Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

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Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before April 15, 2022.

• **5667** *Proposed by Albert Stadler, Herrilberg, Switzerland.*

Prove with at most 10 function evaluations that

$$4 \cdot 10^{-89} < \prod_{k=1}^{89} \tan^2\left(\frac{k\pi}{360}\right) < 5 \cdot 10^{-89}.$$

Editor's note: This problem is engendered by and is a follow-up to Problem #5632.

• **5668** *Proposed by Ovidiu-Gabriel Dinu, Technological High School, Petrache Poenaru, Bălceşti, Vâlcea , România.*

Prove that for x and t in $[0, 1]$ and for any integer $k \geqslant 2$:

$$\left| e^{-x^{2k}} - \int_0^1 e^{-t^{2k}} dt \right| \leqslant 2k \left(\sqrt[2k]{\frac{2k-1}{2k}} \right)^{2k-1} e^{-\frac{2k-1}{2k}}.$$

• **5669** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose a is a real number. Find:

$$\Omega = \lim_{p\to\infty} \frac{1}{p^a} \cdot \sum_{m=1}^{p} \sum_{n=1}^{m} \sum_{k=1}^{n} \frac{k^2}{2k^2 - 2nk + n^2}.$$

• **5670** *Proposed by Kenneth Korbin, New York, NY.*

Find a positive real number x such that

$$\sqrt[4]{\pi + x} = 1 + \sqrt[4]{\pi - x}\,.$$

• **5671** *Proposed by Michael Brozinsky, Central Islip, New York.*

Isosceles triangle $\triangle RST$ with $RS = ST$ has the following property:

There are only three points such that the distances from each of these points to the lines $\overleftrightarrow{RT}$, $\overleftrightarrow{RS}$ and $\overleftrightarrow{ST}$ have, respectively, the same ratios as $1:2:3$.

Determine the angles of triangle $\triangle RST$.

• **5672** *Proposed by Nikos Ntorvas, Athens, Greece.*

Given

$$F(x,y) = (y-x)\left[y\left(3y^2 - 28y + 3xy - 14x + 84\right)\right.$$
$$\left. +x\left(3x^2 - 28x + 3xy - 14y + 84\right) - 96\right],$$

where $x, y \in \mathbb{R}$, with $0 \leqslant x < y$, find $A = \min F(x,y)$ and the corresponding minimizing values for x and y.

Solutions

• **5649** *Proposed by Kenneth Korbin, New York, NY.*

A trapezoid with perimeter $18 + 14\sqrt{2}$ is inscribed in a circle with diameter $7 + 5\sqrt{2}$. Each of the sides of the trapezoid are of the form $a + b\sqrt{2}$, where a and b are positive integers. Find the dimensions of the trapezoid.

Solution 1 by Michael Brozinsky, Central Islip, New York.

Let the bases of the necessarily isosceles trapezoid (being inscribed in a circle) have lengths $a+b\sqrt{2}$ and $c + d\sqrt{2}$ and the length of either leg be $e + f\sqrt{2}$ where a, b, c, d, e, and f are positive integers.

We have thus

$$a + b\sqrt{2} = \frac{18 + 14\sqrt{2} - (c + d\sqrt{2} + 2e + 2f\sqrt{2})}{2} \qquad (1)$$

and

$$c + d\sqrt{2} = \frac{18 + 14\sqrt{2} - (a + b\sqrt{2} + 2e + 2f\sqrt{2})}{2} \tag{2}$$

and so (from 1) c and d must be even and (from 2) a and b must be even as $2e$ and $2f$ are even. Now

$$e + f\sqrt{2} = \frac{18 + 14\sqrt{2}) - (a + b\sqrt{2} + c + d\sqrt{2})}{2} \tag{3}$$

Now since $c + 2e$ is at least 4 we have from (1) that a is at most 6 so that a is 2, 4 or 6 and similarly (also from 1) since $d + 2f$ is at least 4 that b is 2 or 4. Hence there are $3 \cdot 2 = 6$ possibilities for the first base and in fact for any base of the trapezoid. Hence there are $\binom{6}{2} = 15$ combinations for the two base lengths (since the bases must be different) and the resulting leg lengths are found in the 15 rows of the chart. Note $e = \frac{18 - (a + c)}{2}$ and $f = \frac{14 - (b + d)}{2}$. Note, for example, the leg length $5 + 4\sqrt{2}$ can occur in 3 ways i.e., if the bases are $2 + 2\sqrt{2}$ and $6 + 4\sqrt{2}$ or $2 + 4\sqrt{2}$ and $6 + 2\sqrt{2}$ or $4 + 2\sqrt{2}$ and $4 + 4\sqrt{2}$. Note also that all sides determined are less than or equal to the diameter as any chord of a circle would have to be.

a	b	c	d	e	f
2	2	2	4	7	4
2	2	4	2	6	5
2	2	4	4	6	4
2	4	4	2	6	4
2	4	4	4	6	3
2	2	6	2	5	5
2	2	6	4	5	4
2	4	6	2	5	4
2	4	6	4	5	3
4	2	4	4	5	4
4	2	6	2	4	5
4	2	6	4	4	4
4	4	6	2	4	4
4	4	6	4	4	3
6	2	6	4	3	4

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

A cyclic quadrilateral with successive sides a, b, c, d and semiperimeter s has the circumradius (the radius of the circumcircle) given by

$$R = \frac{1}{4}\sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}}.$$

(see for instance https://en.wikipedia.org/wiki/Cyclic_quadrilateral).

By assumption, $d = b$, $s = \dfrac{a+b+c+d}{2} = \dfrac{a+c}{2} + b = 9 + 7\sqrt{2}$, $2R = 7 + 5\sqrt{2}$, and the formula collapses to

$$2R = \frac{(a+c)\,b}{2\,(s-b)}\sqrt{\frac{ac+b^2}{(s-a)\,(s-c)}} = b\sqrt{\frac{ac+b^2}{(s-a)\,(s-c)}} = \left(s - \frac{a+c}{2}\right)\sqrt{\frac{ac + \left(s - \frac{a+c}{2}\right)^2}{(s-a)\,(s-c)}}. \qquad (1)$$

We need to find all solutions of (1) subject to the constraints $s = 9+7\sqrt{2}$, $2R = 7+5\sqrt{2}$, and a and c belong to the ring $\mathbb{Z}\left[\sqrt{2}\right] = \left\{m + n\sqrt{2} \;/\; m, n \in \mathbb{Z}\right\}$. At this point we recollect a few facts about $\mathbb{Z}\left[\sqrt{2}\right]$ (see for instance https://math.stackexchange.com/questions/3062306/primes-in-mathbbz-sqrt2 and https://math.stackexchange.com/questions/150885/proving-that-mathbbz-sqrt2-is-a-euclidean-domain):

$\mathbb{Z}\left[\sqrt{2}\right]$ is a unique factorization domain.

The units in $\mathbb{Z}\left[\sqrt{2}\right]$ are the numbers $\pm\left(1+\sqrt{2}\right)^k$ with k an arbitrary integer.

The prime elements of $\mathbb{Z}\left[\sqrt{2}\right]$ up to associates are

(i) $\sqrt{2}$,

(ii) $\alpha \in \mathbb{Z}\left[\sqrt{2}\right]$ such that N(α)=±p and p is a prime ≡ 1, 7 (mod 8),

(iii) integer primes p ≡ 3, 5 (mod 8).

Note: a unit is an element $w \in \mathbb{Z}\left[\sqrt{2}\right]$ that has a multiplicative inverse $w^{-1} \in \mathbb{Z}\left[\sqrt{2}\right]$. The multiplicative inverse of $\left(\sqrt{2}+1\right)$ is $\left(\sqrt{2}-1\right)$. The associates of a number $w \in \mathbb{Z}\left[\sqrt{2}\right]$ are the numbers $\pm\left(1+\sqrt{2}\right)^k w$. The norm N(.) of an element $m + n\sqrt{2} \in \mathbb{Z}\left[\sqrt{2}\right]$ is defined as

$$N\left(m+n\sqrt{2}\right) = \left(m+n\sqrt{2}\right)\left(m-n\sqrt{2}\right) = m^2 - 2n^2.$$

From $s = \dfrac{a+b+c+d}{2} = \dfrac{a+c}{2} + b = 9 + 7\sqrt{2}$ we deduce that $\dfrac{a+c}{2} \in \mathbb{Z}\left[\sqrt{2}\right]$.

Put x:=s-a, y:=s-c, $u := \dfrac{x+y}{2}$, $v := \dfrac{x-y}{2}$. Then $x, y, u, v \in \mathbb{Z}\left[\sqrt{2}\right]$. Equation (1) is equivalent to each of the following lines:

$$2R = \left(\frac{x+y}{2}\right)\sqrt{\frac{(s-x)\,(s-y) + \left(\frac{x+y}{2}\right)^2}{xy}},$$

$$4R^2 = \left(\frac{x+y}{2}\right)^2 \frac{(s-x)\,(s-y) + \left(\frac{x+y}{2}\right)^2}{xy},$$

$$4R^2 xy = \left(\frac{x+y}{2}\right)^2 \left(s^2 - s\,(x+y) + xy + \left(\frac{x+y}{2}\right)^2\right),$$

$$xy\left(4R^2-\left(\frac{x+y}{2}\right)^2\right)=\left(\frac{x+y}{2}\right)^2\left(s-\frac{x+y}{2}\right)^2,$$

$$xy\left(2R-\frac{x+y}{2}\right)\left(2R+\frac{x+y}{2}\right)=\left(\frac{x+y}{2}\right)^2\left(s-\frac{x+y}{2}\right)^2,$$

$$(u+v)(u-v)(2R-u)(2R+u)=u^2(s-u)^2,$$

$$v^2=\frac{2u^2\left(-40-28\sqrt{2}+u\left(9+7\sqrt{2}-u\right)\right)}{\left(7+5\sqrt{2}\right)^2-u^2}=$$

$$=278+196\sqrt{2}+\left(-18-14\sqrt{2}\right)u+2u^2-\frac{2\left(1+\sqrt{2}\right)^5}{\left(1+\sqrt{2}\right)^3-u}-\frac{16\left(1+\sqrt{2}\right)^7}{\left(1+\sqrt{2}\right)^3+u}. \qquad (2)$$

