Problems and Solutions

Albert Natian, Section Editor

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This section of SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before March 15, 2022.

• **5661** *Proposed by Kenneth Korbin, New York, NY.*

Given positive acute angles A, B, C with $\sin^2(A+B+C) = 1/10$, find two triples of positive integers (x,y,z), with $x<y<z$, such that $\sin^2 A = 1/x$, $\sin^2 B = 1/y$, $\sin^2 C = 1/z$.

• **5662** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral $\displaystyle\int_1^\infty \frac{x\ln x}{(x+1)(x^2+1)}\,dx$.

• **5663** *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Spain.*

Let $n\in\mathbb{N}$ and suppose $|r|<1$. Evaluate the sum $\displaystyle\sum_{i_n=0}^{\infty}\cdots\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty}\left(\sum_{j=1}^{n} i_j\right) r^{\left(\sum_{j=1}^{n} i_j\right)}$.

• **5664** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania.*

Prove that $\forall x,y\in(0,\pi/2):\quad \log_{\sin x}\left(\dfrac{\sin 2x}{\sin x+\cos x}\right)+\log_{\cos x}\left(\dfrac{\sin 2x}{\sin x+\cos x}\right)\geqslant 2.$

• **5665** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

Prove that if $a, b \geqslant 0$, then

$$\int_0^{1/2}\int_0^{1/2}\sqrt{a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2+b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2}\,dxdy \leqslant \frac{\sqrt{9a}+\sqrt{9b}}{64}.$$

• **5666** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $H_n = \sum_{k=1}^{n} 1/k$, which is known as the nth harmonic number. Calculate

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(2)-\sum_{k=1}^{n+1}\frac{1}{k^2}\right).$$

Editor's note: Here ζ denotes the Euler-Riemann zeta function, defined as $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$.

Solutions

• **5643** *Proposed by Kenneth Korbin, New York, NY.*

Let K be the product of three different prime numbers each congruent to 1(mod 4). How many different Pythagorean triangles have hypotenuse K^2?

Solution 1 by David E. Manes, Oneonta, NY.

In Lehman, J. L., (2019), *Quadratic Number Theory*, Providence, RI, American Mathematical Society, the author proves the following theorem using properties of the Gaussian integers:

> Theorem 1.6.3 (P. 50). Let a be a positive integer written as
>
> $$a = 2^e \cdot p_1^{e_1}\dots p_n^{e_n} q_1^{f_1}\dots q_l^{f_l},$$
>
> where each p_i and q_i is a distinct prime with $p_i \equiv 1 \pmod 4$ and $q_i \equiv 3 \pmod 4$, each e_i and f_i is a positive integer, and $e \geqslant 0$. Let $m = (e_1+1)(e_2+1)\cdots(e_n+1)$ if n is positive and let $m_1 = 1$ if $n = 0$. If each f_i is even, then the number of representations of a by x^2+y^2 with $x \geqslant y \geqslant 0$ is $\left\lfloor \frac{m+1}{2} \right\rfloor$. If instead any f_i is odd, then a has no representations by x^2+y^2.

Here $\left\lfloor \frac{m+1}{2} \right\rfloor$ is the floor function; that is, the largest integer less than or equal to $\frac{m+1}{2}$. Note that if $x, y > 0$ and a is a perfect square $(a = t^2, t > 0)$, then $x^2 + y^2 = t^2 = a$ defines a Pythagorean triangle (x, y, t). However, if $y = 0$, then $x = t$ implies $x^2 + y^2 = t^2 + 0 = a$ and no Pythagorean triangle is defined. Therefore, the number of Pythagorean triangles when a is a perfect square is the number of ways of writing $a = x^2 + y^2$ with $x \geqslant y > 0$ which is $\left\lfloor \frac{m+1}{2} \right\rfloor - 1$.

For the proposed problem, let $K = p_1 p_2 p_3$, where each p_i is a prime and congruent to 1 (mod 4) and $p_i \neq p_j$ if $i \neq j$. If (a, b, K^2) is a Pythagorean triangle, then $a^2 + b^2 = (K^2)^2 = K^4 = p_1^4 \cdot p_2^4 \cdot p_3^4$ with $a \geqslant b > 0$. By Theorem 1.6.3, the number of Pythagorean triangles with hypotenuse K^2 is the number of ways of writing $K^4 = x^2 + y^2$ with $x \geqslant y > 0$ which is $\left\lfloor \frac{m+1}{2} \right\rfloor - 1$. In this case, $m = (e_1 + 1)(e_2 + 1)(e_3 + 1) = 5 \cdot 5 \cdot 5 = 125$ so that $\left\lfloor \frac{m+1}{2} \right\rfloor - 1 = \left\lfloor \frac{126}{2} \right\rfloor - 1 = 63 - 1 = 62$. Summarizing then, if K is the product of three different primes each congruent to $1 \pmod 4$, then the number of Pythagorean triangles with hypotenuse K^2 is 62.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We will prove that there are 124 tuples (x,y) such that x>0, y>0 and $K^4 = x^2 + y^2$. Taking into account the symmetry with respect to x and y we see that there are 62 tuples (x,y) such that x>y>0 and $K^4 = x^2 + y^2$.
The arithmetic of Pythagorean triangles is best understood in the context of Gaussian integers. Gaussian integers are the set

$$Z[i] = \{a + ib \mid a,\ b \in Z\},$$

where $i^2 = -1$. Gaussian integers form a unique factorization domain. This means that a Gaussian integer is representable as a product of Gaussian primes, and this representation is unique up to units. The units in Z[i] are the numbers 1, -1, i, -i. The norm of a Gaussian integer a+ib is defined as N(a+ib):=(a+ib)(a-ib). A Gaussian integer a + bi is a Gaussian prime if and only if either:

one of a, b is zero and the absolute value of the other is a prime number p congruent to 3 modulo 4,

N(a+ib)=$a^2 + b^2$=p with p congruent to 1 modulo 4,

a=±1 and b=±1.

In other words, a Gaussian integer a+ib is a Gaussian prime if and only if either its norm N(a+ib) is a prime number, or it is the product of a unit (±1, ±i) and a prime number of the form 4n + 3.
So suppose that K=pqr with primes p, q, r and each prime is congruent to 1 (mod 4). There is a factorization of p, q, r into Gaussian primes: $p = \pi_1 \pi_2$, $q = \theta_1 \theta_2$, $r = \rho_1 \rho_2$. Hence$K^4 = (\pi_1\ \pi_2\ \theta_1\ \theta_2\ \rho_1\ \rho_2)^4$. We need to determine the number of tuples (x,y) with x>0, y>0 and K^4=(x+iy)(x-iy)=$x^2 + y^2$. This is equivalent to the number of ways we can split the product($\pi_1\ \pi_2\ \theta_1\ \theta_2\ \rho_1$

$\rho_2)^4$ into two factors that are conjugate to each other. These two factors are then necessarily of the form $\pi_1^r\pi_2^{4-r}\theta_1^s\theta_2^{4-s}\rho_1^t\rho_2^{4-t}$ and $\pi_1^{4-r}\pi_2^r\theta_1^{4-s}\theta_2^s\rho_1^{4-t}\rho_2^t$ with 0⩽r,s,t⩽4. There are 125 ways of selecting a triple (r,s,t). One single combination, namely r=s=t=2, does not result in a factorization (x+iy)(x-iy) with both x>0, y>0. So the total count of tuples (x,y) with x>0, y>0 and $K^4=x^2+y^2$ is 124.

Solution 3 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

We'll explain why there are 62 different Pythagorean triangles with hypotenuse K^2.

K^2 is a hypotenuse of a Pythagorean triangle if and only if $\left(K^2\right)^2 = b^2 + c^2$ for two (different, positive) integers b and c.

That is, we need to count the number of ways that K^4can be written non-trivially as a sum of squares.

This is a very old problem with a rich history, dating back to Girard (1625) and Fermat (1640).

