b^2c^2}\end{aligned}$$

and the inequality becomes

$$\textbf{(1)}\quad \frac{\left(a^2b^2+b^2c^2+c^2a^2\right)^3}{4a^2b^2c^2} \geqslant 54a^2b^2c^2-162\max\left\{a^4bc, ab^4c, abc^4\right\}.$$

Also note that from three numbers ab, bc, ca two of them are negative and one is positive because $ab \cdot bc \cdot ca = (abc)^2 > 0$ and $ab+bc+ca=0$.

Let it be $ab, ca < 0$ and $bc > 0$. Then $\max\left\{a^4bc, ab^4c, abc^4\right\} = a^4bc$ and denoting

$$x := bc, y := -ca, z := -ab, t := \frac{(y+z)^2}{yz} \geqslant 4,$$

we obtain

$$a^2 = \frac{yz}{x},\ a^2b^2c^2 = xyz,\ a^4bc = \left(\frac{yz}{x}\right)^2 x = \frac{y^2z^2}{x},$$
$$ab+bc+ca=0 \iff x=y+z$$

and **(1)** becomes

$$\frac{\left(z^2+x^2+y^2\right)^3}{4xyz} \geqslant 54xyz - \frac{162y^2z^2}{x} \iff \frac{\left(z^2+x^2+y^2\right)^3}{4xyz} \geqslant 54xyz - \frac{162y^2z^2}{x}$$

$$\iff \left(z^2+x^2+y^2\right)^3 \geqslant 54\cdot 4x^2y^2z^2 - 162\cdot 4y^3z^3$$

$$\iff \left(z^2+(y+z)^2+y^2\right)^3 \geqslant 54\cdot 4\,(y+z)^2\,y^2z^2 - 162\cdot 4y^3z^3$$

$$\iff 8\left((y+z)^2-yz\right)^3 \geqslant 54\cdot 4\,(y+z)^2\,y^2z^2 - 162\cdot 4y^3z^3$$

$$\iff \left((y+z)^2-yz\right)^3 \geqslant 27\,(y+z)^2\,y^2z^2 - 81y^3z^3$$

$$\iff (t-1)^3 \geqslant 27t-81$$

$$\iff (t+5)\,(t-4)^2 \geqslant 0.$$

Solution 2 by Moti Levy, Rehovot, Israel.

The constraint $ab+bc+ca=0$ implies that a,b,c cannot all have the same sign,i.e., either $a>0$, $b>0$, $c<0$ or $a<0$, $b<0$, $c>0$, and that

$$c=-\frac{ab}{a+b}.$$

Case 1: $a>0$, $b>0, c<0$.

$$\max(a^4bc;ab^4c;abc^4) = abc^4 = ab\left(-\frac{ab}{a+b}\right)^4 = \frac{a^5b^5}{(a+b)^4}$$

The original inequality becomes

$$\left(a^2+b^2+\left(-\frac{ab}{a+b}\right)^2\right)\left(a^2b^2+b^2\left(-\frac{ab}{a+b}\right)^2+\left(-\frac{ab}{a+b}\right)^2a^2\right)$$
$$-54a^2b^2\left(-\frac{ab}{a+b}\right)^2+162\frac{a^5b^5}{(a+b)^4}\geqslant 0.$$

After simplification, the inequality becomes

$$2a^2b^2\frac{(a-b)^4}{(a+b)^4}\left(a^2+7ab+b^2\right)\geqslant 0,$$

which is true indeed.

Case 2: $a < 0$, $b < 0$, $c > 0$ is similar to case 1.

Also solved by Albert Stadler, Herrliberg, Switzerland.

• **5641** *Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Find all numbers $x_1, x_2, x_3, \cdots, x_n$ such that

$$\begin{aligned} x_1^4 + 6x_2^2 &< 4x_2^3 + 5x_2 - x_1, \\ x_2^4 + 6x_3^2 &< 4x_3^3 + 5x_3 - x_2, \\ &\cdots \\ x_{n-1}^4 + 6x_n^2 &< 4x_n^3 + 5x_n - x_{n-1}, \\ x_n^4 + 6x_1^2 &< 4x_1^3 + 5x_1 - x_n. \end{aligned}$$

Solution 1 by Titu Zvonaru, Comănești, Romania.