The greatest common divisor of $\left(1+\sqrt{2}\right)^3-u$ and $\left(1+\sqrt{2}\right)^3+u$ is a divisor of $\left(1+\sqrt{2}\right)^3-u+\left(1+\sqrt{2}\right)^3+u=2\left(1+\sqrt{2}\right)^3$, hence of 2. Neither $\left(1+\sqrt{2}\right)^3-u$ nor $\left(1+\sqrt{2}\right)^3+u$ can be divisible by any prime other than $\sqrt{2}$ (and its associates), for suppose that $p\in\mathbb{Z}\left[\sqrt{2}\right]$ is a prime other than $\sqrt{2}$ (and its associates) that divides $\left(1+\sqrt{2}\right)^3-u$. Then p does not divide $\left(1+\sqrt{2}\right)^3+u$ and therefore

$$-\frac{2\left(1+\sqrt{2}\right)^5}{\left(1+\sqrt{2}\right)^3-u}-\frac{16\left(1+\sqrt{2}\right)^7}{\left(1+\sqrt{2}\right)^3+u}=$$

$$=-\frac{2\left(1+\sqrt{2}\right)^5\left(\left(1+\sqrt{2}\right)^3+u\right)+16\left(1+\sqrt{2}\right)^7\left(\left(1+\sqrt{2}\right)^3-u\right)}{\left(\left(1+\sqrt{2}\right)^3-u\right)\left(\left(1+\sqrt{2}\right)^3+u\right)}=$$

$$=-\frac{2\left(1+\sqrt{2}\right)^5\left\{\left(\left(1+\sqrt{2}\right)^3+u\right)+8\left(1+\sqrt{2}\right)^2\left(\left(1+\sqrt{2}\right)^3-u\right)\right\}}{\left(\left(1+\sqrt{2}\right)^3-u\right)\left(\left(1+\sqrt{2}\right)^3+u\right)} \qquad (3)$$

cannot be in $\mathbb{Z}\left[\sqrt{2}\right]$, since p divides the denominator, but not the numerator. We deduce from (**??**) that $\left(1+\sqrt{2}\right)^3-u$ is a divisor of 2 and $\left(1+\sqrt{2}\right)^3+u$ is a divisor of 16. So there are integers i, j, g, h, 0⩽i⩽2, 0⩽g⩽8 such that

$$u = \left(1+\sqrt{2}\right)^3 \pm \left(\sqrt{2}\right)^i \left(1+\sqrt{2}\right)^j = -\left(1+\sqrt{2}\right)^3 \pm \left(\sqrt{2}\right)^g \left(1+\sqrt{2}\right)^h$$

implying

$$2 = \pm\left(\sqrt{2}\right)^i \left(1+\sqrt{2}\right)^{j-3} \pm \left(\sqrt{2}\right)^g \left(1+\sqrt{2}\right)^{h-3}.$$

The solutions of this equation are found by expanding the brackets by means of the binomial theorem. They are

$2 = \left(\sqrt{2}\right)^0\left(1+\sqrt{2}\right)^0 + \left(\sqrt{2}\right)^0\left(1+\sqrt{2}\right)^0$, leading to $u=0,\ v^2=0$,

$2 = \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^{-1} + \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^0$, leading to $u=\left(1+\sqrt{2}\right)^2,\ v^2=-\sqrt{2}<0$,

$2 = \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^{0} + \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^{-1}$, leading to $u=-\left(1+\sqrt{2}\right)^2<0$,

$2 = \left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^{-4} + \left(\sqrt{2}\right)^7\left(1+\sqrt{2}\right)^{-2}$, leading to $u=3\left(3+\sqrt{2}\right),\ v^2=-9\left(1+5\sqrt{2}\right)<0$,

$2 = \left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^{-2} + \left(\sqrt{2}\right)^4\left(1+\sqrt{2}\right)^{-1}$, leading to $u=5+3\sqrt{2},\ v^2=\left(1+2\sqrt{2}\right)^2$,

$2 = \left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^{-1} + \left(\sqrt{2}\right)^3\left(1+\sqrt{2}\right)^{-1}$, leading to $u=1+\sqrt{2},\ v^2=-1-\sqrt{2}<0$,

$2 = -\left(\sqrt{2}\right)^0\left(1+\sqrt{2}\right)^{-1} + \left(\sqrt{2}\right)^0\left(1+\sqrt{2}\right)^1$, leading to $u=\sqrt{2}\left(1+\sqrt{2}\right)^3,\ v^2=5\left(\sqrt{2}\right)^5(1+\sqrt{2})^3$,

$2 = -\left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^0 + \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^1$, leading to $u=\left(1+\sqrt{2}\right)^4, v^2=714+505\sqrt{2}$,
but $714+505\sqrt{2}$ cannot be the square of an element of $\mathbb{Z}\left[\sqrt{2}\right]$,

$2 = -\left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^{-2} + \left(\sqrt{2}\right)^5\left(1+\sqrt{2}\right)^{-1}$, leading to $u=9+7\sqrt{2}, v^2=\left(9+7\sqrt{2}\right)^2$,

$2 = -\left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^{-1} + \left(\sqrt{2}\right)^3\left(1+\sqrt{2}\right)^0$, leading to $u=\left(1+\sqrt{2}\right)^2\left(3+\sqrt{2}\right), v^2=365+257\sqrt{2}$, but $365+257\sqrt{2}$ cannot be the square of an element of $\mathbb{Z}\left[\sqrt{2}\right]$,

$2 = -\left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^0 + \left(\sqrt{2}\right)^4\left(1+\sqrt{2}\right)^0$, leading to $u=3\left(1+\sqrt{2}\right)^3, v^2=9\left(5+2\sqrt{2}\right)\left(\sqrt{2}+1\right)^4$,
and $5+2\sqrt{2}$ is a prime element of $\mathbb{Z}\left[\sqrt{2}\right]$,

$2 = -\left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^1 + \left(\sqrt{2}\right)^3\left(1+\sqrt{2}\right)^1$, leading to $u=\left(1+\sqrt{2}\right)^5,\ v^2=5415+3829\sqrt{2}$,
but $5415+3829\sqrt{2}$ cannot be the square of an element of $\mathbb{Z}\left[\sqrt{2}\right]$,

$2 = -\left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^5\left(1+\sqrt{2}\right)^1$, leading to $u=89+63\sqrt{2},\ v^2=17^2\left(1+\sqrt{2}\right)^6$,

$2 = \left(\sqrt{2}\right)^0\left(1+\sqrt{2}\right)^1 - \left(\sqrt{2}\right)^0\left(1+\sqrt{2}\right)^{-1}$, leading to $u=\left(1+\sqrt{2}\right)^3-\left(1+\sqrt{2}\right)^4<0$,

$2 = \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^1 - \left(\sqrt{2}\right)^1\left(1+\sqrt{2}\right)^0$, leading to $u=\left(1+\sqrt{2}\right)^3-\sqrt{2}\left(1+\sqrt{2}\right)^4<0$,

$2 = \left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^1 - \left(\sqrt{2}\right)^3\left(1+\sqrt{2}\right)^0$, leading to $u=\left(1+\sqrt{2}\right)^3-2\left(1+\sqrt{2}\right)^4<0$,

$2 = \left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^2 - \left(\sqrt{2}\right)^4\left(1+\sqrt{2}\right)^1$, leading to $u=\left(1+\sqrt{2}\right)^3-2\left(1+\sqrt{2}\right)^5<0$,

$2 = \left(\sqrt{2}\right)^2\left(1+\sqrt{2}\right)^4 - \left(\sqrt{2}\right)^7\left(1+\sqrt{2}\right)^2$, leading to $u = \left(1+\sqrt{2}\right)^3 - 2\left(1+\sqrt{2}\right)^7 < 0$.
Taking into account that u>0, $v \in \mathbb{Z}\left[\sqrt{2}\right]$ this leaves us with the following solutions for (u,v):

$$(u,v) \in \left\{\left(5+3\sqrt{2}, \pm\left(1+2\sqrt{2}\right)\right), \left(9+7\sqrt{2}, \pm\left(9+7\sqrt{2}\right)\right), \left(89+63\sqrt{2}, \pm\left(119+85\sqrt{2}\right)\right)\right\}.$$

We have x=u+v=s-a, y=u-v=s-c. Taking into account that x>0, y>0, we get

$$(x,y) \in \left\{\left(6+5\sqrt{2}, 4+\sqrt{2}\right), \left(4+\sqrt{2}, 6+5\sqrt{2}\right)\right\}$$

as *only* solutions in x, y. These solutions in turn imply

$$(a,b,c,d) = \left(3+2\sqrt{2}, 5+3\sqrt{2}, 5+6\sqrt{2}, 5+3\sqrt{2}\right)$$

or

$$(a,b,c,d) = \left(5+6\sqrt{2}, 5+3\sqrt{2}, 3+2\sqrt{2}, 5+3\sqrt{2}\right)$$

It's easily verified that these two tuples indeed satisfy (1).

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We show that the dimensions are $5+3\sqrt{2}$, $3+2\sqrt{2}$, $5+3\sqrt{2}$, and $5+6\sqrt{2}$.

Since an inscribed trapezoid must be isosceles, we label the side lengths (in order around the trapezoid) by w, x, w, and $p-2w-x$, where $p = 18+14\sqrt{2}$. Using the circumradius formula of Parameshvara with $s = p/2$, we have

$$16\left(\frac{7+5\sqrt{2}}{2}\right)^2 = \frac{[wx+w(p-2w-x)]^2[w^2+x(p-2w-x)]}{(s-w)^2(s-x)(-s+2w+x)},$$

or equivalently $(396+280\sqrt{2})(-x^2+2sx-2wx+2sw-s^2) = 4w^2(w^2+2sx-2wx-x^2)$. Next, we write $w = a+b\sqrt{2}$ and $x = c+d\sqrt{2}$ for positive integers a, b, c, and d with $1 \leqslant a \leqslant 8$, $1 \leqslant b \leqslant 6$, $1 \leqslant c \leqslant 15$, and $1 \leqslant d \leqslant 11$. A computer search yields the unique solution $(a,b) = (5,3)$, which produces $(c,d) = (3,2)$ or $(c,d) = (5,6)$. Hence the dimensions of the trapezoid are $5+3\sqrt{2}$, $3+2\sqrt{2}$, $5+3\sqrt{2}$, and $5+6\sqrt{2}$ as claimed.

Also solved by David Stone, Georgia Southern University, Statesboro, GA.

Proposer Ken Korbin's comments: In this problem we have a circle with diameter $7+5\sqrt{2}$. It is possible to inscribe in this circle infinitely many different trapezoids with sides of the form $a+b\sqrt{2}$, where a and b are integers. Included among these are infinitely many different pairs of trapezoids that both have the same perimeter.