At Wolfram MathWorld [1] we find the following "formula" for calculating the number of ways that an integer N can be written as the sum of the squares of two different, positive integers, denoted $r_2'(N)$.

Let $N = 2^{a_0}\left(p_1^{2a_1}p_2^{2a_2}...p_r^{2a_r}\right)\left(q_1^{b_1}q_2^{b_2}....q_s^{b_s}\right)$, where each $p_i \equiv 3\ mod 4$and each $q_j \equiv 1\ mod 4$.

Let $B = (b_1+1)(b_2+1)...(b_s+1)$.

$$\text{Then } r_2'(N) = \begin{cases} 0 & \text{if any } \mathrm{a_i} \text{is a half} - \text{integer} \\ \frac{1}{2}B & \text{if all } \mathrm{a_i} \text{ are integers and B is even} \\ \frac{1}{2}\left(B-(-1)^{a_0}\right) & \text{if all } \mathrm{a_i} \text{ are integers and B is odd} \end{cases}$$

In our problem, $N = K^4 = q_1^4q_2^4q_3^4$and all of the $a_i = 0$.

Thus B = 125 so $r_2'\left(K^4\right) = \frac{1}{2}(125-1) = 62$.

<u>Comment 1</u> The first result in the formula is Euler's well-known result: if some p_i has an odd exponent, then $r_2'(N) = 0$.

<u>Comment 2</u> It would be interesting to know how many of these Pythagorean triples are primitive.

We know of several which are not primitive. For instance, each q_j can be expressed as the sum of two squares in exactly one way; each of these three Pythagorean triangles can then be "scaled up" to have hypotenuseK^2.

[1] Weisstein, Eric W. "Sums of Squares Function";
https://mathworld.wolfram.com/SumofSquaresFunction.html

Solutions were also received from Michael Brozinsky and the proposer of the problem.

• **5644** *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Pete, Paul, and Tom are three sides of a triangle. Tom is originally a perfect 10. Tom becomes 8 when 1 is borrowed from Pete and paid Paul. However, Tom becomes 4 when 1 is borrowed from Paul and paid Pete, instead. It turns out none of this borrowing and paying makes any difference on the area of the triangle. What is the area? And who is Pete and Paul?

Solution 1 by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

Let x and y represent Pete and Paul, respectively, in the original triangle. Then the three triangles have side lengths $(x, y, 10), (x-1, y+1, 8)$, and $(x+1, y-1, 4)$. We use Heron's theorem to calculate the areas, which should all be equal. The semi-perimeters are given by $s_1 = \dfrac{x+y}{2} + 5$, $s_2 = \dfrac{x+y}{2} + 4$, and $s_3 = \dfrac{x+y}{2} + 2$, and the squares of the areas are

$$\begin{aligned}
A_1^2 &= \left(\frac{x+y}{2}+5\right)\left(\frac{x+y}{2}-5\right)\left(\frac{x-y}{2}+5\right)\left(\frac{y-x}{2}+5\right)\\
&= -\frac{(x^2-y^2)^2}{16} + \frac{25}{4}\left((x+y)^2+(x-y)^2\right) - 625,\\
A_2^2 &= \left(\frac{x+y}{2}+4\right)\left(\frac{x+y}{2}-4\right)\left(\frac{x-y}{2}+3\right)\left(\frac{y-x}{2}+5\right)\\
&= -\frac{(x^2-y^2)^2}{16} + 4(x+y)^2 + \frac{15}{4}(x-y)^2 + \frac{1}{4}(x+y)^2(x-y) - 16(x-y) - 240,\\
A_3^2 &= \left(\frac{x+y}{2}+2\right)\left(\frac{x+y}{2}-2\right)\left(\frac{x-y}{2}+3\right)\left(\frac{y-x}{2}+1\right)\\
&= -\frac{(x^2-y^2)^2}{16} + \frac{3}{4}(x+y)^2 + (x-y)^2 - \frac{1}{4}(x+y)^2(x-y) + 4(x-y) - 12.
\end{aligned}$$

Thus,

$$\begin{aligned}
A_1^2 - A_2^2 &= \frac{10}{4}(x+y)^2 + \frac{9}{4}(x-y)^2 - \frac{1}{4}(x+y)^2(x-y) + 16(x-y) - 385\\
&= \frac{1}{4}(x+y)^2(y-x+10) + \frac{9}{4}(y-x)(y-x+10) - \frac{154}{4}(y-x+10)\\
&= \frac{1}{4}(y-x+10)\left[(x+y)^2 + 9(y-x) - 154\right] = 0
\end{aligned}$$

and

$$\begin{aligned}
A_2^2 - A_3^2 &= 3(x+y)^2 + 3(x-y)^2 - \frac{1}{2}(x+y)^2(x-y) - 20(x-y) - 228 \\
&= \frac{1}{2}(x+y)^2(x-y+6) + \big(3(x-y) - 38\big)(x-y+6) \\
&= \frac{1}{2}(x-y+6)\left[(x+y)^2 + 6(x-y) - 76\right] = 0.
\end{aligned}$$

Since $A_1 = A_2$, then either $x - y = 10$ or $(x+y)^2 + 9(y-x) = 154$; similarly, since $A_2 = A_3$, then either $y - x = 6$ or $(x+y)^2 + 6(x-y) = 76$. If $x - y = 10$, then $y - x \neq 6$, so $(x+y)^2 + 6(x-y) = 76$ and $(2y+10)^2 = 16$. Thus, $2y + 10 = \pm 4$ and $y = -3$ or $y = -7$, neither of which can be a triangle side length, so we must have $(x+y)^2 + 9(y-x) = 154$. If $y - x = 6$, then $(2x+6)^2 = 100$, $2x + 6 = \pm 10$, and $x = -8$ or $x = 2$. We cannot have a negative side length, so $x \neq -8$; if $x = 2$, then $y = 8$, but this violates the triangle inequality for the original $(x, y, 10)$ triangle.

Therefore, we must have $(x+y)^2 + 9(y-x) = 154$ and $(x+y)^2 + 6(x-y) = 76$. Subtracting gives $15(y-x) = 78$ and $y - x = \dfrac{26}{5}$. Thus,

$$\begin{aligned}
(x+y)^2 + 9 \cdot \frac{26}{5} &= 154 \\
(x+y)^2 &= \frac{770 - 234}{5} \\
x + y &= \sqrt{\frac{536}{5}} \\
\frac{x+y}{2} &= \sqrt{\frac{134}{5}}.
\end{aligned}$$

Thus, Pete is $x = \sqrt{\dfrac{134}{5}} - \dfrac{13}{5} \approx 2.57687$ and Paul is $y = \sqrt{\dfrac{134}{5}} + \dfrac{13}{5} \approx 7.77687$. The approximate side lengths of the three triangles are $(2.577, 7.777, 10)$, $(1.577, 8.777, 8)$, and $(3.577, 6.777, 4)$ so that all three satify the triangle inequality. In addition,

$$\begin{aligned}
A^2 &= \left[\frac{134}{5} - 25\right]\left[25 - \frac{169}{25}\right] \\
&= \frac{9}{5} \cdot \frac{456}{25}, \\
A &= \frac{6}{5}\sqrt{\frac{114}{5}},
\end{aligned}$$

so the area is $A = \dfrac{6}{5}\sqrt{\dfrac{114}{5}} \approx 5.7299$.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX.

If we let a denote Pete and b denote Paul, then the sides of the three triangles expressed as ordered triples are $(a,b,10)$, $(a-1,b+1,8)$, and $(a+1,b-1,4)$.