Adding up we obtain

$$\left(x_1^4 + 6x_1^2 - 4x_1^3 - 4x_1\right) + \left(x_2^4 + 6x_2^2 - 4x_2^3 - 4x_2\right) + \cdots + \left(x_n^4 + 6x_n^2 - 4x_n^3 - 4x_n\right) < 0.$$

It is easy to see that $x_1 = x_2 = x_3 = \cdots = x_n = 1$ is a solution. We shall prove there are no other solutions.

If $a \leqslant 0$, then $a^4 + 6a^2 \geqslant 0 \geqslant 4a^3 + 4a^2$. If $a \geqslant 2$, then $a^4 + 6a^2 = a^4 + 4a^2 + 2a^2 \geqslant 4a^3 + 4a$. It therefore follows that if a is an integer and $a^4 + 6a^2 < 4a^3 + 4a^2$, then $a = 1$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By assumption,

$$x_i^4 + x_i < 4x_{i+1}^3 - 6x_{i+1}^2 + 5x_{i+1} \quad \text{for} \quad 1 \leqslant i \leqslant n \qquad (\star)$$

where $x_{n+1} = x_1$. The function $f(x) = x^4 + x$ is non-negative for all integers x and monotonically increasing for $x \geqslant 0$. The function $g(x) = 4x^3 - 6x^2 + 5x$ is monotonically increasing since $g'(x) = 12x^2 - 12x + 5 = 3(2x-1)^2 + 2 > 0$ for all x. We have $g(x) > 0$ for $x > 0$ and $g(x) < 0$ for $x < 0$. We conclude by induction that $x_i > 0$ for all i. Indeed, $x_2 > 0$ since $f(x_1) \geqslant 0$, and $x_{i-1} > 0$ implies $x_i > 0$.

Furthermore if $k \geqslant 2$, then $x_i \geqslant k$ implies $x_{i+1} \geqslant k+1$ since if we had $x_{i+1} \leqslant k$, then

$$g(x_{i+1}) - f(x_i) \leqslant g(k) - f(k) = 4k^3 - 6k^2 + 5k - (k^4 + k) = k(2-k)(1+(k-1)^2) \leqslant 0$$

for $k \geqslant 2$, in contradiction to $(\star)$. So $x_i = 1$ for all i. We check that $x_i = 1$ for all i satisfies all inequalities.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC.

Given a real number x, we let $f(x) = x^4 + x$ and $g(x) = 4x^3 - 6x^2 + 5x$. Then we seek all positive integers n and all integers $x_1, x_2, \ldots, x_n$ with

$$f(x_i) < g(x_{i+1}) \text{ for each } i \text{ in } \{1, 2, \ldots, n-1\} \quad \text{and} \quad f(x_n) < g(x_1).$$

It is straightforward to verify the following properties: (i) $g(x)$ is increasing on $\mathbb{R}$, with $g(x) < 0$ for each $x < 0$; (ii) $f(x) < 0$ if and only if $-1 < x < 0$; (iii) $f(x) < g(x)$ if and only if $0 < x < 2$. In particular, for each integer m, $f(m) < g(m)$ if and only if $m = 1$; also, $f(m) \geqslant g(x)$ for every real $x \leqslant 0$, so each x_i must be positive, with $f(x_i) < g(1) = 3$ if and only if $x_i = 1$. We thus assume $x_i > 0$ for each i and apply the above observations.

If $n = 1$, then we require $f(x_1) < g(x_1)$, so $x_1 = 1$. If $n > 1$, then we show that for each i, $x_i = 1$, so this case in essence collapses back to the case when $n = 1$.

(a) If $x_i \neq 1$ for each i, then $f(x_i) \geqslant g(x_i)$ for each i and hence

$$f(x_1) < g(x_2) \leqslant f(x_2) < g(x_3) \leqslant f(x_3) < \cdots < g(x_{n-1}) \leqslant f(x_{n-1}) < g(x_n) \leqslant f(x_n) < g(x_1),$$

a contradiction.

(b) If $x_1 = 1$, then $f(x_n) < g(x_1) = 3$ implies $x_n = 1$. Next, $f(x_{n-1}) < g(x_n) = 3$, so $x_{n-1} = 1$. Continuing inductively, we conclude that $1 = x_1 = x_n = x_{n-1} = \cdots = x_2$.