• **5650** *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Spain.*

Let L_n be the nth Lucas number defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geqslant 2$. Prove

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}.$$

Solution 1 by Ajay Srinivasan, University of Southern California, Los Angeles.

Re-indexing the sum on the LHS of the inequality gives:

$$\sum_{j=0}^{n-1} \sqrt{L_jL_{j+3}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$$

Note that $\sqrt{L_0L_3} = \sqrt{8}$. This yields:

$$\sum_{j=1}^{n-1} \sqrt{L_jL_{j+3}} \leqslant \sqrt{L_{n+1}L_{n+4}} - \sqrt{33}$$

Now one can rewrite the RHS terms as the result of a telescoping sum.

$$\sum_{j=1}^{n-1} \sqrt{L_jL_{j+3}} \leqslant \sum_{j=1}^{n-1} \left(\sqrt{L_{j+2}L_{j+5}} - \sqrt{L_{j+1}L_{j+4}} \right)$$

Evidently if the following inequality is true, then the problem is solved.

$$\sqrt{L_jL_{j+3}} \leqslant \sqrt{L_{j+2}L_{j+5}} - \sqrt{L_{j+1}L_{j+4}} \tag{4}$$

Rewriting the involved Lucas Numbers in terms of L_j, L_{j+1}, one gets
$L_{j+2} = L_{j+1} + L_j$, $L_{j+3} = 2L_{j+1} + L_j$, $L_{j+4} = 3L_{j+1} + 2L_j$, $L_{j+5} = 5L_{j+1} + 3L_j$.

Using the substitution $r = \dfrac{L_{j+1}}{L_j}$ in equation (4) and rearranging a little bit, we get:

$$\sqrt{(1+r)(3+5r)} \geqslant \sqrt{1+2r} + \sqrt{r(2+3r)}$$

Squaring both sides of the inequality (given that both sides are always positive):

$$3 + 8r + 5r^2 \geqslant 1 + 4r + 3r^2 + 2\sqrt{r(1+2r)(2+3r)}$$

Rearranging so that the square root is one side:

$$1 + 2r + r^2 \geqslant \sqrt{r(1+2r)(2+3r)}$$

Squaring both sides (given that both sides are always positive):

$$r^4 + 4r^3 + 6r^2 + 4r + 1 \geqslant 2r + 7r^2 + 6r^3$$

$$r^4 - 2r^3 - r^2 + 2r + 1 \geqslant 0$$

Evidently, $r^4 - 2r^3 - r^2 + 2r + 1 = (r^2 - r - 1)^2$. This is a non-negative quantity for all $r \in \mathbb{R}$. The proof is now complete.

Solution 2 by Péter Fülöp, Gyömrő, Hungary.

1 Complete induction

1.1 Induction Step 1

Let's check the validity of the inequality (2) in the case of $n = 2$ and $n = 3$

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + \sqrt{L_0L_3} - \sqrt{L_2L_5}. \tag{5}$$

n=2

LHS of (2):

$$\sqrt{L_0L_3} + \sqrt{L_1L_4} = 2\sqrt{2}\sqrt{7} \simeq 5,474148 \tag{6}$$

RHS of (2):

$$\sqrt{L_3L_6} + 2\sqrt{2} - \sqrt{33} = 6\sqrt{2} + 2\sqrt{2} - \sqrt{33} \simeq 5,5691458 \tag{7}$$

n=3

LHS of (2):

$$\sqrt{L_0L_3} + \sqrt{L_1L_4} + \sqrt{L_2L_5} = 2\sqrt{2} + \sqrt{7} + \sqrt{33} \simeq 11,218741 \tag{8}$$

RHS of (2):

$$\sqrt{L_4L_7} + 2\sqrt{2} - \sqrt{33} = \sqrt{7 * 29} + 2\sqrt{2} - \sqrt{33} \simeq 11,331671 \tag{9}$$

The inequality is true for both cases (n=2,3)

1.2 Induction Step 2

Let's check: if the inequality (2) is true for $k = n$ then will it also be true for $k = n + 1$. Based on (2) for

n+1

$$\sum_{k=1}^{n+1} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+2}L_{n+5}} + \sqrt{L_0L_3} - \sqrt{L_2L_5}. \tag{10}$$

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} + \sqrt{L_nL_{n+3}} \leqslant \sqrt{L_{n+2}L_{n+5}} + \sqrt{L_0L_3} - \sqrt{L_2L_5}. \tag{11}$$

LHS of (8) can be majorized by RHS of (2) we get:

$$\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_nL_{n+3}} \leqslant \sqrt{L_{n+2}L_{n+5}} \tag{12}$$

If (9) is true then the statement (1) is proved.

- Express (9)with L_n and L_{n+1}

Applying the following relation between Lucas and Fibonacci numbers in (9):

$$L_{n+m} = L_{n+1}F_m + L_nF_{m-1} \tag{13}$$

If $x = \dfrac{L_n}{L_{n+1}}$ then LHS is equal to:

$$\sqrt{L_{n+1}}\left(\sqrt{3+2x} + \sqrt{x(x+2)}\right) \tag{14}$$

and RHS is equal to:

$$\sqrt{L_{n+1}}\sqrt{3x^2+8x+5} \tag{15}$$

Using the complete induction again for (9)

Step 1

n=2

$$LHS = \sqrt{4}\left(\sqrt{3+2(\frac{3}{4})} + \sqrt{(2+\frac{3}{4})\frac{3}{4}}\right) \simeq 7.114922$$

$$RHS = \sqrt{4}\Big(\sqrt{3(\frac{3}{4})^2 + 8(\frac{3}{4}) + 5}\Big) \simeq 7.123903$$

n=3

$$LHS = \sqrt{7}\Big(\sqrt{3 + 2(\frac{4}{7})} + \sqrt{(2 + \frac{4}{7})\frac{4}{7}}\Big) \simeq 8.592300$$

$$RHS = \sqrt{7}\Big(\sqrt{3(\frac{4}{7})^2 + 8(\frac{4}{7}) + 5}\Big) \simeq 8.594020$$

The inequality (9) is true for both cases (n=2,3).

Step 2

Provided the inequality is true for n:

$$\sqrt{L_{n+1}}\Big(\sqrt{3 + 2x} + \sqrt{x(x+2)}\Big) \leqslant \sqrt{L_{n+1}}\sqrt{3x^2 + 8x + 5} \tag{16}$$

Both sides are positive because $0 < x < 1$ we can square them:

$$\sqrt{(2x+3)(x^2+2x)} \leqslant x^2 + 2x + 1$$

Square both sides again and perform the possible cancellations we get:

$$0 \leqslant x^4 + 2x^3 - x^2 - 2x + 1 \tag{17}$$

Inequality (14) is true in the range $0 < x < 1$, because the values of the function $y = x^4 + 2x^3 - x^2 - 2x + 1$ are positives. It is zero at $x = \frac{\sqrt{5}-1}{2}$, where the function has the local minimum value.

In case of n+1 we get:

$$LHS = \sqrt{L_{n+2}L_{n+5}} + \sqrt{L_{n+1}L_{n+4}}, \qquad RHS = \sqrt{L_{n+3}L_{n+6}} \tag{18}$$

- Express (15)with L_{n+1} and L_{n+2}

Applying (10) and let $x = \frac{L_{n+1}}{L_{n+2}}$ we get the $LHS \leqslant RHS$ for n+1, because of (13):

$$\sqrt{L_{n+2}}\Big(\sqrt{3 + 2x} + \sqrt{x(x+2)}\Big) \leqslant \sqrt{L_{n+2}}\sqrt{3x^2 + 8x + 5} \tag{19}$$

We assumend that (13) was true, it follows that (16) is true.

So 1.2 Induction Step 2 (9) is also true. The statement (1) is proved.

Solution 3 by Moti Levy, Rehovot, Israel.

We rearrange the original inequality,

$$\sqrt{L_{n+1}L_{n+4}} - \sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \geqslant \sqrt{33} - \sqrt{8}, \tag{20}$$

and define the sequence $(a_n)_{n\geqslant 1}$ as follows:

$$a_n := \sqrt{L_{n+1}L_{n+4}} - \sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}}, \quad n = 1, 2, \ldots.$$

Clearly

$$a_1 = \sqrt{L_2L_5} - \sqrt{L_0L_3} = \sqrt{33} - \sqrt{8},$$

hence the original inequality (20) can be rephrased as

$$a_n \geqslant a_1.$$

Now we show that the sequence $(a_n)_{n\geqslant 1}$ is increasing for $n \geqslant 1$.

Let

$$\begin{aligned} d_n := a_{n+1} - a_n &= \sqrt{L_{n+2}L_{n+5}} - \sum_{k=1}^{n+1} \sqrt{L_{k-1}L_{k+2}} - \sqrt{L_{n+1}L_{n+4}} + \sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \\ &= \sqrt{L_{n+2}L_{n+5}} - \sqrt{L_{n+1}L_{n+4}} - \sqrt{L_nL_{n+3}}. \end{aligned} \tag{21}$$

The function $f(x, y) := \sqrt{xy}$ is bivariate concave function over the convex region

$$\left\{(x, y) \middle| \left\{(x, y) : \ x \geqslant 0, \ y \geqslant 0\right\}\right. .$$

By Jensen's inequality for multivariate functions,

$$\sqrt{\frac{x_1 + x_2}{2}\frac{y_1 + y_2}{2}} \geqslant \frac{\sqrt{x_1y_1} + \sqrt{x_2y_2}}{2},$$

or

$$\sqrt{(x_1 + x_2)(y_1 + y_2)} \geqslant \sqrt{x_1y_1} + \sqrt{x_2y_2}. \tag{22}$$

Now we substitute in (22)

$$x_1 = L_{n+1}, \ y_1 = L_{n+4}, \ x_2 = L_n, \ y_2 = L_{n+3}.$$

The right-hand side becomes

$$\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_nL_{n+3}}$$

and the left-hand side becomes

$$\sqrt{\left(L_{n+1}+L_n\right)\left(L_{n+4}+L_{n+3}\right)} = \sqrt{L_{n+2}L_{n+5}}.$$

It follows that $d_n \geqslant 0$ for $n \geqslant 1$, which implies that $a_n \geqslant a_1$ for $n \geqslant 1$, and the proof is complete.

Solution 4 by Michel Bataille, Rouen, France.