By Heron's Formula, the areas of the three triangles are:

$$\begin{aligned}
A_1 &= \frac{1}{4}\sqrt{(a+b+10)(-a+b+10)(a-b+10)(a+b-10)}\\
A_2 &= \frac{1}{4}\sqrt{(a+b+8)(-a+b+10)(a-b+6)(a+b-8)}\\
A_3 &= \frac{1}{4}\sqrt{(a+b+4)(-a+b+2)(a-b+6)(a+b-4)}
\end{aligned}$$

Since $A_1 = A_2$, it follows that either

$$(a+b+10)(a-b+10)(a+b-10) = (a+b+8)(a-b+6)(a+b-8) \tag{1}$$

or

$$-a+b+10 = 0. \tag{2}$$

Since $A_2 = A_3$, it also follows that either

$$(a+b+8)(-a+b+10)(a+b-8) = (a+b+4)(-a+b+2)(a+b-4) \tag{3}$$

or

$$a-b+6 = 0. \tag{4}$$

Equations (1) and (2) simplify to

$$a^2+2ab-9a+b^2+9b-154 = 0 \tag{5}$$
$$b = a-10, \tag{6}$$

and equations (3) and (4) simplify to

$$a^2+2ab+6a+b^2-6b-76 = 0 \tag{7}$$
$$b = a+6. \tag{8}$$

Subtracting equation (7) from equation (5) yields $-15a+15b-78=0$, whence $b = a+\dfrac{26}{5}$. Substituting this value for b into equation (5) yields the quadratic equation $25a^2+130a-501=$

0, whose solutions are $a = \dfrac{-13 \pm \sqrt{670}}{5}$. Since $a > 0$, we have $a = \dfrac{-13 + \sqrt{670}}{5}$, so that $b = a + \dfrac{26}{5} = \dfrac{13 + \sqrt{670}}{5}$.

Now we check the other three combinations of one equation from (5) and (6) along with one equation from (7) and (8).

Equations (5) and (8) yield the quadratic equation $a^2 + 6a - 16 = 0$, whose solutions are $a = -8$ and $a = 2$. Since $a > 0$, we consider only $a = 2$. But then $b = a + 6 = 2 + 6 = 8$ yielding a triangle with sides $(2, 8, 10)$, which is impossible by the triangle inequality.

Equations (6) and (7) yield the quadratic equation $a^2 - 10a + 21 = 0$, whose solutions are $a = 3$ and $a = 7$. Neither of these values is possible: Since $b = a - 10$, in either case $b < 0$.

Clearly equations (6) and (8) cannot simultaneously be true.

So Pete is $a = \dfrac{-13 + \sqrt{670}}{5}$ and Paul is $b = \dfrac{13 + \sqrt{670}}{5}$. Substituting these values into the formula for A_1 above yields $A_1 = \dfrac{6\sqrt{570}}{25}$ as the area of each of the three triangles.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

Let Pete = x and Paul = y. By the triangle inequality, x+y⩾10. Based on Heron's formula for the area of a triangle we find the following equations for x and y

$$\sqrt{(10 + x + y)(-10 + x + y)(10 - x + y)(10 + x - y)} =$$

$$= \sqrt{(8 + x - 1 + y + 1)(-8 + x - 1 + y + 1)(8 - x + 1 + y + 1)(8 + x - 1 - y - 1)} =$$
$$= \sqrt{(4 + x + 1 + y - 1)(-4 + x + 1 + y - 1)(4 - x - 1 + y - 1)(4 + x + 1 - y + 1)}$$

which are equivalent to

$$(10 + x + y)(-10 + x + y)(10 - x + y)(10 + x - y)$$
$$= (8 + x + y)(-8 + x + y)(8 - x + y + 2)(8 + x - y - 2)$$

which in turn is equivalent to

$$(-10 + x - y)\left(-154 - 9x + x^2 + 9y + 2xy + y^2\right) = 0$$

and

$$(8 + x + y)(-8 + x + y)(8 - x + y + 2)(8 + x - y - 2)$$
$$= (4 + x + y)(-4 + x + y)(4 - x + y - 2)(4 + x - y + 2)$$

which in turn is equivalent to

$$(6+x-y)\left(-76+6x+x^2-6y+2xy+y^2\right)=0$$

So there are three equations we need to consider:

$x-y=10$ & $(x+y)^2+6(x-y)=76$**: then x+y=4, contradicting x+y⩾10.**

$x-y=-6$ & $(x+y)^2-9(x-y)=154$**: then x+y=10, x=2, y=8. The triangle with side lengths 2, 8, 10 is degenerate with area 0.**

$(x+y)^2+6(x-y)=76$ & $(x+y)^2-9(x-y)=154$**: then -15(x-y)=78 and finally** $x=\frac{1}{5}\left(-13+\sqrt{670}\right)\approx 2.57687, y=\frac{1}{5}(13+\sqrt{670})\approx 7.77687$**. The area of the triangle equals** $\frac{6\sqrt{570}}{25}\approx 5.729921465430396$**.**

According to Wikipedia (https://en.wikipedia.org/wiki/To_rob_Peter_to_pay_Paul) the phrase "to borrow from Peter to pay Paul" means to take from one person or thing to give to another, especially when it results in the elimination of one debt by incurring another. To try to borrow your way out of debt. English folklore has it that the phrase alludes to an event in mid-16^{th} century England in which the abbey church of Saint Peter, Westminster was deemed a cathedral by letters patent; but ten years later it was absorbed into the diocese of London when the diocese of Westminster was dissolved, and a few years after that many of its assets were expropriated for repairs to Saint Paul's Cathedral. However, the phrase was popular even before that, dating back to at least the late 14^{th} century.

Solution 4 by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA.

We shall show that Pete $=\frac{\sqrt{670}-13}{5}$, Paul $=\frac{\sqrt{670}+13}{5}$ and Area $=\frac{6\sqrt{570}}{25}$.

We'll be using Heron's Formula, which tells us that if a, b, c are the sides of a triangle and K is the area, then $K^2=\frac{1}{16}\left[c^2-(a-b)^2\right]\left[(a+b)^2-c^2\right]$.

For convenience let T = Pete and L = Paul.

In the original triangle, the third side (Tom) is 10, so

$$K^2=\frac{1}{16}\left[10^2-(T-L)^2\right]\left[(T+L)^2-10^2\right].$$

After robbing Peter to pay Paul, we have a triangle with the same area of sides T-1, L+1, 8, so we have

$$K^2=\frac{1}{16}\left[8^2-\left((T-1)-(L+1)\right)^2\right]\left[(T-1+L+1)^2-8^2\right]=\frac{1}{16}\left[64-(T-L)^2\right]\left[(T+L)^2-64\right].$$

If we instead rob Paul to pay Peter, we have a triangle with the same area of sides T+1, L-1, 4, so we have

$$K^2=\frac{1}{16}\left[4^2-\left((T+1)-(L-1)\right)^2\right]\left[\left(T+1+(L-1)\right)^2-4^2\right]=\frac{1}{16}\left[16-(T-L+2)^2\right]\left[(T+L)^2-16\right]$$

For convenience, let $x = T + L$ and $y = T - L$. Then we have

$$\begin{aligned}
16K^2 &= \left[100 - y^2\right]\left[x^2 - 100\right] = (10 - y)(10 + y)\left(x^2 - 100\right)\\
16K^2 &= \left[64 - (y-2)^2\right]\left[x^2 - 64\right] = (10 - y)(6 + y)\left(x^2 - 64\right)\\
16K^2 &= \left[16 - (y+2)^2\right]\left[x^2 - 16\right] = (2 - y)(6 + y)\left(x^2 - 16\right)
\end{aligned}$$

Combining the first and second expressions and using the fact that y $\neq$ 10 (else the third triangle fails the triangle inequality), we have

$$(10 + y)\left(x^2 - 100\right) = (6 + y)\left(x^2 - 64\right) \text{ so } y = \frac{x^2 - 154}{9}.$$

Combining the second and third expressions and using the fact that y $\neq$ -6, we have

$$(10 - y)\left(x^2 - 64\right) = (2 - y)\left(x^2 - 16\right) \text{ so } y = \frac{-x^2 + 76}{6}.$$

Eliminating y, we have $\dfrac{x^2 - 154}{9} = \dfrac{-x^2 + 76}{6} \Rightarrow x^2 = \dfrac{536}{5} \Rightarrow x = \dfrac{2}{5}\sqrt{670}.$