(c) If $x_i = 1$ for $i > 1$, then $f(x_{i-1}) < g(x_i) = 3$ implies $x_{i-1} = 1$. Again continuing inductively, we conclude that $1 = x_i = x_{i-1} = \cdots = x_1$; then, as seen in case (a), we have $1 = x_1 = x_n = x_{n-1} = \cdots = x_{i+1}$ as well.

Solution 4 by David E. Manes, Oneonta, NY.

For each positive integer n, the only solution for the system of n inequalities in n variables is the n-tuple $(1, 1, \ldots, 1)$ consisting of all ones. If $n = 2$, then the system of inequalities is

$$\begin{aligned} x_1^4 + 6x_2^2 &< 4x_2^3 + 5x_2 - x_1, \\ x_2^4 + 6x_1^2 &< 4x_1^3 + 5x_1 - x_2. \end{aligned}$$

If $x_1 = -1$, then the second inequality reduces to $x_2^4 + x_2 < -15$, a contradiction since $x_2^4 + x_2 > -1$ for all values of x_2. If $x_1 = 0$, then $x_2^4 < -x_2$, another contradiction. Therefore, $x_1 > 0$ and $x_2 > 0$. If $x_1 = 1$, then the second inequality reads $x_2^4 + x_2 < 3$. The only integers satisfying this inequality are $x_2 = -1, 0$ or 1. If $x_2 = -1$, then the first inequality reduces to $7 < -10$, a contradiction. If $x_2 = 0$, then $1 < -1$ from the first inequality, another contradiction. If $x_2 = 1$, then both inequalities are satisfied. Therefore, the only solution for the case $n = 2$ is $x_1 = x_2 = 1$ or the ordered pair $(1, 1)$. Inductively, assume that the n-tuple $(1, 1, \ldots, 1)$ is the only solution for the system of n inequalities in n variables for any positive integer $n \geqslant 2$. Then the system of inequalities for the positive integer $n + 1$ is

$$\begin{aligned}
x_1^4 + 6x_2^2 &< 4x_2^3 + 5x_2 - x_1, \\
x_2^4 + 6x_3^2 &< 4x_3^3 + 5x_3 - x_2, \\
&\cdots \\
x_{n-1}^4 + 6x_n^2 &< 4x_n^3 + 5x_n - x_{n-1}, \\
x_n^4 + 6x_{n+1}^2 &< 4x_{n+1}^3 + 5x_{n+1} - x_n, \\
x_{n+1}^4 + 6x_1^2 &< 4x_1^3 + 5x_1 - x_{n+1}.
\end{aligned}$$

Using the induction hypothesis that the n-tuple $(1, 1, \ldots, 1)$ satisfies the first $n-1$ inequalities, the system reduces to

$$\begin{aligned}
1 + 6x_{n+1}^2 &< 4x_{n+1}^3 + 5x_{n+1} - 1, \\
x_{n+1}^4 + 6 &< 4 + 5 - x_{n+1}.
\end{aligned}$$

From the case $n = 2$, the only solution for this last system is $x_{n+1} = 1$. Therefore, the $(n + 1)$-tuple consisting of all ones satisfies the system of $(n + 1)$ inequalities in $(n + 1)$ variables, Hence, the result now follows by induction.

Solution 5 by Michel Battaille, Rouen, France.

A solution is obviously provided by $x_1 = x_2 = \cdots = x_n = 1$. We show that there is no other solution.

Let $x_1, x_2, \ldots, x_n$ be integers satisfying the given inequalities and let $S_k = x_1^k + x_2^k + \cdots + x_n^k$. By addition of the inequalities we obtain

$$S_4 + 6S_2 < 4S_3 + 5S_1 - S_1,$$

that is,

$$S_4 - 4S_3 + 6S_2 - 4S_1 < 0,$$

which can be written as

$$[(x_1 - 1)^4 - 1] + [(x_2 - 1)^4 - 1] + \cdots + [(x_n - 1)^4 - 1] < 0$$

or

$$(x_1 - 1)^4 + (x_2 - 1)^4 + \cdots + (x_n - 1)^4 < n.$$

This cannot be satisfied if $|x_i - 1| \geqslant 1$ for $i = 1, 2, \ldots, n$, hence we must have $x_i = 1$ for some i. Without loss of generality, we suppose that $i = 1$ so that $x_1 = 1$. Then we have