We use induction on n. Since $L_2 = 3, L_3 = 4, L_5 = 11$, we have $\sqrt{L_0 L_3} = 2\sqrt{2}$ and $\sqrt{L_2 L_5} = \sqrt{33}$, hence equality holds for $n = 1$. Now, assume that the proposed inequality holds for some integer $n \geqslant 1$. Then we have

$$\sum_{k=1}^{n+1} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33} + \sqrt{L_n L_{n+3}}$$

and it is sufficient to prove that

$$\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_n L_{n+3}} \leqslant \sqrt{L_{n+2}L_{n+5}}$$

or, squaring and arranging, that

$$2\sqrt{L_{n+1}L_{n+4}L_n L_{n+3}} \leqslant L_{n+2}L_{n+5} - L_{n+1}L_{n+4} - L_n L_{n+3}. \tag{1}$$

Since

$$L_{n+2}L_{n+5} = (L_{n+1}+L_n)(L_{n+4}+L_{n+3}) = L_{n+1}L_{n+4} + L_n L_{n+3} + L_{n+1}L_{n+3} + L_n L_{n+4},$$

(1) can be written as

$$2\sqrt{(L_n L_{n+4})(L_{n+1}L_{n+3})} \leqslant L_n L_{n+4} + L_{n+1}L_{n+3},$$

that is,

$$\left(\sqrt{L_n L_{n+4}} - \sqrt{L_{n+1}L_{n+3}}\right)^2 \geqslant 0.$$

The latter clearly holds and the induction step follows.

Solution 5 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

We first collect the following well-known identities involving the Lucas numbers. For the first identity, see sequence A0000032 of the Online Encyclopedia of Integer Sequences, with a comment by John Blythe Dobson, 2007. The next two identities are immediate applications of the first, while the last two follow from the recurrence relation for the Lucas sequence.

Lemma 1 *For nonnegative integers n and m with $n \leqslant m$,*

1. $L_m \cdot L_n = L_{m+n} + (-1)^n L_{m-n}$;

2. $L_{n+3} \cdot L_n = L_{2n+3} + (-1)^n \cdot 4$*;*

3. $L_n^2 = L_{2n} + (-1)^n \cdot 2$*;*

4. $2L_{n+4} = 3L_{n+3} + L_n$*;*

5. $L_{n+6} = 4L_{n+3} + L_n$*.*

We prove the inequality using induction on n. When $n = 1$, then

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} = \sqrt{L_0 \cdot L_3} = 2\sqrt{2} = \sqrt{L_2 \cdot L_5} + 2\sqrt{2} - \sqrt{33}.$$

We assume

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$$

for some positive integer n. Then

$$\sum_{k=1}^{n+1} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_nL_{n+3}} + \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33},$$

so it suffices to show that

$$\sqrt{L_nL_{n+3}} + \sqrt{L_{n+1}L_{n+4}} \leqslant \sqrt{L_{n+2}L_{n+5}},$$

or, equivalently, using part (b) of the Lemma,

$$\sqrt{L_{2n+3} + (-1)^n \cdot 4} + \sqrt{L_{2n+5} + (-1)^{n+1} \cdot 4} \leqslant \sqrt{L_{2n+7} + (-1)^{n+2} \cdot 4}.$$

Since the terms are positive, this is equivalent to proving

$$\left(\sqrt{L_{2n+3} + (-1)^n \cdot 4} + \sqrt{L_{2n+5} + (-1)^{n+1} \cdot 4}\right)^2 \leqslant L_{2n+7} + (-1)^n \cdot 4.$$

The left side is

$$L_{2n+3} + (-1)^n \cdot 4 + L_{2n+5} + (-1)^{n+1} \cdot 4 + 2\sqrt{L_{2n+3}L_{2n+5} + (-1)^{n+1}4L_{2n+3} + (-1)^n 4L_{2n+5} - 16},$$

which using part (1) of the lemma, is equal to

$$L_{2n+3} + L_{2n+5} + 2\sqrt{L_{4n+8} + (-1)^n \cdot 4L_{2n+4} - 19}.$$

Meanwhile, using parts (4) and (5) of the lemma, the right side is

$$\begin{aligned} L_{2n+7} + (-1)^n \cdot 4 &= L_{2n+5} + L_{2n+6} + (-1)^n \cdot 4 \\ &= L_{2n+5} + 4L_{2n+3} + L_{2n} + (-1)^n \cdot 4 \\ &= L_{2n+5} + L_{2n+3} + 2L_{2n+4} + (-1)^n \cdot 4 \end{aligned}$$

Thus, it is sufficient to prove that

$$\begin{aligned} L_{2n+3} + L_{2n+5} + 2\sqrt{L_{4n+8} + (-1)^n \cdot 4L_{2n+4} - 19} &\leqslant L_{2n+5} + L_{2n+3} + 2L_{2n+4} + (-1)^n \cdot 4 \\ 2\sqrt{L_{4n+8} + (-1)^n \cdot 4L_{2n+4} - 19} &\leqslant 2L_{2n+4} + (-1)^n \cdot 4 \\ \sqrt{L_{4n+8} + (-1)^n \cdot 4L_{2n+4} - 19} &\leqslant L_{2n+4} + (-1)^n \cdot 2 \end{aligned}$$

The square of the right side is

$$\begin{aligned} L_{2n+4}^2 + (-1)^n \cdot 4L_{2n+4} + 4 &= L_{4n+8} + (-1)^n \cdot 4L_{2n+4} + 6 \\ &\geqslant L_{4n+8} + (-1)^n \cdot 4L_{2n+4} - 19, \end{aligned}$$

which is the square of the left side.

Solution 6 by David E. Manes, Oneonta, NY.

For each positive integer n, let $P(n)$ be the following statement:

$$P(n) : \sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}.$$

If $n = 1$, then $\sqrt{L_{k-1}L_{k+2}} = \sqrt{L_0L_3} = \sqrt{2 \cdot 4} = 2\sqrt{2}$ and $\sqrt{L_{n+1}L_{n+4}} = \sqrt{L_2L_5} = \sqrt{3 \cdot 11} = \sqrt{33}$. Therefore,

$$\sum_{k=1}^{1} \sqrt{L_{k-1}L_{k+2}} = 2\sqrt{2} \leqslant \sqrt{L_2L_5} + 2\sqrt{2} - \sqrt{33} = 2\sqrt{2}.$$

Hence, the statement $P(1)$ is true. Assume inductively that the positive integer $n \geqslant 1$ and the statement $P(n)$ is true. Then the statement for the integer $n + 1$ reads

$$P(n+1) : \sum_{k=1}^{n+1} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+2}L_{n+5}} + 2\sqrt{2} - \sqrt{33}.$$

To show that this inequality is true, note that

$$\begin{aligned} \sum_{k=1}^{n+1} \sqrt{L_{k-1}L_{k+2}} &= \sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} + \sqrt{L_nL_{n+3}} \\ &\leqslant \left(\sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}\right) + \sqrt{L_nL_{n+3}} \end{aligned}$$

by the induction hypothesis. Therefore, the statement $P(n + 1)$ is true if and only if

$$\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_nL_{n+3}} + 2\sqrt{2} - \sqrt{33} \leqslant \sqrt{L_{n+2}L_{n+5}} + 2\sqrt{2} - \sqrt{33}$$

or

$$\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_nL_{n+3}} \leqslant \sqrt{L_{n+2}L_{n+5}}$$

for each integer $n \geqslant 1$. Since all terms are nonnegative, it follows that this inequality is true if and only if

$$\left(\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_nL_{n+3}}\right)^2 \leqslant \left(\sqrt{L_{n+2}L_{n+5}}\right)^2$$

or $L_{n+1}L_{n+4} + L_nL_{n+3} + 2\sqrt{L_{n+1}L_{n+4}L_nL_{n+3}} \leqslant L_{n+2}L_{n+5}$. By definition of the Lucas numbers for $n \geqslant 1$, we get $L_{n+2} = L_n + L_{n+1}$ and $L_{n+5} = L_{n+3} + L_{n+4}$. Then substituting these values, expanding the square, and simplifying the inequality, one obtains that $P(n+1)$ is valid if and only if

$$2\sqrt{L_{n+1}L_{n+4}L_nL_{n+3}} \leqslant L_nL_{n+4} + L_{n+1}L_{n+3}$$

if and only if

$$0 \leqslant \left(\sqrt{L_nL_{n+4}} - \sqrt{L_{n+1}L_{n+3}}\right)^2.$$

Hence, by induction

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$$

for each positive integer n.

Solution 7 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

Routine arithmetic shows that equality holds for $n = 1$ since $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, and $L_5 = 11$. It suffices to prove that

$$\sqrt{L_{k-1}L_{k+2}} < \sqrt{L_{k+1}L_{k+4}} - \sqrt{L_kL_{k+3}} \tag{23}$$

for $k \geqslant 2$ as then

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} < \sqrt{L_2L_5} + 2\sqrt{2} - \sqrt{33} + \sum_{k=2}^{n}\left(\sqrt{L_{k+1}L_{k+4}} - \sqrt{L_kL_{k+3}}\right) = \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$$

for $n \geqslant 2$. Squaring both sides of inequality (23) yields

$$L_{k-1}L_{k+2} < L_{k+1}L_{k+4} + L_kL_{k+3} - 2\sqrt{L_kL_{k+1}L_{k+3}L_{k+4}},$$

which in turn is equivalent to

$$\begin{aligned}2\sqrt{L_kL_{k+1}L_{k+3}L_{k+4}} &< L_{k+1}L_{k+4} + L_kL_{k+3} - L_{k-1}L_{k+2} \\ &= (L_{k+2} - L_k)(L_{k+2} + L_{k+3}) + L_kL_{k+3} - L_{k-1}L_{k+2} \\ &= L_{k+2}(L_{k+2} + L_{k+3} - L_k - L_{k-1}) \\ &= L_{k+2}(L_{k+2} + L_{k+3} - L_{k+1}) = 2L_{k+2}^2.\end{aligned}$$

Canceling out the 2 and squaring both sides of the above inequality one more time yields

$$L_kL_{k+1}L_{k+3}L_{k+4} < L_{k+2}^4. \tag{24}$$

To see that (24) is true, apply the Cassini-type identity (see equation 3.3 of [1])

$$L_{m+i}L_{m+j-1} - L_{m-1}L_{m+i+j} = F_{i+1}(-1)^{m-1}(L_j - 2L_{j+1}),$$

where F_n is the n^{th} Fibonacci number, to get that

$$\begin{aligned}
L_kL_{k+1}L_{k+3}L_{k+4} &= (L_kL_{k+4})(L_{k+1}L_{k+3}) \\
&= \left(L_{k+2}^2 - F_2(-1)^k(L_2 - 2L_3)\right)\left(L_{k+2}^2 - F_1(-1)^{k+1}(L_1 - 2L_2)\right) \\
&= \left(L_{k+2}^2 + (-1)^k 5\right)\left(L_{k+2}^2 - (-1)^k 5\right) \\
&= L_{k+2}^4 - 25 < L_{k+2}^4.
\end{aligned}$$

References

[1] Voll, Nils Gaute, The Cassini identity and its relatives, *Fibonacci Quart.*, **48**, (2010), 197-201.