Hence $y = \dfrac{x^2 - 154}{9} = \dfrac{\frac{536}{5} - 154}{9} = -\dfrac{26}{5}.$

At long last,

$$\text{Pete} = T = \frac{x + y}{2} = \frac{\frac{2}{5}\sqrt{670} - \frac{26}{5}}{2} = \frac{\sqrt{670} - 13}{5} \approx 2.57686$$

and

$$\text{Paul} = L = \frac{x - y}{2} = \frac{\frac{2}{5}\sqrt{670} + \frac{26}{5}}{2} = \frac{\sqrt{670} + 13}{5} \approx 7.77687.$$

The three triangles are (approximately)
(2.57686, 7.77687, 10), (1.57686, 8.77687, 8), (3.57686, 6.77687, 4);
each of these satisfies the triangle inequality.
Finally

$$16K^2 = \frac{1}{16}\left[100 - y^2\right]\left[x^2 - 100\right] = \frac{1}{16}\left[100 - \left(-\frac{26}{5}\right)\right]\left[\frac{536}{5} - 100\right] = \frac{36}{25}\frac{114}{5},$$

so

$$\text{Area} = K = \frac{6}{5}\sqrt{\frac{114}{5}} = \frac{6}{25}\sqrt{570} \approx 5.73.$$

Solution 5 by Bruno Salgueiro Fanego, Viveiro, Spain.

Let x and y be the respective values of Pete and Paul originally. From Heron´s formula, if $s = x + y$ and $t = x - y$, the area of the triangle with sides $(x, y, 10)$ is

$$\frac{1}{4}\sqrt{(s + 10)(s - 10)(10 + t)(10 - t)},$$

the area of the triangle with sides $(x-1, y+1, 8)$ is

$$\frac{1}{4}\sqrt{(s+8)(s-8)(6+t)(10-t)}$$

and the area of the triangle with sides $(x+1, y-1, 4)$ is

$$\frac{1}{4}\sqrt{(s+4)(s-4)(6+t)(2-t)}.$$

Hence, both triangles have the same area if and only if

$$\begin{aligned}(s+10)(s-10)(10+t)(10-t)& \\ &= (s+8)(s-8)(6+t)(10-t) \\ &\qquad = (s+4)(s-4)(6+t)(2-t)\,.\end{aligned}$$

Since $t=10$ would imply $s=4$, which is impossible because $s \geqslant t$, then $t=-6$ or $s^2+6t-76=0$. If $t=-6$, then $s=10$, so $(x,y)=(2,8)$, which provides three triangles which degenerate at a segment and whose areas are, hence, 0 and Pete and Paul would be 2 and 8, respectively; and if $t \neq -6$ then $(s,t) = \left(\frac{2\sqrt{670}}{5}, -\frac{26}{5}\right)$, that is, the only non-degenerate solution is that Pete is $x = \frac{\sqrt{670}-13}{5}$ and Paul is $y = \frac{\sqrt{670}+13}{5}$ and the common value of the area is $\frac{6\sqrt{570}}{25}$.

Solutions were also received from Michael Brozinsky, Brian Beasley and the proposer of the problem.

• **5645** *Proposed by Daniel Sitaru, National Economic College,"Theodor Costescu" Drobeta Turna-Severin, Romania.*

Prove: If $a, b, c > 1$ and for all $x > 1$,

$$\log_a x^2 + \log_b x^2 + \log_c x^2 \geqslant \log_a 2^x + \log_b 2^x + \log_c 2^x,$$

then

$$\log_a\left(\frac{2}{e}\right) + \log_b\left(\frac{2}{e}\right) + \log_c\left(\frac{2}{e}\right) = 0.$$

Editor's note: Daniel Sitaru's problem has raised some mathematical eyebrows. Some readers of the Column have written to me about the falsity of the hypothesis of the above problem. And, indeed the hypothesis is false, as clearly demonstrated by Henry Ricardo of Westchester Area Math Circle in an email in which he wrote

(1) The hypothesis inequality can't be true for $a, b, c > 1$ and all $x > 1$. It is, in fact, true for $2 \leqslant x \leqslant 4$. There is equality for $x = 2$ and $x = 4$.

(2) If $a, b, c > 1$ and $2/e < 1$, how can $\log_a(2/e) + \log_b(2/e) + \log_c(2/e)$ equal 0? Each summand must be negative.

As accurate as Henry Ricardo's observations are, it must nonetheless be said in no uncertain terms that Daniel Sitaru's problem (as well as the solution he provided along with the submission of the problem) — understood as a problem of mathematical implication — is impeccable! It's important to keep in proper view the context of Daniel's result. His result is a consequence of his stated hypothesis. His hypothesis is not true for all $x > 1$. For example, let $a = b = c = e$. Then from the hypothesis we can conclude $x^2 \geqslant 2^x$, which is false for $x = 9$. To get a result from a false hypothesis that is itself false is logical and mathematically valid (though not *sound*, technically speaking). So, one way to prove Daniel's result is to completely bypass all normal 'substantive' mathematical implications ensuing from the hypothesis and instead, just say because the hypothesis is false, then not only Daniel's stated conclusion is 'true' (as the result of an implicative process), but so is its negation.

Let's imagine for a moment that no one had become aware of the falsity of the hypothesis in Daniel's problem and that every solution, put forth, traversed through all valid steps until it reached at the stated conclusion, meanwhile having produced a substantial amount of significant math and insight. The take-home lesson is that even a 'wrong' problem has the potential to produce significant math and insight. In fact, some mathematicians have produced noteworthy results based on the assumption of certain unsolved conjectures being true.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

As the problem is stated, the final result is impossible. Because $2/e < 1$, if $a, b, c > 1$, then each of the logarithms

$$\log_a\left(\frac{2}{e}\right), \quad \log_b\left(\frac{2}{e}\right), \quad \text{and} \quad \log_c\left(\frac{2}{e}\right)$$

is negative. Consequently, the sum of these logarithms cannot be equal to zero. If the condition $a, b, c > 1$ is replaced with $a, b, c > 0$, then the final result is possible.

So, suppose $a, b, c > 0$ and for all $x > 1$,

$$\log_a x^2 + \log_b x^2 + \log_c x^2 \geqslant \log_a 2^x + \log_b 2^x + \log_c 2^x.$$

This inequality can be rewritten as

$$\log_a \frac{x^2}{2^x} + \log_b \frac{x^2}{2^x} + \log_c \frac{x^2}{2^x} = \left(\frac{1}{\ln a} + \frac{1}{\ln b} + \frac{1}{\ln c}\right) \ln \frac{x^2}{2^x} \geqslant 0.$$

Let
$$f(x) = \frac{x^2}{2^x}.$$
Then
$$f'(x) = \frac{x(2 - x\ln 2)}{2^x},$$
and f is increasing for $1 < x < 2/\ln 2$ and is decreasing for $x > 2/\ln 2$. With $f(2) = f(4) = 1$, it follows that $f(x) < 1$ for $1 < x < 2$ and for $x > 4$ and $f(x) > 1$ for $2 < x < 4$. This implies $\ln(x^2/2^x)$ takes on both positive and negative values for $x > 1$, so the inequality
$$\left(\frac{1}{\ln a} + \frac{1}{\ln b} + \frac{1}{\ln c}\right) \ln \frac{x^2}{2^x} \geqslant 0$$
can only be satisfied for all $x > 1$ if
$$\frac{1}{\ln a} + \frac{1}{\ln b} + \frac{1}{\ln c} = 0.$$
Finally,
$$\log_a\left(\frac{2}{e}\right) + \log_b\left(\frac{2}{e}\right) + \log_c\left(\frac{2}{e}\right) = \left(\frac{1}{\ln a} + \frac{1}{\ln b} + \frac{1}{\ln c}\right) \ln \frac{2}{e} = 0.$$

Solution 2 by Michel Bataille, Rouen, France.