$$2 = x_1^4 + x_1 < 4x_2^3 - 6x_2^2 + 5x_2,$$

hence (since $4x_2^3 - 6x_2^2 + 5x_2$ is an integer) $p(x_2) \geqslant 0$ where

$$p(x) = 4x^3 - 6x^2 + 5x - 3.$$

Noticing that $p(x) = (x-1)(4x^2 - 2x + 3)$ and $4x^2 - 2x + 3 > 0$ for all real x, we see that $x_2 \geqslant 1$. Using the next inequality, we obtain

$$2 \leqslant x_2^4 + x_2 < 4x_3^3 - 6x_3^2 + 5x_3,$$

hence $p(x_3) \geqslant 0$ and so $x_3 \geqslant 1$. Repeating the process, we obtain $x_4, \ldots, x_n \geqslant 1$. The last inequality then gives

$$2 \leqslant x_n^4 + x_n < 4x_1^3 - 6x_1^2 + 5x_1 = 3,$$

and since x_n is a positive integer, we obtain $x_n = 1$. The last but one inequality now gives $x_{n-1} = 1$ and, continuing upwards to the top inequality, $x_{n-2} = \cdots = x_2 = 1$. Finally $x_1 = x_2 = \cdots = x_n = 1$ and we are done.

Solution 6 by Moti Levy, Rehovot, Israel.

We rewrite the original problem:

Find all integer numbers $x_1, x_2, x_3, \ldots, x_n$ such that

$$x_k^4 + 6x_{k+1}^2 < 4x_{k+1}^3 + 5x_{k+1} - x_k, \qquad k = 1, 2, \ldots, n \qquad x_{n+1} = x_1. \tag{9}$$

If $x_{k+1} \leqslant 0$, then the inequality in (9) cannot hold, hence $x_1, x_2, x_3, \ldots, x_n$ must be positive integers.

Let $M_m := \left(\frac{1}{n}\sum_{j=1}^{n} x_j^m\right)^{\frac{1}{m}}$ be the m-power mean of $x_1, x_2, x_3, \ldots, x_n$. Summing the inequalities in (9), we get

$$M_4^4 + 6M_2^2 < 4M_3^3 + 5M_1 \tag{10}$$

By the power mean inequality,

$$M_3 \leqslant M_4, \qquad M_1 \leqslant M_3. \tag{11}$$

It follows from (10) and (11) that

$$M_3^4 < M_4^4 + 6M_2^2 < 4M_3^3 + 5M_1 < 4M_3^3 + 5M_3$$

But

$$M_3^4 < 4M_3^3 + 5M_3,$$

implies that

$$1 \leqslant M_3 < 5. \tag{12}$$

Without loss of generality we can assume that $1 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant \ldots \leqslant x_n$, so that

$$x_1 = \min\{x_k, k = 1, 2, \ldots, n\} \leqslant M_3,$$

hence

$$1 \leqslant x_1 \leqslant 5.$$

Setting $k = 1$ in (9) we get (13):

$$x_1^4 + 6x_2^2 + x_1 < 4x_2^3 + 5x_2$$

Since $x_2 \geqslant x_1$ and $x_2 \leqslant x_n$,

$$x_1^4 + 6x_1^2 + x_1 < x_1^4 + 6x_2^2 + x_1 < 4x_2^3 + 5x_2 < 4x_n^3 + 5x_n \tag{13}$$

Setting $k = n$ in (9) we get

$$x_n^4 + 6x_1^2 < 4x_1^3 + 5x_1 - x_n. \tag{14}$$

After slight rearranging of (13) and (14), we see that x_1 and x_n must satisfy the following two inequalities,

$$4x_n^3 + 5x_n - \left(x_1^4 + 6x_1^2 + x_1\right) > 0 \tag{15}$$

$$x_n^4 + x_n + 6x_1^2 - 4x_1^3 - 5x_1 < 0 \tag{16}$$

If $x_1 = 1$ then (15) and (16) becomes

$$4x_n^3 + 5x_n - 8 > 0,$$
$$x_n^4 + x_n - 3 > 0.$$

One can verify that they are satisfied only when $x_n = 1$. If $x_1 = 2$ then (15) and (16) become

$$4x_n^3 + 5x_n - 42 > 0,$$
$$x_n^4 + x_n - 18 < 0.$$

One can verify that they cannot be satisfied by positive integer. Similar checks for $x_1 = 3, 4, 5$ give the same conclusion. We conclude that only the integers $x_1 = x_2 = \cdots = x_n = 1$ satisfy the inequalities in (9).