Solution 8 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

We proceed by induction on n. With $n = 1$,

$$\sum_{k=1}^{1} \sqrt{L_{k-1}L_{k+2}} = \sqrt{L_0L_3} = 2\sqrt{2}$$

and

$$\sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33} = \sqrt{L_2L_5} + 2\sqrt{2} - \sqrt{33} = 2\sqrt{2},$$

so

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$$

holds for $n = 1$. Now, suppose

$$\sum_{k=1}^{N} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{N+1}L_{N+4}} + 2\sqrt{2} - \sqrt{33}$$

holds for some positive integer N. Then

$$\sum_{k=1}^{N+1} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_NL_{N+3}} + \sqrt{L_{N+1}L_{N+4}} + 2\sqrt{2} - \sqrt{33}.$$

By Mahler's inequality,

$$\sqrt{L_NL_{N+3}} + \sqrt{L_{N+1}L_{N+4}} \leqslant \sqrt{(L_N + L_{N+1})(L_{N+3} + L_{N+4})} = \sqrt{L_{N+2}L_{N+5}},$$

so

$$\sum_{k=1}^{N+1} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{N+2}L_{N+5}} + 2\sqrt{2} - \sqrt{33}.$$

Thus,

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$$

holds for all positive integers n.

Solution 9 by Brian D. Beasley, Presbyterian College, Clinton, SC.

For each positive integer n, let $a_n = \sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}}$ and $b_n = \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}$. We apply induction on n and use the following identities:

$$L_{n+2}L_{n+5} - L_{n+1}L_{n+4} - L_nL_{n+3} = 2L_{n+2}^2 \tag{25}$$

$$L_nL_{n+1}L_{n+3}L_{n+4} + 25 = L_{n+2}^4 \tag{26}$$

First, we check that $a_1 = b_1 = 2\sqrt{2}$. Next, assume $a_n \leqslant b_n$ for some positive integer n. Then $a_{n+1} \leqslant b_{n+1}$ will follow provided that

$$\sqrt{L_{n+1}L_{n+4}} + \sqrt{L_nL_{n+3}} \leqslant \sqrt{L_{n+2}L_{n+5}}.$$

This inequality in turn is equivalent to $L_{n+1}L_{n+4} + 2\sqrt{L_nL_{n+1}L_{n+3}L_{n+4}} + L_nL_{n+3} \leqslant L_{n+2}L_{n+5}$. By (25), the latter inequality holds if and only if $\sqrt{L_nL_{n+1}L_{n+3}L_{n+4}} \leqslant L_{n+2}^2$, which follows immediately from (26).

Addendum. (i) We note the similarity of this problem to Problem 12213 in *The American Mathematical Monthly* (127:9, Nov. 2020, p. 853), which gives the corresponding result for the Fibonacci sequence:

> **12213.** *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$. Prove
>
> $$\sum_{k=1}^{n} \sqrt{F_{k-1}F_{k+2}} \leqslant \sqrt{F_{n+1}F_{n+4}} - \sqrt{5}.$$

(ii) To establish (25), we have

$$\begin{aligned} L_{n+2}L_{n+5} &= (L_{n+1} + L_n)(L_{n+4} + L_{n+3}) \\ &= (L_{n+1}L_{n+4} + L_nL_{n+3}) + (L_nL_{n+4} + L_{n+1}L_{n+3}), \end{aligned}$$

with

$$\begin{aligned} L_nL_{n+4} + L_{n+1}L_{n+3} &= (L_{n+2} - L_{n+1})(L_{n+2} + L_{n+3}) + L_{n+1}L_{n+3} \\ &= L_{n+2}^2 + L_{n+2}(L_{n+3} - L_{n+1}) \\ &= 2L_{n+2}^2. \end{aligned}$$

To establish (26), it is straightforward to apply Binet's formula $L_n = \alpha^n + \beta^n$ with $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ to show that $L_n L_{n+4} - 5(-1)^n = L_{n+2}^2$ and $L_{n+1}L_{n+3} + 5(-1)^n = L_{n+2}^2$.

Solution 10 Albert Stadler, Herrliberg, Switzerland.

We will prove that

$$\sqrt{L_{k-1}L_{k+2}} \leqslant \sqrt{L_{k+1}L_{k+4}} - \sqrt{L_k L_{k+3}},\ k \geqslant 1. \qquad (1)$$

The claim then follows from (1), since

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} = \sqrt{L_0 L_3} + \sum_{k=2}^{n} \left(\sqrt{L_{k+1}L_{k+4}} - \sqrt{L_k L_{k+3}} \right) = 2\sqrt{2} + \sqrt{L_{n+1}L_{n+4}} - \sqrt{L_2 L_5} =$$

$$= \sqrt{L_{n+1}L_{n+4}} + 2\sqrt{2} - \sqrt{33}.$$

(**??**) is equivalent to each of the following lines:

$$\sqrt{L_{k-1}L_{k+2}} + \sqrt{L_k L_{k+3}} \leqslant \sqrt{L_{k+1}L_{k+4}},$$

$$L_{k-1}L_{k+2} + 2\sqrt{L_{k-1}L_k L_{k+2}L_{k+3}} + L_k L_{k+3} \leqslant L_{k+1}L_{k+4},$$

$$L_{k-1}L_{k+2} + 2\sqrt{L_{k-1}L_k L_{k+2}L_{k+3}} + L_k L_{k+3} \leqslant L_{k+1}\left(L_{k+2} + L_{k+3}\right),$$

$$2\sqrt{L_{k-1}L_k L_{k+2}L_{k+3}} \leqslant \left(L_{k+1}L_{k+2} - L_{k-1}L_{k+2}\right) + \left(L_{k+1}L_{k+3} - L_k L_{k+3}\right),$$

$$2\sqrt{L_{k-1}L_k L_{k+2}L_{k+3}} \leqslant L_k L_{k+2} + L_{k-1}L_{k+3},$$

and the last inequality is true by the AM-GM inequality.
Note: a slightly better estimate is obtained by applying the Cauchy-Schwarz inequality. We have

$$\sum_{k=1}^{n} \sqrt{L_{k-1}L_{k+2}} = \sqrt{L_0 L_3} + \sum_{k=2}^{n} \sqrt{L_{k-1}L_{k+2}} \leqslant 2\sqrt{2} + \sqrt{\left(\sum_{k=2}^{n} L_{k-1}\right)\left(\sum_{k=2}^{n} L_{k+2}\right)} =$$

$$= 2\sqrt{2} + \sqrt{\left(\sum_{k=2}^{n} \left(L_{k+1} - L_k\right)\right)\left(\sum_{k=2}^{n} \left(L_{k+4} - L_{k+3}\right)\right)} = 2\sqrt{2} + \sqrt{\left(L_{n+1} - L_2\right)\left(L_{n+4} - L_5\right)} =$$

$$= 2\sqrt{2} + \sqrt{\left(L_{n+1} - 3\right)\left(L_{n+4} - 11\right)}.$$

Then

$$\sqrt{\left(L_{n+1} - 3\right)\left(L_{n+4} - 11\right)} \leqslant \sqrt{L_{n+1}L_{n+4}} - \sqrt{33},$$

since the last inequality is equivalent to

$$\left(L_{n+1} - 3\right)\left(L_{n+4} - 11\right) \leqslant L_{n+1}L_{n+4} - 2\sqrt{33 L_{n+1}L_{n+4}} + 33$$

which in turn is equivalent to

$$2\sqrt{33 L_{n+1}L_{n+4}} \leqslant 11 L_{n+1} + 3 L_{n+4}$$

which is true by the AM-GM inequality.

Also solved by Daniel Văcaru, Pitești, Romania; and the proposer.

• **5651** *Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Let $f:[0,+\infty)\to(0,+\infty)$ be a concave function. Show that

$$20\int_0^{1/4} f(x)\,dx + 15\int_0^{1/3} f(x)\,dx + 24\int_0^{5/12} f(x)\,dx \geqslant 75\int_0^{1/6} f(x)\,dx + 10\int_0^{1/2} f(x)\,dx.$$

Editor's note: A real-valued function f defined over an interval I is said to be *concave* (*down*) if and only if $\forall\alpha,\beta\in[0,1]$, with $\alpha+\beta=1$, $\forall a,b\in I:\ \alpha f(a)+\beta f(b)\leqslant f(\alpha a+\beta b)$.

Solution 1 by Michel Bataille, Rouen, France.

For any real $k\neq 0$, the change of variables $x=ku$ gives $\int_0^k f(x)\,dx = k\int_0^1 f(ku)\,du$. It readily follows that the required inequality is equivalent to

$$\int_0^1 f\left(\frac{x}{4}\right)dx + \int_0^1 f\left(\frac{x}{3}\right)dx + 2\int_0^1 f\left(\frac{5x}{12}\right)dx \geqslant \frac{5J}{2}+K \tag{1}$$

where we set $J=\int_0^1 f\left(\frac{x}{6}\right)dx$ and $K=\int_0^1 f\left(\frac{x}{2}\right)dx$.