For $u > 0,\ u \neq 1$, we have $\log_u\left(\frac{2}{e}\right) = \frac{\ln(2/e)}{\ln(u)}$. It follows that the equality to be proved is equivalent to
$$\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)} = 0.$$
The hypothesis can be written as
$$(2\ln(x))\left(\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)}\right) \geqslant (x\ln(2))\left(\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)}\right),$$
that is,
$$f(x) \cdot \left(\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)}\right) \geqslant 0 \tag{1}$$
where $f(x) = 2\ln(x) - x\ln(2)$ for $x > 1$. Taking the limit when $x \to 1^+$ in (1), we see that $(-\ln(2))\left(\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)}\right) \geqslant 0$, hence we must have
$$\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)} \leqslant 0. \tag{2}$$
In addition we have $f(2/\ln(2)) = 2\ln\left(\frac{2}{\ln(2)}\right) - 2 > 0$ (since $\frac{2}{\ln(2)} > \frac{2}{0.7} > 2.8 > e$). Therefore, taking $x = \frac{2}{\ln(2)}$ in (1), we obtain that
$$\frac{1}{\ln(a)} + \frac{1}{\ln(b)} + \frac{1}{\ln(c)} \geqslant 0. \tag{3}$$

The desired result follows from (2) and (3).

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

The hypothesis is equivalent to

$$(2\log x\ - x\log 2\ )\left(\frac{1}{\log a}+\frac{1}{\log b}+\frac{1}{\log c}\right)\geqslant 0.$$

We note that
$2\log x\ - x\log 2\ > 0$ for 2<x<4,
$2\log x\ - x\log 2\ = 0$ for x∈{2,4},
$2\log x\ - x\log 2\ < 0$ for 1<x<2 or x>4.
Indeed, let $f\left(x\right) = 2\log x\ - x\log 2$. Then $f'\left(x\right) = \dfrac{2}{x} - \log 2$ and f(x) is monotonically increasing for 0<x<2/log2 and monotonically decreasing in x>2/log2.
Hence $\dfrac{1}{\log a}+\dfrac{1}{\log b}+\dfrac{1}{\log c} = 0$ which is equivalent to $\log_a\left(\dfrac{2}{e}\right) + \log_b\left(\dfrac{2}{e}\right) + \log_c\left(\dfrac{2}{e}\right) = 0.$

Solutions were also received from Henry Ricardo, David Stone, John Hawkins, Michael Brozinsky and the proposer of the problem.

• **5646** *Proposed by Michel Bataille, Rouen, France.*

If a, b are real numbers such that $b > a > 0$, then find the minimal value of MN if M is a point of the parabola $y = ax^2$ and N is a point of the parabola $y = bx^2 + 1$.

Solution 1 by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

The minimal value of MN is 1 if $2ab \leqslant b - a$ and $\dfrac{\sqrt{(b-a)(4ab-b+a)}}{2ab}$ if $2ab > b - a$.

We first prove the following lemma.

Lemma 1. *Let C be a curve in the xy-plane, and let P be a point in the xy-plane that is not on C. If Q is the point on C that is closest to P, then the vector $\overrightarrow{PQ}$ is normal to C at Q.*

Proof. Let $\vec{r}(t)$ parametrize C, with $\vec{r}(a) = \vec{Q}$, and let $\vec{s}(t) = \vec{r}(t) - \vec{P}$. Then $f(t) = \vec{s}(t) \cdot \vec{s}(t)$ is minimized when $t = a$, since Q is the point of C closest to P. Thus,

$$0 = f'(a) = 2\vec{r}\,'(a) \cdot \left(\vec{r}(a) - \vec{P}\right),$$

so that $\overrightarrow{PQ}$ is normal to the tangent vector to C at the point Q. □

Let $M=(x,ax^2)$ and $N=(w,bw^2+1)$. By the lemma, $\overrightarrow{MN}$ is normal to the parabolas at M and N, so that the derivatives $2ax$ and $2bw$ are equal, and $w=\left(\dfrac{a}{b}\right)x$. We seek to minimize the square of the distance MN, which is given by

$$\begin{aligned}(x-w)^2+\left(bw^2-ax^2+1\right)^2 &= \left(x-\frac{ax}{b}\right)^2+\left(\frac{a^2x^2}{b}-ax^2+1\right)^2\\ f(x) &= \frac{a^2(b-a)^2}{b^2}x^4+\frac{(b-a)(b-a-2ab)}{b^2}x^2+1.\end{aligned}$$

Thus,

$$\begin{aligned}f'(x) &= \frac{4a^2(b-a)^2}{b^2}x^3+\frac{2(b-a)(b-a-2ab)}{b^2}x\\ &= \frac{2(b-a)x}{b^2}\left(2a^2(b-a)x^2+b-a-2ab\right).\end{aligned}$$

If $2ab\leqslant b-a$, then $x=0$ is the only critical point of f; by the first derivative test, the absolute minimum of f is $f(0)=1$.

If $2ab>b-a$ then f has additional critical points at

$$\begin{aligned}x^2 &= \frac{2ab-b+a}{2a^2(b-a)}\\ x &= \pm\sqrt{\frac{2ab-b+a}{2a^2(b-a)}}\end{aligned}$$

Since $f'(x)<0$ for $0<x<\sqrt{\dfrac{2ab-b+a}{2a^2(b-a)}}$ and $f'(x)>0$ for $x>\sqrt{\dfrac{2ab-b+a}{2a^2(b-a)}}$, then by the first derivative test, and symmetry, the absolute minimum value of f is

$$\begin{aligned}f\left(\pm\sqrt{\frac{2ab-b+a}{2a^2(b-a)}}\right) &= \frac{a^2(b-a)^2(2ab-b+a)^2}{4a^4b^2(b-a)^2}-\frac{(b-a)(2ab-b+a)^2}{2a^2b^2(b-a)}+1\\ &= \frac{(2ab-b+a)^2}{4a^2b^2}-\frac{(2ab-b+a)^2}{2a^2b^2}+1\\ &= \frac{4a^2b^2-(2ab-b+a)^2}{4a^2b^2}\\ &= \frac{(b-a)(4ab-b+a)}{4a^2b^2},\end{aligned}$$

and the minimum value of MN, when $2ab>b-a$, is

$$\frac{\sqrt{(b-a)(4ab-b+a)}}{2ab}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $(\xi, a\xi^2)$ denote the point M of the parabola $y = ax^2$ and $(\eta, 1 + b\eta^2)$ denote the point N of the parabola $y = 1 + bx^2$. Because minimizing distance is equivalent to minimizing the square of the distance, let

$$f(\xi, \eta) = (\eta - \xi)^2 + (1 + b\eta^2 - a\xi^2)^2.$$

By symmetry, we can restrict attention to $\eta \geqslant 0$ and $\xi \geqslant 0$. The critical points of f satisfy the equations

$$\frac{\partial f}{\partial \xi} = -2(\eta - \xi) - 4a\xi(1 + b\eta^2 - a\xi^2) = 0, \tag{9}$$

$$\frac{\partial f}{\partial \eta} = 2(\eta - \xi) + 4b\eta(1 + b\eta^2 - a\xi^2) = 0. \tag{10}$$

Multiply (1) by $b\eta$ and (2) by $a\xi$ and then add to obtain

$$(\eta - \xi)(a\xi - b\eta) = 0.$$

Thus, either $\eta = \xi$ or $\eta = a\xi/b$. Substituting $\eta = \xi$ into (1) yields $\eta = \xi = 0$; thus, $(0, 0)$ is a critical point of f. On the other hand, substituting $\eta = a\xi/b$ into (1) yields

$$\xi\left[\frac{a-b}{b} + 2a + \frac{2a^2}{b}(a-b)\xi^2\right] = 0.$$

It follows that $(0, 0)$ is the only critical point of f if

$$\frac{a-b}{b} + 2a \leqslant 0; \quad \text{that is, if} \quad a \leqslant \frac{b}{2b+1}.$$