• **5642** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Compute

$$\sum_{n=1}^{\infty} (-1)^{n-1} H_n \left(\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right)$$

where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$ denotes the nth harmonic number.

Solution 1 by Narendra Bhandari, Bajura, Nepal.

Since

$$\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} + \cdots = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{n+2k+1}}_{\mathcal{S}_1} + \underbrace{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{n+2k}}_{\mathcal{S}_2}$$

Now we find the integral representation of the last two obtained series, namely

$$\mathcal{S}_1 = \sum_{k=0}^{\infty} (-1)^k \int_0^1 x^{n+2k} dx = \int_0^1 x^n \left(\sum_{k=0}^{\infty} (-1)^k x^{2k} \right) dx = \int_0^1 \frac{x^n}{1+x^2} dx$$

and similarly, we get $\mathcal{S}_2 = \int_0^1 \frac{x^{n+1}}{1+x^2} dx$ and hence on plugging $\mathcal{S}_1$ and $\mathcal{S}_2$ in the original problem, we get

$$\sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(\int_0^1 \frac{x^n}{1+x^2} + \int_0^1 \frac{x^{n+1}}{1+x^2} \right) dx = -\int_0^1 \frac{dx}{1+x^2} \sum_{n=1}^{\infty} (-1)^n \left(x^n + x^{n+1} \right) H_n$$

Using the generating function of harmonic nuembers $\sum_{n \geqslant 1} x^n H^n = -\frac{\log(1-x)}{1-x}$, the last expression boils down to (by replacing x by $-x$) the following integrals

$$\int_0^1 \frac{\log(1+x)}{(1+x^2)(1+x)} dx + \int_0^1 \frac{x \log(1+x)}{(1+x^2)(1+x)} dx = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

We evaluate the latter integral by subbing $x = \tan y$ and using the reflection property of integral $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ gives us

$$\int_0^{\frac{\pi}{4}} \log(1+\tan y)\, dy = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - y\right)\right) dy = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan y}\right) dy$$

So, the integral we have

$$\int_0^{\frac{\pi}{4}} \log(1+\tan y)\, dy = \int_0^{\frac{\pi}{4}} \log(2) dy - \int_0^{\frac{\pi}{4}} \log(1+\tan y)\, dy$$

and therefore, $\int_0^{\frac{\pi}{4}} \log(1+\tan y) dy = \frac{\pi}{8} \log(2)$. We use $\tan\left(\frac{\pi}{4} - y\right) = \frac{1-\tan y}{1+\tan y}$.

So, we can conclude that

$$\sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} + \cdots \right) = \frac{\pi}{8} \log(2)$$

Solution 2 by Michel Battaille, Rouen, France.

Since $\lim_{k\to\infty} \frac{1}{n+k} = 0$, we can consider the series $\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} + \cdots$ as $S_n = \sum_{k=0}^{\infty} u_k$ with

$$u_k = \frac{1}{n+4k+1} + \frac{1}{n+4k+2} - \frac{1}{n+4k+3} - \frac{1}{n+4k+4}.$$

Since $x^{n+4k} + x^{n+4k+1} - x^{n+4k+2} - x^{n+4k+3} \geqslant 0$ for $x \in [0,1]$, we have

$$\begin{aligned} S_n &= \sum_{k=0}^{\infty} \int_0^1 (x^{n+4k} + x^{n+4k+1} - x^{n+4k+2} - x^{n+4k+3})\, dx \\ &= \int_0^1 \left(\sum_{k=0}^{\infty} (x^{n+4k} + x^{n+4k+1} - x^{n+4k+2} - x^{n+4k+3}) \right) dx \\ &= \int_0^1 (x^n + x^{n+1} - x^{n+2} - x^{n+3}) \left(\sum_{k=0}^{\infty} x^{4k} \right) dx \\ &= \int_0^1 x^n (1+x)(1-x^2) \cdot \frac{dx}{1-x^4} = \int_0^1 \frac{x^n + x^{n+1}}{1+x^2}\, dx = I_n + I_{n+1} \end{aligned}$$

where $I_n = \int_0^1 \frac{x^n}{1+x^2}\, dx = \int_0^{\pi/4} (\tan\theta)^n\, d\theta$.