Using the concavity of f and the identities

$$\frac{x}{4}=\frac{3}{4}\cdot\frac{x}{6}+\frac{1}{4}\cdot\frac{x}{2},\quad \frac{x}{3}=\frac{1}{2}\cdot\frac{x}{6}+\frac{1}{2}\cdot\frac{x}{2},\quad \frac{5x}{12}=\frac{1}{4}\cdot\frac{x}{6}+\frac{3}{4}\cdot\frac{x}{2},$$

we obtain

$$f\left(\frac{x}{4}\right)\geqslant\frac{3}{4}f\left(\frac{x}{6}\right)+\frac{1}{4}f\left(\frac{x}{2}\right),\quad f\left(\frac{x}{3}\right)\geqslant\frac{1}{2}f\left(\frac{x}{6}\right)+\frac{1}{2}f\left(\frac{x}{2}\right),\quad f\left(\frac{5x}{12}\right)\geqslant\frac{1}{4}f\left(\frac{x}{6}\right)+\frac{3}{4}f\left(\frac{x}{2}\right)$$

and we deduce that

$$\int_0^1 f\left(\frac{x}{4}\right)dx\geqslant\frac{3}{4}\cdot J+\frac{1}{4}\cdot K,\quad \int_0^1 f\left(\frac{x}{3}\right)dx\geqslant\frac{1}{2}\cdot J+\frac{1}{2}\cdot K,\quad \int_0^1 f\left(\frac{5x}{12}\right)dx\geqslant\frac{1}{4}\cdot J+\frac{3}{4}\cdot K.$$

As a result, (1) will be satisfied if

$$\frac{3}{4}\cdot J+\frac{1}{4}\cdot K+\frac{1}{2}\cdot J+\frac{1}{2}\cdot K+2\left(\frac{1}{4}\cdot J+\frac{3}{4}\cdot K\right)\geqslant\frac{5J}{2}+K$$

or, arranging,

$$K\geqslant\frac{3J}{5}. \tag{2}$$

Again, since $\dfrac{x}{2}=\dfrac{3}{5}\cdot\dfrac{x}{6}+\dfrac{2}{5}\cdot x$, we have $f\left(\dfrac{x}{2}\right)\geqslant\dfrac{3}{5}f\left(\dfrac{x}{6}\right)+\dfrac{2}{5}f(x)>\dfrac{3}{5}f\left(\dfrac{x}{6}\right)$ (the last inequality because f is positive). The inequality (2) immediately follows by integration between 0 and 1.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We will prove more precisely that

$$20\int_0^{\frac{1}{4}} f(x)\,dx+15\int_0^{\frac{1}{3}} f(x)\,dx+24\int_0^{\frac{5}{12}} f(x)\,dx \geqslant 75\int_0^{\frac{1}{6}} f(x)\,dx+10\int_0^{\frac{1}{2}} f(x)\,dx+\frac{5}{2}\int_0^1 f(x)\,dx. \quad (1)$$

Inequality (1) is sharper than the claimed inequality, because by assumption f(x)>0 for x⩾0. We perform a change of variables and obtain the equivalent inequality

$$5\int_0^1 f\left(\frac{x}{4}\right)dx+5\int_0^1 f\left(\frac{x}{3}\right)dx+10\int_0^1 f\left(\frac{5x}{12}\right)dx \geqslant \frac{25}{2}\int_0^1 f\left(\frac{x}{6}\right)dx+5\int_0^1 f\left(\frac{x}{2}\right)dx+\frac{5}{2}\int_0^1 f(x)\,dx. \quad (2)$$

By assumption,

$$f\left(\frac{x}{4}\right)\geqslant\frac{3}{4}f\left(\frac{x}{6}\right)+\frac{1}{4}f\left(\frac{x}{2}\right),$$

$$f\left(\frac{x}{3}\right)\geqslant\frac{1}{5}f(x)+\frac{4}{5}f\left(\frac{x}{6}\right),$$

$$f\left(\frac{5x}{12}\right)\geqslant\frac{5}{8}f\left(\frac{11x}{30}\right)+\frac{3}{8}f\left(\frac{x}{2}\right)\geqslant\frac{5}{8}\left(\frac{6}{25}f(x)+\frac{19}{25}f\left(\frac{x}{6}\right)\right)+\frac{3}{8}f\left(\frac{x}{2}\right)=$$

$$=\frac{3}{20}f(x)+\frac{19}{40}f\left(\frac{x}{6}\right)+\frac{3}{8}f\left(\frac{x}{2}\right).$$

Hence

$$5\int_0^1 f\left(\frac{x}{4}\right)dx+5\int_0^1 f\left(\frac{x}{3}\right)dx+10\int_0^1 f\left(\frac{5x}{12}\right)dx\geqslant$$

$$\geqslant 5\int_0^1\left(\frac{3}{4}f\left(\frac{x}{6}\right)+\frac{1}{4}f\left(\frac{x}{2}\right)\right)dx+5\int_0^1\left(\frac{1}{5}f(x)+\frac{4}{5}f\left(\frac{x}{6}\right)\right)dx$$

$$+10\int_0^1\left(\frac{3}{20}f(x)+\frac{19}{40}f\left(\frac{x}{6}\right)+\frac{3}{8}f\left(\frac{x}{2}\right)\right)dx=$$

$$=\frac{25}{2}\int_0^1 f\left(\frac{x}{6}\right)dx+5\int_0^1 f\left(\frac{x}{2}\right)dx+\frac{5}{2}\int_0^1 f(x)\,dx,$$

as claimed in (2).

Solution 3 by David Huckaby, Angelo State University, San Angelo, TX.

With the substitutions $x=\dfrac{1}{4}u$, $x=\dfrac{1}{3}u$, $x=\dfrac{5}{12}u$, $x=\dfrac{1}{6}u$, and $x=\dfrac{1}{2}u$ in the five integrals from left to right, the desired inequality becomes

$$5\int_0^1 f\left(\frac{1}{4}u\right)\,du + 5\int_0^1 f\left(\frac{1}{3}u\right)\,du + 10\int_0^1 f\left(\frac{5}{12}u\right)\,du \geqslant \frac{25}{2}\int_0^1 f\left(\frac{1}{6}u\right)\,du + 5\int_0^1 f\left(\frac{1}{2}u\right)\,du.$$

Since $f(0) > 0$, the following two properties hold:

1. For $t \in [0,1]$, $f(tx) \geqslant tf(x)$.
2. For $a, b \in [0,\infty)$, $f(a) + f(b) \geqslant f(a+b)$.

(See `https://en.wikipedia.org/wiki/Concave_function`.)

Therefore

$$\begin{aligned}
&5\int_0^1 f\left(\frac{1}{4}u\right)\,du + 5\int_0^1 f\left(\frac{1}{3}u\right)\,du + 10\int_0^1 f\left(\frac{5}{12}u\right)\,du \\
&\quad = 5\int_0^1 \left[f\left(\frac{1}{4}u\right) + f\left(\frac{1}{3}u\right) + f\left(\frac{5}{12}u\right)\right]\,du + 5\int_0^1 f\left(\frac{5}{12}u\right)\,du \\
&\quad \geqslant 5\int_0^1 \left[f\left(\frac{1}{4}u + \frac{1}{3}u + \frac{5}{12}u\right)\right]\,du + 5\int_0^1 f\left(\frac{5}{12}u\right)\,du \\
&\quad = 5\int_0^1 f(u)\,du + 5\int_0^1 f\left(\frac{5}{12}u\right)\,du \\
&\quad = 5\int_0^1 \left[f(u) + f\left(\frac{5}{12}u\right)\right]\,du \\
&\quad \geqslant 5\int_0^1 f\left(\frac{17}{12}u\right)\,du \\
&\quad = 5\int_0^1 f\left(\frac{11}{17}\cdot\frac{17}{12}u + \frac{6}{17}\cdot\frac{17}{12}u\right)\,du \\
&\quad \geqslant 5\int_0^1 \left[\frac{11}{17}f\left(\frac{17}{12}u\right) + \frac{6}{17}f\left(\frac{17}{12}u\right)\right]\,du \\
&\quad \geqslant 5\int_0^1 \left[\frac{11}{17}\cdot\frac{17}{2}f\left(\frac{1}{6}u\right) + \frac{6}{17}\cdot\frac{17}{6}f\left(\frac{1}{2}u\right)\right]\,du \\
&\quad = \frac{55}{2}\int_0^1 f\left(\frac{1}{6}u\right)\,du + 5\int_0^1 f\left(\frac{1}{2}u\right)\,du \\
&\quad > \frac{25}{2}\int_0^1 f\left(\frac{1}{6}u\right)\,du + 5\int_0^1 f\left(\frac{1}{2}u\right)\,du.
\end{aligned}$$

(The first and second inequalities follow from the second property stated above, the third inequality follows from the editor's note given after the problem statement, and the fourth inequality follows from the first property stated above.)

Also solved by Péter Fülöp, Gyömrő, Hungary; and the proposer.

• **5652** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania.*

Prove:

$$1 \leqslant a \leqslant b \leqslant c \implies a^a \cdot e^{c-a} \cdot \left(\sqrt{ab}\right)^{b-a} \cdot \left(\sqrt{bc}\right)^{c-b} \leqslant c^c.$$

Solution by Michel Bataille, Rouen, France.

Suppose that $1 \leqslant a \leqslant b \leqslant c$. Taking logarithms, the equality to be proved is equivalent to

$$c - a \leqslant c \ln c - a \ln a - \frac{b-a}{2}(\ln a + \ln b) - \frac{c-b}{2}(\ln b + \ln c)$$

or

$$c - a \leqslant \frac{b+c}{2} \ln \frac{c}{b} - \frac{a+b}{2} \ln \frac{a}{b}.$$

Writing $c - a$ as $(b + c) - (a + b)$, the latter is easily transformed into

$$f\left(\frac{c}{b}\right) \geqslant f\left(\frac{a}{b}\right) \qquad (1)$$

where f is the function defined on $(0, \infty)$ by $f(x) = (1 + x)(\ln(x) - 2)$.

The derivative of f satisfies $f'(x) = \ln(x) - 2 + \frac{1}{x} \cdot (1 + x) = \frac{1}{x} - 1 - \ln\left(\frac{1}{x}\right)$, hence $f'(x) \geqslant 0$ (since $\ln u \leqslant u - 1$ for all positive u) and therefore the function f is nondecreasing on $(0, \infty)$. Since $\frac{c}{b} \geqslant 1 \geqslant \frac{a}{b} > 0$, (1) follows.

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

• **5653** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

Evaluate the following limit:

$$\lim_{n\to\infty} \frac{\sin\left(F_n^{-1}\right)}{\sin\left(L_n^{-1}\right)} \cdot \left(1 + \frac{1}{F_n}\right)^{L_n} \cdot \sum_{y=1}^{n} \sum_{x=1}^{\infty} \frac{1}{x^{2\alpha} y^{2\beta} \left(x^{2\alpha} y^{2\beta} + x^{2\beta} y^{2\alpha}\right)}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Editor's note: Here, F_n and L_n respectively denote the Fibonacci and Lucas numbers.

Solution 1 by Moti Levy, Rehovot, Israel.