If $a > b/(2b + 1)$, then there is a second critical point with

$$\hat{\xi}^2 = -\frac{1}{2a^2} + \frac{b}{a(b-a)} \quad \text{and} \quad \hat{\eta}^2 = -\frac{1}{2b^2} + \frac{a}{b(b-a)}.$$

Note that

$$\hat{\eta}^2\hat{\xi}^2 = \left(\frac{1}{b-a} - \frac{1}{2ab}\right)^2$$

and

$$\frac{1}{b-a} - \frac{1}{2ab} > 0 \quad \text{for} \quad a > \frac{b}{2b+1}.$$

Thus,

$$\begin{aligned} f(\hat{\xi}, \hat{\eta}) &= \hat{\eta}^2 - 2\hat{\eta}\hat{\xi} + \hat{\xi}^2 + (1 + b\hat{\eta}^2 - a\hat{\xi}^2)^2 \\ &= 1 - \left(\frac{2ab + a - b}{2ab}\right)^2. \end{aligned}$$

Because $f(0,0)=1$, the minimal value of MN if M is a point of the parabola $y=ax^2$ and N is a point of the parabola $y=bx^2+1$ is

$$\begin{cases} 1, & 0<a\leqslant \dfrac{b}{2b+1} \\ \sqrt{1-\left(\dfrac{2ab+a-b}{2ab}\right)^2}, & \dfrac{b}{2b+1}<a<b \end{cases}$$

Solutions were also received from Henry Ricardo, David Stone, John Hawkins, Michael Brozinsky, Albert Stadler, Kee-Wai Lau and the proposer of the problem.

Editor's note regarding #5646: One approach to solving the problem is by taking advantage of the fact that a shortest line segment $\overline{MN}$ must be (simultaneously) normal to both curves. Letting $M=\left(m,am^2\right)$ and $N=\left(n,bn^2+1\right)$, and assuming $\overline{MN}$ is non-vertical, we have

$$\frac{bn^2+1-am^2}{n-m}=-\frac{1}{2bn}=-\frac{1}{2am}$$

which yields

$$n^2=\frac{a\,(2b+1)-b}{2b^2\,(b-a)}$$

which holds only for $a\geqslant b/\,(2b+1)$. For $a<b/\,(2b+1)$, the shortest line segment $\overline{MN}$ is vertical, in which case $n=0$.

• **5647** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Find all real solutions to the following system of equations

$$\begin{cases} 3\left(\dfrac{1}{16}\right)^x+2\left(\dfrac{1}{27}\right)^y & =\dfrac{13}{6}, \\ 3\log_{\frac{1}{16}}x-2\log_{\frac{1}{27}}y & =\dfrac{5}{6}. \end{cases}$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle.

Solving each equation for y yields the equivalent system

$$\begin{cases} y \;=\; f(x) \;=\; \log_{\frac{1}{27}}\dfrac{1}{2}\left(\dfrac{13}{6}-3\left(\dfrac{1}{16}\right)^x\right) \\ \\ y \;=\; g(x) \;=\; \left(\dfrac{1}{27}\right)^{\frac{1}{2}\left(3\log_{\frac{1}{16}}x\;-\;\frac{5}{6}\right)} \;=\; 3^{\frac{9}{2}\log_{16}x+\frac{5}{4}}, \end{cases}$$

where the domain of f is $\left(\log_{\frac{1}{16}}(13/18), \infty\right)$ and the domain of g is $(0, \infty)$. Since $(1/16)^x$ is decreasing and $1/27 < 1$, $f(x)$ is strictly decreasing on its domain, and $g(x)$ is clearly strictly increasing for $x > 0$. Since $f(1/4) = g(1/4) = 1/3$, it follows that $(x, y) = (1/4, 1/3)$ is the unique solution of the system.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let

$$\xi = \left(\frac{1}{16}\right)^x \quad \text{and} \quad \eta = \left(\frac{1}{27}\right)^y.$$

From the second equation, x and y must both be positive, so $0 < \xi, \eta < 1$. With this change of variables, the original system of equations becomes

$$\begin{cases} 3\xi + 2\eta = \dfrac{13}{6}, \\ 3\log_{\frac{1}{16}}(\log_{\frac{1}{16}} \xi) - 2\log_{\frac{1}{27}}(\log_{\frac{1}{27}} \eta) = \dfrac{5}{6}. \end{cases}$$

The equation $3\xi + 2\eta = \dfrac{13}{6}$ combined with the constraint $0 < \eta < 1$ yields

$$\frac{1}{18} < \xi < \frac{13}{18}.$$

Consider the function

$$f(\xi) = 3\log_{\frac{1}{16}}(\log_{\frac{1}{16}} \xi) - 2\log_{\frac{1}{27}}\left(\log_{\frac{1}{27}}\left(\frac{13}{12} - \frac{3\xi}{2}\right)\right).$$

Then

$$f'(\xi) = \frac{3}{\left(\ln \frac{1}{16}\right)^2 \xi \log_{\frac{1}{16}} \xi} + \frac{3}{\left(\ln \frac{1}{27}\right)^2 \left(\frac{13}{12} - \frac{3\xi}{2}\right) \log_{\frac{1}{27}}\left(\frac{13}{12} - \frac{3\xi}{2}\right)} > 0$$

for $\dfrac{1}{18} < \xi < \dfrac{13}{18}$. Thus, f is one-to-one for $\dfrac{1}{18} < \xi < \dfrac{13}{18}$. Now,

$$f\left(\frac{1}{2}\right) = 3\log_{\frac{1}{16}}\left(\log_{\frac{1}{16}} \frac{1}{2}\right) - 2\log_{\frac{1}{27}}\left(\log_{\frac{1}{27}} \frac{1}{3}\right) = 3\log_{\frac{1}{16}} \frac{1}{4} - 2\log_{\frac{1}{27}} \frac{1}{3} = \frac{5}{6}.$$

Because f is one-to-one for $\dfrac{1}{18} < \xi < \dfrac{13}{18}$, $\xi = \dfrac{1}{2}$ is the only solution to $f(\xi) = \dfrac{5}{6}$ for $\dfrac{1}{18} < \xi < \dfrac{13}{18}$. Thus, $(\xi, \eta) = \left(\dfrac{1}{2}, \dfrac{1}{3}\right)$ is the only solution to the transformed system of equations. This yields $(x, y) = \left(\dfrac{1}{4}, \dfrac{1}{3}\right)$ as the only solution to the original system of equations.

Solution 3 by Eagle Problem Solvers, Georgia Southern University, Savannah and Statesboro, GA.

There is one real solution: $(x, y) = (1/4, 1/3)$. For the second equation to be defined, x and y must be positive real numbers. For each equation, we can explicitly write y as a function of x. We write

$$y = \frac{\ln\left(\frac{13}{12} - \frac{3}{2}\left(\frac{1}{16}\right)^x\right)}{-3\ln 3} = f(x)$$

and

$$y = 3^{\frac{5}{4} - \frac{9}{2}\log_{\frac{1}{16}} x} = g(x).$$

We use implicit differentiation to find $\dfrac{dy}{dx}$ for each of these functions. For the first equation,

$$\begin{aligned} -12\ln 2\left(\frac{1}{16}\right)^x - 6\ln 3\left(\frac{1}{27}\right)^y \cdot \frac{dy}{dx} &= 0 \\ -6\ln 3\left(\frac{1}{27}\right)^y \cdot \frac{dy}{dx} &= 12\ln 2\left(\frac{1}{16}\right)^x \\ f'(x) &= -\frac{2\ln 2 \cdot 27^y}{\ln 3 \cdot 16^x} < 0. \end{aligned}$$

For the second equation

$$\begin{aligned} -\frac{3}{x\ln 16} + \frac{2}{y\ln 27} \cdot \frac{dy}{dx} &= 0 \\ \frac{2}{3y\ln 3} \cdot \frac{dy}{dx} &= \frac{3}{4x\ln 2} \\ g'(x) &= \frac{3}{4x\ln 2} \cdot \frac{3y\ln 3}{2} \\ &= \frac{9y\ln 3}{8x\ln 2} > 0. \end{aligned}$$

Let $h(x) = g(x) - f(x)$. Since g and f are differentiable for all $x > 0$, the same is true for h. Notice that $h'(x) = g'(x) - f('x) > 0$ for all $x > 0$, and that

$$h\left(\frac{1}{4}\right) = g\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right) = \frac{1}{3} - \frac{1}{3} = 0.$$

By Rolle's Theorem, there can be no other real number b for which $h(b) = 0$, because that would mean $h'(c) = 0$ for some real number c between $\frac{1}{4}$ and b. Thus, $x = \frac{1}{4}$ is the only zero of h, and hence the only real number x for which $f(x) = g(x)$. Consequently, $(x, y) = (1/4, 1/3)$ is the only real solution to the system.