If $\theta \in [0, \pi/4]$, then $(\tan\theta)^n \geqslant (\tan\theta)^{n+1}$, hence $I_n \geqslant I_{n+1}$. In addition, we have $I_n + I_{n+2} = \int_0^{\pi/4} (\tan\theta)^n d(\tan\theta) = \frac{1}{n+1}$. It follows that $2I_{n+2} \leqslant I_n + I_{n+2} \leqslant 2I_n$ so that $\frac{1}{2(n+1)} \leqslant I_n \leqslant \frac{1}{2(n-1)}$. We deduce that $I_n \sim \frac{1}{2n}$ as $n \to \infty$.

To calculate the required sum $S = \sum_{n=1}^{\infty} (-1)^{n-1} H_n S_n$, we first observe that $S_n = I_n + I_{n+1} \sim \frac{1}{n}$, hence $H_n S_n \sim \frac{\ln(n)}{n}$ so that $\lim_{n\to\infty} H_n S_n = 0$ and therefore $S = \lim_{N\to\infty} \sum_{n=1}^{2N} (-1)^{n-1} H_n S_n$.

Now, we have

$$\begin{aligned} \sum_{n=1}^{2N} (-1)^{n-1} H_n S_n &= \sum_{n=1}^{2N} (-1)^{n-1} H_n (I_n + I_{n+1}) = I_1 H_1 - \sum_{n=2}^{2N} (-1)^n I_n (H_n - H_{n-1}) - I_{2N+1} H_{2N} \\ &= \sum_{n=1}^{2N} (-1)^{n-1} \frac{I_n}{n} - I_{2N+1} H_{2N}. \end{aligned}$$

Since $\lim_{N\to\infty} I_{2N+1} H_{2N} = 0$ (because $H_{2N} \sim \ln(2N)$ as $N \to \infty$), we obtain

$$S=\lim_{N\to\infty}\int_0^{\pi/4}\left(\sum_{n=1}^{2N}\frac{(-1)^{n-1}(\tan\theta)^n}{n}\right)d\theta=\int_0^{\pi/4}\ln(1+\tan\theta)d\theta$$

(since the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$ is uniformly convergent toward $\ln(1+x)$ on $[0,1]$).

It remains to calculate the integral $I=\displaystyle\int_0^{\pi/4}\ln(1+\tan\theta)d\theta$:

The change of variables $x=\dfrac{\pi}{4}-u$ gives

$$\int_0^{\pi/4}\ln\left(\cos\left(\frac{\pi}{4}-x\right)\right)\,dx=\int_0^{\pi/4}\ln(\cos u)\,du$$

so that

$$I=\int_0^{\pi/4}\ln\left(\frac{\cos x+\sin x}{\cos x}\right)\,dx=\int_0^{\pi/4}\ln\left(\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)\right)\,dx-\int_0^{\pi/4}\ln(\cos x)\,dx=\int_0^{\pi/4}(\ln\sqrt{2})\,dx$$

and we conclude $S=I=\dfrac{\pi\ln(2)}{8}$.

Solution 3 by G. C. Greubel, Newport News, VA.

The series in question is given by

$$S=\sum_{n=1}^{\infty}(-1)^{n-1}\,H_n\,\left(\sum_{j=1}^{\infty}\frac{i^{(j-1)(j-2)}}{n+j}\right)$$

which can be seen as

$$S=\sum_{j=1}^{\infty}i^{(j-1)(j-2)}\,\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,H_n}{n+j}\right).$$

By using

$$\frac{1}{n+j}=\int_0^1 t^{n+j-1}\,dt$$

then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,H_n}{n+j}&=(-1)\,\int_0^1 t^{j-1}\,\left(\sum_{n=1}^{\infty}H_n\,(-t)^n\right)\,dt\\&=\int_0^1\frac{t^{j-1}\,\ln(1+t)}{1+t}\,dt.\end{aligned}$$

Since

$$\sum_{j=1}^{\infty} i^{(j-1)(j-2)}\, t^{j-1} = \sum_{j=0}^{\infty} i^{j\,(j-1)}\, t^{j} = \frac{1+t}{1+t^2}$$

then

$$S = \sum_{j=1}^{\infty} i^{(j-1)(j-2)} \int_0^1 \frac{t^{j-1}\, \ln(1+t)}{1+t}\, dt = \int_0^1 \frac{\ln(1+t)}{1+t^2}\, dt.$$