The limit in the problem statement is equal to the product of three limits:

$$\lim_{n\to\infty}\frac{\sin\left(\frac{1}{F_n}\right)}{\sin\left(\frac{1}{L_n}\right)}\left(1+\frac{1}{F_n}\right)^{L_n}\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}$$
$$=\left(\lim_{n\to\infty}\frac{\sin\left(\frac{1}{F_n}\right)}{\sin\left(\frac{1}{L_n}\right)}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{F_n}\right)^{L_n}\right)\left(\lim_{n\to\infty}\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}\right).$$

The first limit is

$$\lim_{n\to\infty}\frac{\sin\left(\frac{1}{F_n}\right)}{\sin\left(\frac{1}{L_n}\right)}=\lim_{n\to\infty}\frac{\frac{\sin\left(\frac{1}{F_n}\right)}{\frac{1}{F_n}}}{\frac{\sin\left(\frac{1}{L_n}\right)}{\frac{1}{L_n}}}\frac{L_n}{F_n}=\lim_{n\to\infty}\frac{L_n}{F_n}$$

The following formulas are well known,

$$L_n=\alpha^n+\beta^n,$$
$$F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}.$$

$$\lim_{n\to\infty}\frac{L_n}{F_n}=\sqrt{5}\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^n-\beta^n}=\sqrt{5}\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^n}{1-\left(\frac{\beta}{\alpha}\right)^n}$$

$$\left|\frac{\beta}{\alpha}\right|=\left|\frac{1-\sqrt{5}}{1+\sqrt{5}}\right|<1$$

$$\lim_{n\to\infty}\frac{L_n}{F_n}=\sqrt{5}$$

The second limit is

$$\lim_{n\to\infty}\left(1+\frac{1}{F_n}\right)^{L_n}=\lim_{n\to\infty}\left(\left(1+\frac{1}{F_n}\right)^{F_n}\right)^{\frac{L_n}{F_n}}=e^{\lim_{n\to\infty}\frac{L_n}{F_n}}=e^{\sqrt{5}}.$$

The third limit is

$$
\begin{aligned}
&\lim_{n\to\infty}\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}\\
&=\frac{1}{2}\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\left(\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}+\frac{1}{x^{2\beta}y^{2\alpha}\left(x^{2\beta}y^{2\alpha}+x^{2\alpha}y^{2\beta}\right)}\right)\\
&=\frac{1}{2}\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha+2\beta}y^{2\alpha+2\beta}}=\frac{1}{2}\left(\sum_{y=1}^{n}\frac{1}{y^{2\alpha+2\beta}}\right)\left(\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha+2\beta}}\right)\\
&=\frac{1}{2}\zeta^{2}\left(2\left(\alpha+\beta\right)\right)=\frac{1}{2}\zeta^{2}\left(2\right)=\frac{\pi^{4}}{72}.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{\sin\left(\frac{1}{F_n}\right)}{\sin\left(\frac{1}{L_n}\right)}\left(1+\frac{1}{F_n}\right)^{L_n}\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}\\
&=\frac{\sqrt{5}}{72}\pi^{4}e^{\sqrt{5}}\cong 28.305.
\end{aligned}
$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

Clearly,

$$
\lim_{n\to\infty}\frac{\sin\left(\frac{1}{F_n}\right)}{\sin\left(\frac{1}{L_n}\right)}=\lim_{n\to\infty}\frac{L_n}{F_n}=\sqrt{5}\lim_{n\to\infty}\frac{(\alpha^n+\beta^n)}{(\alpha^n-\beta^n)}=\sqrt{5},
$$

$$
\lim_{n\to\infty}\left(1+\frac{1}{F_n}\right)^{L_n}=\lim_{n\to\infty}\left(\left(1+\frac{1}{F_n}\right)^{F_n}\right)^{\frac{L_n}{F_n}}=e^{\sqrt{5}},
$$

$$
\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}=\frac{1}{x^{4\alpha}y^{4\beta}+x^{2}y^{2}}=\frac{1}{x^{2}y^{2}\left(x^{4\alpha-2}y^{4\beta-2}+1\right)}=
$$

$$
=\frac{1}{x^{2}y^{2}\left(x^{2\sqrt{5}}y^{-2\sqrt{5}}+1\right)}=\frac{y^{2\sqrt{5}}}{x^{2}y^{2}\left(x^{2\sqrt{5}}+y^{2\sqrt{5}}\right)}.
$$

Hence, by interchanging the summation variables,

$$
\sum_{y=1}^{n}\sum_{x=1}^{n}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}=\sum_{y=1}^{n}\sum_{x=1}^{n}\frac{y^{2\sqrt{5}}}{x^{2}y^{2}\left(x^{2\sqrt{5}}+y^{2\sqrt{5}}\right)}=
$$

$$
=\frac{1}{2}\sum_{y=1}^{n}\sum_{x=1}^{n}\frac{y^{2\sqrt{5}}+x^{2\sqrt{5}}}{x^{2}y^{2}\left(x^{2\sqrt{5}}+y^{2\sqrt{5}}\right)}=\frac{1}{2}\left(\sum_{x=1}^{n}\frac{1}{x^{2}}\right)^{2}=\frac{1}{2}\left(\frac{\pi^{2}}{6}+O\left(\frac{1}{n}\right)\right)^{2},
$$

and

$$\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}=\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{y^{2\sqrt{5}}}{x^2y^2\left(x^{2\sqrt{5}}+y^{2\sqrt{5}}\right)}=$$

$$=\sum_{y=1}^{n}\sum_{x=1}^{n}\frac{y^{2\sqrt{5}}}{x^2y^2\left(x^{2\sqrt{5}}+y^{2\sqrt{5}}\right)}+\sum_{y=1}^{n}\sum_{x=n+1}^{\infty}\frac{y^{2\sqrt{5}}}{x^2y^2\left(x^{2\sqrt{5}}+y^{2\sqrt{5}}\right)}=$$

$$=\frac{1}{2}\left(\frac{\pi^2}{6}+O\left(\frac{1}{n}\right)\right)^2+O\left(\sum_{y=1}^{\infty}\sum_{x=n+1}^{\infty}\frac{1}{x^2y^2}\right)=\frac{\pi^4}{72}+O\left(\frac{1}{n}\right),$$

as n tends to infinity. Finally,

$$\lim_{n\to\infty}\frac{\sin\left(\frac{1}{F_n}\right)}{\sin\left(\frac{1}{L_n}\right)}\cdot\left(1+\frac{1}{F_n}\right)^{L_n}\cdot\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}\left(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}\right)}=\sqrt{5}\cdot e^{\sqrt{5}}\cdot\frac{\pi^4}{72}.$$

Solution 3 by Michel Bataille, Rouen, France.

Recall that $F_n=\dfrac{\alpha^n-\beta^n}{\sqrt{5}}$ and $L_n=\alpha^n+\beta^n$ for $n\in\mathbb{N}$. Since $|\beta/\alpha|<1$, it follows that $F_n\sim\dfrac{\alpha^n}{\sqrt{5}}$ and $L_n\sim\alpha^n$ as $n\to\infty$. Thus $\lim\limits_{n\to\infty}F_n^{-1}=\lim\limits_{n\to\infty}L_n^{-1}=0$ and therefore

$$\frac{\sin(F_n^{-1})}{\sin(L_n^{-1})}\sim\frac{F_n^{-1}}{L_n^{-1}}\sim\frac{\alpha^n}{\alpha^n/\sqrt{5}}$$

as $n\to\infty$. As a result,

$$\lim_{n\to\infty}\frac{\sin(F_n^{-1})}{\sin(L_n^{-1})}=\sqrt{5}. \tag{1}$$

We have $\ln\left(\left(1+\dfrac{1}{F_n}\right)^{L_n}\right)=L_n\ln\left(1+\dfrac{1}{F_n}\right)\sim L_n\cdot\dfrac{1}{F_n}$ as $n\to\infty$. It follows that

$$\lim_{n\to\infty}\ln\left(\left(1+\frac{1}{F_n}\right)^{L_n}\right)=\sqrt{5}\quad\text{and therefore}\quad\lim_{n\to\infty}\left(1+\frac{1}{F_n}\right)^{L_n}=e^{\sqrt{5}}. \tag{2}$$

Since $u(x,y):=\dfrac{1}{x^{2\alpha}y^{2\beta}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})}>0$, we have $\lim\limits_{n\to\infty}\sum\limits_{y=1}^{n}\sum\limits_{x=1}^{\infty}u(x,y)=S$ where

$$S=\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}u(x,y)=\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}u(x,y),$$

a positive real number or ∞. We remark that $S = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} u(y,x)$, hence

$$2S = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty}(u(y,x)+u(x,y)) = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{1}{x^{2\alpha}x^{2\beta}y^{2\alpha}y^{2\beta}} = \sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\frac{1}{x^2y^2}$$

(the last equality because $\alpha+\beta=1$).
Thus,

$$2S = \left(\sum_{y=1}^{\infty}\frac{1}{y^2}\right)\left(\sum_{x=1}^{\infty}\frac{1}{x^2}\right) = \frac{\pi^4}{36}. \tag{3}$$

From (1), (2), (3), we conclude that the required limit is

$$\frac{\pi^4\sqrt{5}\,e^{\sqrt{5}}}{72}.$$

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

The Binet forms for the nth Fibonacci number and the nth Lucas number are

$$F_n = \frac{\alpha^n-\beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n+\beta^n,$$

respectively. Because $|\alpha|>1$ and $|\beta|<1$, it follows that

$$F_n \sim \frac{\alpha^n}{\sqrt{5}} \quad \text{and} \quad L_n \sim \alpha^n$$

as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\frac{\sin(F_n^{-1})}{\sin(L_n^{-1})} = \lim_{n\to\infty}\frac{L_n}{F_n} = \sqrt{5},$$

and

$$\lim_{n\to\infty}\left(1+\frac{1}{F_n}\right)^{L_n} = \lim_{n\to\infty}\left(1+\frac{\sqrt{5}}{\alpha^n}\right)^{\alpha^n} = e^{\sqrt{5}}.$$

Now,

$$\begin{aligned}
\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})} &= \sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^{2\beta}y^{2\alpha}}\left(\frac{1}{x^{2\alpha}y^{2\beta}}-\frac{1}{x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha}}\right)\\
&= \sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^2y^2}-\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^{2\beta}y^{2\alpha}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})}\\
&= \sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^2y^2}-\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})},
\end{aligned}$$

so

$$\sum_{y=1}^{\infty}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})}=\frac{1}{2}\left(\sum_{y=1}^{\infty}\frac{1}{y^2}\right)\left(\sum_{x=1}^{\infty}\frac{1}{x^2}\right)=\frac{\pi^4}{72}.$$

Thus,

$$\lim_{n\to\infty}\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})}=\frac{\pi^4}{72},$$

and

$$\lim_{n\to\infty}\frac{\sin(F_n^{-1})}{\sin(L_n^{-1})}\cdot\left(1+\frac{1}{F_n}\right)^{L_n}\cdot\sum_{y=1}^{n}\sum_{x=1}^{\infty}\frac{1}{x^{2\alpha}y^{2\beta}(x^{2\alpha}y^{2\beta}+x^{2\beta}y^{2\alpha})}=\sqrt{5}e^{\sqrt{5}}\cdot\frac{\pi^4}{72}.$$

Also solved by the proposer.