Solution 4 by Michel Bataille, Rouen, France.

It is readily checked that a solution for (x, y) is $(\frac{1}{4}, \frac{1}{3})$. We show that there is no other solution. Suppose that (x, y) is a solution. Then $x, y > 0$ and from the second equation

$$\frac{3\ln x}{4\ln 2} - \frac{2\ln y}{3\ln 3} = \frac{2\ln 3}{3\ln 3} - \frac{3\ln 4}{4\ln 2}$$

so that

$$\frac{3\ln(4x)}{4\ln 2} = \frac{2\ln(3y)}{3\ln 3}. \tag{1}$$

Similarly, the first equation can be written as

$$3(e^{-4x\ln 2} - e^{-\ln 2}) + 2(e^{-3y\ln 3} - e^{-\ln 3}) = 0. \tag{2}$$

Now, if $x < \frac{1}{4}$, then we must have $y < \frac{1}{3}$ (from (1)) and it follows that $e^{-4x\ln 2} > e^{-\ln 2}$ and $e^{-3y\ln 3} > e^{-\ln 3}$, contradicting (2). Similarly, if $x > \frac{1}{4}$, then $y > \frac{1}{3}$, leading to a contradiction of (2). Thus $x = \frac{1}{4}$ and, from (1), $y = \frac{1}{3}$.

Solution 5 by Seán M. Stewart, Bomaderry, NSW, Australia.

A trivial solution to the system of equations is $(x, y) = \left(\frac{1}{4}, \frac{1}{3}\right)$. We show this is the only real solution to the system of equations.

From the first of the equations, namely

$$3\left(\frac{1}{16}\right)^x + 2\left(\frac{1}{27}\right)^y = \frac{13}{6},$$

on making y the subject of this equation we have

$$y = f_1(x) = -\frac{1}{\ln(27)} \ln\left(\frac{13}{12} - \frac{3}{2^{4x+1}}\right),$$

where $x > x_0 = \ln\left(\frac{18}{13}\right)/\ln(16)$. Observe that $\lim_{x\to x_0^+} f_1(x) = \infty$ and $\lim_{x\to\infty} f_1(x) = -\ln\left(\frac{13}{12}\right)/\ln(27) < 0$. Also

$$f_1'(x) = -\frac{24\ln(2)}{(13\cdot 16^x - 18)\ln(3)} < 0,$$

for all $x > x_0$. Thus $f_1(x)$ is a decreasing function.

From the second of the equations, namely

$$3\log_{\frac{1}{16}} x - 2\log_{\frac{1}{27}} y = \frac{5}{6},$$

on making y the subject of this equation we have

$$y = f_2(x) = \exp\left[\ln(27)\left(\frac{5}{12} + \frac{3\ln x}{\ln(256)}\right)\right],$$

where $x > 0$. Observe $\lim\limits_{x\to 0^+} f_2(x) = 0$ and $\lim\limits_{x\to\infty} f_2(x) = \infty$. Also

$$f_2'(x) = \frac{\ln(27)}{x\ln(256)} 3^{\frac{9\ln(4x)}{\ln(256)}} > 0,$$

for all $x > 0$. Thus $f_2(x)$ is an increasing function.

As we have a decreasing function $f_1(x)$ with

$$\lim_{x\to x_0^+} f_1(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} f_1(x) = -\ln\left(\frac{13}{12}\right)/\ln(27) < 0,$$

and an increasing function $f_2(x)$ with

$$\lim_{x\to 0^+} f_2(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} f_2(x) = \infty,$$

as both functions are continuous they accordingly only intersect at a single point for $x > x_0$. That point is the trivial point already found and represents all real solutions to the given system of equations.

Solution 6 by Albert Stadler, Herrliberg, Switzerland.

The logarithms are defined for x, y>0.

$3\log_{\frac{1}{16}} x \ - 2\log_{\frac{1}{27}} y \ = \dfrac{5}{6}$ is equivalent to each of the following lines:

$$\frac{3\log x}{-4\log 2} - \frac{2\log y}{-3\log 3} = \frac{5}{6},$$

$$-9\frac{\log x}{\log 2} + 8\frac{\log y}{\log 3} = 10,$$

$$y^{\frac{8}{\log 3}} = e^{10} x^{\frac{9}{\log 2}},$$

$$y = e^{\frac{10\log 3}{8}} x^{\frac{9\log 3}{8\log 2}}.$$

The function

$$f(x) = 3\left(\frac{1}{16}\right)^x + 3\left(\frac{1}{27}\right)^{e^{\frac{10\log 3}{8}} x^{\frac{9\log 3}{8\log 2}}}$$

is monotonically decreasing. So there is a unique value x such that f(x)=13/6. We find for x=1/4:

$$3\left(\frac{1}{16}\right)^{(1/4)} + 2\left(\frac{1}{27}\right)^{e^{\frac{10\log 3}{8}}(1/4)^{\frac{9\log 3}{8\log 2}}} = \frac{3}{2} + 2\left(\frac{1}{27}\right)^{\frac{1}{3}} = \frac{13}{6}.$$

If x=1/4 then y=1/3. We verify that (x,y) = (1/4,1/3) is indeed a solution and it is unique.

Solutions were also received from Michael Brozinsky, Brian Beasley and the proposer of the problem.

• **5648** *Problem proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $a, b > 0$ and $\alpha \in (0,1]$ be real numbers. Calculate:

$$\int_0^1 \frac{x^a - x^b}{\left(-\ln(x)\right)^\alpha} dx.$$

Solution 1 by Moti Levy, Rehovot, Israel.

$$I(a,b,\alpha) := \int_0^1 \frac{x^a - x^b}{\left(-\ln(x)\right)^\alpha} dx$$

After change of variable $-\ln(x) = t$,

$$I(a,b,\alpha) = \int_0^\infty \left(t^{-\alpha}e^{-t}e^{-at} - t^{-\alpha}e^{-t}e^{-bt}\right) dt$$

For $0 < \alpha < 1$, $\displaystyle\int_0^\infty t^{-\alpha}e^{-t}e^{-st}dt = \mathcal{L}\left(t^{-\alpha}e^{-t}\right)$, where $F(s) = \mathcal{L}\left(f(t)\right)$ is the Laplace transform.

$$\mathcal{L}\left(t^{-\alpha}e^{-t}\right) = \frac{\Gamma(1-\alpha)}{(1+s)^{1-\alpha}}, \quad 0 \leqslant \alpha < 1$$

$$\int_0^1 \frac{x^a - x^b}{\left(-\ln(x)\right)^\alpha} dx = \frac{\Gamma(1-\alpha)}{(1+a)^{\alpha+1}} - \frac{\Gamma(1-\alpha)}{(1+b)^{\alpha+1}}.$$

For $\alpha = 1$

$$\begin{aligned} I(a,b,1) &= \int_0^\infty \frac{e^{-t(1+a)} - e^{-t(1+b)}}{t} dt \\ &= \int_0^\infty \int_{1+a}^{1+b} e^{-ut} du dt = \int_{1+a}^{1+b} \int_0^\infty e^{-ut} dt du = \int_{1+a}^{1+b} \frac{1}{u} du = \ln(1+b) - \ln(1+a). \end{aligned}$$

We conclude that

$$\int_0^1 \frac{x^a - x^b}{\left(-\ln(x)\right)^\alpha} dx = \begin{cases} \dfrac{\Gamma(1-\alpha)}{(1+a)^{\alpha+1}} - \dfrac{\Gamma(1-\alpha)}{(1+b)^{\alpha+1}}, & 0 \leqslant \alpha < 1 \\ \ln \dfrac{1+b}{1+a}, & \alpha = 1. \end{cases}$$

Solution 2 by Michel Bataille, Rouen, France.