Integration yields:

$$\begin{aligned} S &= \int_0^1 \frac{\ln(1+t)}{1+t^2}\, dt \\ &= \frac{1}{2\,i} \left[\mathrm{Li}_2\left(\frac{(1-i)}{2}\,(1+x) \right) - \mathrm{Li}_2\left(\frac{(1+i)}{2}\,(1+x) \right) + \ln(1+x)\, \ln\left(i\, \frac{1+i\,x}{1-i\,x} \right) \right]_0^1 \\ &= \frac{1}{2i} \left[\pi\, i\, \ln 2 + \mathrm{Li}_2(1+i) - \mathrm{Li}_2(1-i) - \mathrm{Li}_2\left(\frac{1-i}{2} \right) + \mathrm{Li}_2\left(\frac{1+i}{2} \right) \right]. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{Li}_2(1-i) &= -i\,\gamma + \frac{3\pi^2}{16} - \frac{\pi i}{2} \left(\frac{\ln 2}{2} - \frac{\pi i}{4} \right) \\ \mathrm{Li}_2(1+i) &= i\,\gamma + \frac{3\pi^2}{16} + \frac{\pi i}{2} \left(\frac{\ln 2}{2} + \frac{\pi i}{4} \right) \\ \mathrm{Li}_2\left(\frac{1-i}{2} \right) &= -i\,\gamma + \frac{\pi^2}{48} - \frac{1}{2} \left(\frac{\ln 2}{2} - \frac{\pi i}{4} \right)^2 \\ \mathrm{Li}_2\left(\frac{1+i}{2} \right) &= i\,\gamma + \frac{\pi^2}{48} - \frac{1}{2} \left(\frac{\ln 2}{2} + \frac{\pi i}{4} \right)^2 \end{aligned}$$

then

$$\begin{aligned} \mathrm{Li}_2(1-i) - \mathrm{Li}_2(1+i) &= -2\,i\,\gamma - \frac{\pi i}{2}\, \ln 2 \\ \mathrm{Li}_2\left(\frac{1-i}{2} \right) - \mathrm{Li}_2\left(\frac{1+i}{2} \right) &= -2\,i\,\gamma - \frac{\pi i}{4}\, \ln 2 \end{aligned}$$

and $S = \dfrac{\pi}{8}\, \ln 2$, or

$$\sum_{n=1}^{\infty} (-1)^{n-1}\, H_n \left(\sum_{j=1}^{\infty} \frac{i^{(j-1)(j-2)}}{n+j} \right) = \frac{\pi}{8}\, \ln 2.$$

Solution 4 by Brian Bradie, Christopher Newport University, Newport News, VA.

Note

$$\begin{aligned}
\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}+\cdots &= \sum_{j=0}^{\infty}(-1)^j\left(\frac{1}{n+2j+1}+\frac{1}{n+2j+2}\right)\\
&= \sum_{j=0}^{\infty}(-1)^j\int_0^1\left(x^{n+2j}+x^{n+2j+1}\right)dx\\
&= \int_0^1\sum_{j=0}^{\infty}(-1)^j\left(x^{n+2j}+x^{n+2j+1}\right)dx\\
&= \int_0^1 x^n(1+x)\sum_{j=0}^{\infty}(-1)^j x^{2j}\,dx\\
&= \int_0^1\frac{x^n(1+x)}{1+x^2}\,dx.
\end{aligned}$$

Next, the generating function for the harmonic numbers is

$$\sum_{n=1}^{\infty}H_nx^n=-\frac{\ln(1-x)}{1-x},$$

so

$$\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{n-1}H_n\left(\frac{1}{n+1}+\frac{1}{n+2}-\frac{1}{n+3}-\frac{1}{n+4}+\cdots\right)\\
&= \sum_{n=1}^{\infty}(-1)^{n-1}H_n\int_0^1\frac{x^n(1+x)}{1+x^2}\,dx\\
&= \int_0^1\frac{1+x}{1+x^2}\sum_{n=1}^{\infty}(-1)^{n-1}H_nx^n\,dx\\
&= \int_0^1\frac{\ln(1+x)}{1+x^2}\,dx.
\end{aligned}$$

Let

$$I=\int_0^1\frac{\ln(1+x)}{1+x^2}\,dx.$$

With the substitution $x=\tan\theta$, this becomes

$$I=\int_0^{\pi/4}\ln(1+\tan\theta)\,d\theta,$$

and changing θ to $\frac{\pi}{4} - \theta$ then yields

$$\begin{aligned} I &= \int_0^{\pi/4} \ln\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta = \int_0^{\pi/4} \ln\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) d\theta \\ &= \int_0^{\pi/4} \ln\left(\frac{2}{1 + \tan\theta}\right) d\theta. \end{aligned}$$