We observe that if $x > 0, x \neq 1$ and $a, b \in \mathbb{R}$, then

$$\int_a^b x^u\,du = \int_a^b e^{u\ln x}\,du = \left[\frac{e^{u\ln x}}{\ln x}\right]_a^b = \frac{x^b - x^a}{\ln x}$$

It follows that

$$\frac{x^a - x^b}{(-\ln x)^\alpha} = (-\ln x)^{1-\alpha}\int_a^b x^u\,du$$

so that

$$I(\alpha) = \int_0^1 \left(\int_a^b x^u(-\ln x)^{1-\alpha}\,du\right)\,dx = \int_a^b \left(\int_0^1 x^u(-\ln x)^{1-\alpha}\,dx\right)\,du$$

(since $x^u(-\ln x)^{1-\alpha} \geqslant 0$ for $x \in (0,1],\ u \in [a,b]$).
The change of variables $x = e^{(-y)/(1+u)}$ leads to

$$\int_0^1 x^u(-\ln x)^{1-\alpha}\,dx = \frac{1}{(1+u)^{2-\alpha}}\int_0^\infty e^{-y}y^{1-\alpha}\,dy = \frac{\Gamma(2-\alpha)}{(1+u)^{2-\alpha}}.$$

We deduce that $I(\alpha) = \Gamma(2-\alpha)\displaystyle\int_a^b \frac{du}{(1+u)^{2-\alpha}}$ and finally:

$$I(1) = \ln\left(\frac{1+b}{1+a}\right) \quad \text{and} \quad I(\alpha) = \frac{\Gamma(2-\alpha)}{1-\alpha}\left((1+a)^{\alpha-1} - (1+b)^{\alpha-1}\right) \quad (\alpha \in (0,1)).$$

Solution 3 by Seán M. Stewart, Bomaderry, NSW, Australia.

For $a, b > 0$ and $\alpha \in (0,1]$ we show that

$$I(\alpha) = \begin{cases} \dfrac{\Gamma(2-\alpha)}{\alpha-1}\left((b+1)^{\alpha-1} - (a+1)^{\alpha-1}\right), & 0 < \alpha < 1, \\ \log\left(\dfrac{b+1}{a+1}\right), & \alpha = 1. \end{cases}$$

Here Γ denotes the gamma function.

To show this we begin by observing

$$x^a - x^b = -\log(x)\int_a^b x^t\,dt.$$

So writing $I(\alpha)$ as a double integral we have

$$I(\alpha) = \int_a^b \int_0^1 \frac{x^t}{(-\log x)^{\alpha-1}}\,dx\,dt.$$

Here the order of the integration has been changed and is permissible due to Fubini's theorem for double integrals. In the inner integration let $x = e^{-u}$. Then

$$I(\alpha) = \int_a^b \int_0^\infty \frac{e^{-u(t+1)}}{u^{\alpha-1}}\, du\, dt.$$

Next, substituting $z = u(t+1)$ yields

$$I(\alpha) = \int_0^\infty z^{1-\alpha} e^{-z}\, dz \int_a^b (t+1)^{\alpha-2}\, dt.$$

The z-integral is related to the gamma function. Here

$$\int_0^\infty z^{1-\alpha} e^{-z}\, dz = \Gamma(2-\alpha).$$

The t-integral is elementary. Here

$$\int_a^b (t+1)^{\alpha-2}\, dt = \begin{cases} \dfrac{1}{\alpha-1}\left((b+1)^{\alpha-1} - (a+1)^{\alpha-1}\right), & 0 < \alpha < 1, \\ \log\left(\dfrac{b+1}{a+1}\right), & \alpha = 1. \end{cases}$$

Since $\Gamma(2-\alpha) = \Gamma(1) = 1$ when $\alpha = 1$, the desired result then follows.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Solution: With the substitution $u = -\ln x$,

$$I(\alpha) = \int_0^\infty \frac{e^{-(a+1)u} - e^{(b+1)u}}{u^\alpha}\, du.$$

For $\alpha \in (0,1)$, the substitution $t = (a+1)u$ yields

$$\int_0^\infty \frac{e^{-(a+1)u}}{u^\alpha}\, du = (a+1)^{\alpha-1} \int_0^\infty t^{-\alpha} e^{-t}\, dt = (a+1)^{\alpha-1}\Gamma(1-\alpha),$$

where $\Gamma(z)$ is the gamma function. Similarly,

$$\int_0^\infty \frac{e^{-(b+1)u}}{u^\alpha}\, du = (b+1)^{\alpha-1}\Gamma(1-\alpha),$$

so

$$I(\alpha) = \left((a+1)^{\alpha-1} - (b+1)^{\alpha-1}\right)\Gamma(1-\alpha).$$

For $\alpha = 1$,

$$\int_0^\infty \frac{e^{-(a+1)u} - e^{(b+1)u}}{u}\, du = \lim_{t\to 0^+} \left(\Gamma(0,(a+1)t) - \Gamma(0,(b+1)t)\right),$$

where $\Gamma(s,z)$ is the incomplete gamma function

$$\Gamma(s,z) = \int_z^\infty t^{s-1}e^{-t}\,dt.$$

With

$$\Gamma(0,z) = -\gamma - \ln z - \sum_{k=1}^{\infty}(-1)^k\frac{z^k}{kk!},$$

it follows that

$$\lim_{t\to 0^+}\big(\Gamma(0,(a+1)t) - \Gamma(0,(b+1)t)\big) = \ln\frac{b+1}{a+1}.$$

Thus,

$$I(1) = \ln\frac{b+1}{a+1}.$$

In summary,

$$I(\alpha) = \begin{cases} \big((a+1)^{\alpha-1} - (b+1)^{\alpha-1}\big)\,\Gamma(1-\alpha), & 0<\alpha<1 \\ \ln\dfrac{b+1}{a+1}, & \alpha = 1 \end{cases}.$$

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Germany.

Let $a,b>0$ and $\alpha\in(0,1]$. Using

$$x^a - x^b = (-\ln x)\int_a^b x^t dt$$

we obtain

$$I(\alpha) = \int_0^1\int_a^b (-\ln x)^{1-\alpha}\,x^t dt\ dx.$$

Interchanging the order of integration and a change of variable $x = e^{-u/(1+t)}$ $(t\geqslant 0)$ yield

$$\begin{aligned} I(\alpha) &= \int_a^b \frac{1}{1+t}\int_0^\infty u^{1-\alpha}e^{-ut/(1+t)}e^{-u/(1+t)}du\ dt \\ &= \int_0^\infty u^{1-\alpha}e^{-u}du\int_a^b\frac{1}{1+t}dt \\ &= \Gamma(2-\alpha)\ln\frac{1+b}{1+a}. \end{aligned}$$

Solutions were also received from Kee-Wai Lau, Ankush Kumar parcha, Henry Ricardo, Team inspired, Albert Stadler, Michael Brozinsky and the proposer of

the problem.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributers. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following recommendations. As you peruse below, you may construe that the recommendations amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "Statement of the Problem".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "Solution of the Problem".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Proposed problem to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "Statement of the Problem".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "Solution of the Problem".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Proposed problem to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica ($\longleftarrow$ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