Thus,

$$2I = \int_0^{\pi/4} \ln(1 + \tan\theta)\, d\theta + \int_0^{\pi/4} \ln\left(\frac{2}{1 + \tan\theta}\right) d\theta = \int_0^{\pi/4} \ln 2\, d\theta = \frac{\pi}{4} \ln 2.$$

Finally,

$$\sum_{n=1}^{\infty} (-1)^{n-1} H_n \left(\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right) = \frac{\pi}{8} \ln 2.$$

Solution 5 by Seán M. Stewart, Bomaderry, NSW, Australia.

Denote the sum to be found by S. We show that $S = \frac{\pi}{8} \log(2)$. Now

$$\begin{aligned} S &= \sum_{n=1}^{\infty} (-1)^{n+1} H_n \left(\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} + \cdots\right) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} H_n \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{2k+n-1} + \frac{1}{2k+n}\right). \end{aligned}$$

Define

$$S_n = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{2k+n-1} + \frac{1}{2k+n}\right).$$

Noting that for $k, n \in \mathbb{N}$

$$\int_0^1 x^{2k+n-1}\, dx = \frac{1}{2k+n} \quad \text{and} \quad \int_0^1 x^{2k+n-2}\, dx = \frac{1}{2k+n-1},$$

the term for S_n can be expressed as

$$\begin{aligned} S_n &= \sum_{k=1}^{\infty} (-1)^{k+1} \left(\int_0^1 x^{2k+n-2}\, dx + \int_0^1 x^{2k+n-1}\, dx\right) \\ &= -\int_0^1 x^{n-2}(1+x) \sum_{k=1}^{\infty} (-1)^k x^{2k}\, dx = \int_0^1 x^{n-2}(1+x) \cdot \frac{x^2}{1+x^2}\, dx \\ &= \int_0^1 \frac{(1+x)x^n}{1+x^2}\, dx. \end{aligned}$$

Note the interchange made between the summation and integral can be justified by the dominated convergence theorem. Returning to the expression for S we have

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} H_n \int_0^1 \frac{(1+x)x^n}{1+x^2}\, dx = -\int_0^1 \frac{1+x}{1+x^2} \sum_{n=1}^{\infty} (-1)^n H_n x^n \, dx. \tag{17}$$

Note again the interchange made between the summation and integral can be justified by the dominated convergence theorem. From the well-known generating function for the harmonic numbers, namely

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\log(1-x)}{1-x}, \quad |x| < 1,$$

replacing x with $-x$ gives

$$\sum_{n=1}^{\infty} (-1)^n H_n x^n = -\frac{\log(1+x)}{1+x}, \quad |x| < 1.$$

Thus (17) becomes

$$S = \int_0^1 \frac{1+x}{1+x^2} \cdot \frac{\log(1+x)}{1+x}\, dx = \int_0^1 \frac{\log(1+x)}{1+x^2}\, dx.$$

The integral that has appeared is quite famous. In the literature it is known as *Serret's* integral [1, pp. 53–54]. Its value can be most readily found using a so-called *self-similar* substitution of $x = (1-u)/(1+u)$. Doing so produces

$$\begin{aligned} S &= \int_0^1 \frac{(1+u)^2}{2(1+u^2)} \log\left(\frac{2}{1+u}\right) \frac{2}{(1+u)^2}\, du = \int_0^1 \log\left(\frac{2}{1+u}\right) \frac{du}{1+u^2} \\ &= \log(2) \int_0^1 \frac{du}{1+u^2} - S, \end{aligned}$$

or

$$S = \frac{\pi}{8} \log(2),$$

as claimed.

References

[1] P. J. Nahin (2015). *Inside Interesting Integrals* (Springer: New York).

Solutions were also received from Moti Levy, Rehovot, Israel; Peter Fulop, Gyomro, Hungary; Albert Stadler, Herrliberg, Switzerland.

***********   ***********   ***********

***********   ***********   ***********

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributers. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following recommendations. As you peruse below, you may construe that the recommendations amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

If possible, when submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s).

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "Statement of the Problem".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "Solution of the Problem".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Proposed problem to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "Statement of the Problem".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "Solution of the Problem".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Proposed problem to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