• **5654** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{mn^2(n+m)^2}$$

where $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ denotes the nth harmonic number.

Solution 1 by Moti Levy, Rehovot, Israel.

We apply the Borwein's trick to get symmetric double sum:

$$\begin{aligned} S &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{H_{m+n}}{mn^2\left(m+n\right)^2}=\frac{1}{2}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{H_{m+n}}{mn^2\left(m+n\right)^2}+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{H_{m+n}}{m^2n\left(m+n\right)^2}\right)\\ &=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{H_{m+n}}{m^2n^2\left(m+n\right)}. \end{aligned} \tag{27}$$

Integral representation of the harmonic number is well known:

$$H_n=-n\int_0^1 z^{n-1}\ln\left(1-z\right)dz \tag{28}$$

We plug (28) into (27) and change the order of summation and integration,

$$\begin{aligned} S &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{-(n+m)\int_0^1 z^{n+m-1}\ln(1-z)}{m^2n^2(m+n)}dz \\ &= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\int_0^1 z^{n+m-1}\ln(1-z)}{m^2n^2}dz \\ &= -\frac{1}{2}\int_0^1\frac{\ln(1-z)}{z}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{z^{n+m}}{m^2n^2}dz. \end{aligned} \tag{29}$$

By definition of the dilogarithm $\mathbf{Li}_2(z)$,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{z^{n+m}}{m^2n^2} = \left(\sum_{n=1}^{\infty}\frac{z^n}{n^2}\right)\left(\sum_{n=1}^{\infty}\frac{z^m}{m^2}\right) = \left(\mathbf{Li}_2(z)\right)^2. \tag{30}$$

Substitution of (30) into (29) and integration produce the required result:

$$S = -\frac{1}{2}\int_0^1\frac{\ln(1-z)}{z}\left(\mathbf{Li}_2(z)\right)^2dz = \frac{1}{2}\int_0^1\left(\frac{d}{dz}\mathbf{Li}_2(z)\right)\left(\mathbf{Li}_2(z)\right)^2dz = \frac{1}{6}\left(\mathbf{Li}_2(1)\right)^3,$$

$$S = \frac{\pi^6}{1296} \cong 0.741\,81..$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We reduce this double sum to known evaluations of Euler sums. All involved terms are positive. We may therefore rearrange the double sum by grouping all tuples (m,n) for which $m+n=k+1$ and then sum over k from 1 to infinity:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{mn^2(n+m)^2} = \sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^2}\sum_{n=1}^{k}\frac{1}{(k+1-n)n^2} =$$

$$= \sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^2}\sum_{n=1}^{k}\left(\frac{1}{(k+1)n^2}+\frac{1}{(k+1)^2n}+\frac{1}{(k+1)^2(k+1-n)}\right) =$$

$$= \sum_{k=1}^{\infty}\frac{H_{k+1}H_k^{(2)}}{(k+1)^3} + 2\sum_{k=1}^{\infty}\frac{H_{k+1}H_k}{(k+1)^4},$$

where $H_k^{(2)} = \sum_{n=1}^{k}\frac{1}{n^2}$. We have

$$\sum_{k=1}^{\infty}\frac{H_{k+1}H_k^{(2)}}{(k+1)^3} = \sum_{k=1}^{\infty}\frac{H_{k+1}H_{k+1}^{(2)}}{(k+1)^3} - \sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^5} = \sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3} - \sum_{k=1}^{\infty}\frac{H_k}{k^5},$$

$$\sum_{k=1}^{\infty}\frac{H_{k+1}H_k}{(k+1)^4} = \sum_{k=1}^{\infty}\frac{H_{k+1}^2}{(k+1)^4} - \sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^5} = \sum_{k=1}^{\infty}\frac{H_k^2}{k^4} - \sum_{k=1}^{\infty}\frac{H_k}{k^5}.$$

We will use the following results:

$$\sum_{k=1}^{\infty}\frac{H_k}{k^5}=\frac{7}{2}\zeta(6)-\zeta(2)\zeta(4)-\frac{1}{2}\zeta^2(3)=\frac{1}{540}\pi^6-\frac{1}{2}\zeta^2(3), \qquad [1],\ Theorem\ 2.2\ \text{(Euler)}$$

$$\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}=\frac{97}{24}\zeta(6)-2\zeta^2(3)=\frac{97\pi^6}{22680}-2\zeta^2(3), \qquad [1],\ (4-1)$$

$$\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}=\frac{5}{2}\zeta^2(3)-\frac{101}{45360}\pi^6. \qquad [2],\ (4.11)$$

Hence

$$\sum_{k=1}^{\infty}\frac{H_{k+1}H_k^{(2)}}{(k+1)^3}=\sum_{k=1}^{\infty}\frac{H_{k+1}H_{k+1}^{(2)}}{(k+1)^3}-\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^5}=\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}-\sum_{k=1}^{\infty}\frac{H_k}{k^5},$$
$$\sum_{k=1}^{\infty}\frac{H_{k+1}H_k}{(k+1)^4}=\sum_{k=1}^{\infty}\frac{H_{k+1}^2}{(k+1)^4}-\sum_{k=1}^{\infty}\frac{H_{k+1}}{(k+1)^5}=\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}-\sum_{k=1}^{\infty}\frac{H_k}{k^5},$$

and finally

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{mn^2(n+m)^2}=\sum_{k=1}^{\infty}\frac{H_{k+1}H_k^{(2)}}{(k+1)^3}+2\sum_{k=1}^{\infty}\frac{H_{k+1}H_k}{(k+1)^4}=\sum_{k=1}^{\infty}\frac{H_kH_k^{(2)}}{k^3}-3\sum_{k=1}^{\infty}\frac{H_k}{k^5}+2\sum_{k=1}^{\infty}\frac{H_k^2}{k^4}=$$

$$=\frac{5}{2}\zeta^2(3)-\frac{101}{45360}\pi^6-3\left(\frac{1}{540}\pi^6-\frac{1}{2}\zeta^2(3)\right)+2\left(\frac{97\pi^6}{22680}-2\zeta^2(3)\right)=\frac{\pi^6}{1296}.$$

References

[1] Philippe Flajolet and Bruno Salvy. Euler sums and contour integral representations. Experimental Mathematics, 1998, 7(1): 15–35, available at https://hal.inria.fr/inria-00073780/document.[2] Junesang Choi, Hari Mohan Srivastava. Explicit Evaluation of Euler and Related Sums. August 2005 The Ramanujan Journal 10(1):51-70, available at https://www.researchgate.net/publication/225506416_Explicit_Evaluation_of_Euler_and_Related_Sums.

Solution 3 by Michel Bataille, Rouen, France.

We show that the required sum S is

$$\frac{\pi^6}{1296}.$$

Since $\dfrac{H_{n+m}}{mn^2(n+m)^2}, \dfrac{H_{n+m}}{n^2(n+m)^3}, \dfrac{H_{n+m}}{mn(n+m)^3}$ are positive and the first one is the sum of the other two, we may write

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{n^2(n+m)^3} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{mn(n+m)^3}.$$

Since

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{mn(n+m)^3} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{(n+m)^4}\left(\frac{n+m}{mn}\right) = 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{n(n+m)^4}$$

we see that

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{n^2(n+m)^3} + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{n(n+m)^4}.$$

Now, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{n(n+m)^4} = \sum_{k=2}^{\infty}\sum_{n=1}^{k-1}\frac{H_k}{k^4}\cdot\frac{1}{n} = \sum_{k=2}^{\infty}\frac{H_k^2}{k^4} - \sum_{k=2}^{\infty}\frac{H_k}{k^5}$$

and similarly

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{n+m}}{n^2(n+m)^3} = \sum_{k=2}^{\infty}\frac{H_kH_k^{(2)}}{k^3} - \sum_{k=2}^{\infty}\frac{H_k}{k^5}$$

(where $H_k^{(2)} = \sum_{j=1}^{k}\frac{1}{j^2}$). Note that the series on the right are convergent since $H_k \sim \ln(k)$ as $k \to \infty$ and $\sum_{k\geqslant 1}\frac{(\ln(k))^{\beta}}{k^{\alpha}}$ is convergent if $\alpha > 1$. Finally, we obtain

$$S = \sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3} + 2\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} - 3\sum_{n=1}^{\infty}\frac{H_n}{n^5}.$$

To finish the calculation, we use some known results:

First, a result of Euler's: for $p \in \mathbb{N}$ with $p \geqslant 2$, $2\sum_{n=1}^{\infty}\frac{H_n}{n^p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2}\zeta(p-j)\zeta(j+1)$,

which readily leads to

$$\sum_{n=1}^{\infty}\frac{H_n}{n^5} = \frac{7}{2}\zeta(6) - \zeta(4)\zeta(2) - \frac{1}{2}(\zeta(3))^2 = \frac{\pi^6}{540} - \frac{(\zeta(3))^2}{2}$$

(since $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(6) = \frac{\pi^6}{945}$).

Second, we have

$$\sum_{n=1}^{\infty}\frac{H_nH_n^{(2)}}{n^3} = \frac{5(\zeta(3))^2}{2} - \frac{101\pi^6}{45360}$$

(see J. Choi, H.M. Srivastava, Explicit Evaluation of Euler and Related Sums, *Ramanujan Journal*, Vol 10, 2005, formula 4-11, p. 63) and

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^4} = \frac{97\pi^6}{22680} - 2(\zeta(3))^2$$

(see De-Yin Zheng, Further summation formulae related to generalized harmonic numbers, *J. Math. Anal. Appl.* 335, 2007, formula 3.4b p. 698).
All these results yield

$$S = \frac{2\times 97\pi^6}{22680} - \frac{101\pi^6}{45360} - \frac{3\pi^6}{540} = \frac{\pi^6}{1296}.$$

Solution 4 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

Denote the sum to be found by S. We claim $S = \dfrac{\pi^6}{1296}$.

As $m, n \in \mathbb{N}$, observer that

$$\int_0^1 t^{m+n-1}\,dt = \frac{1}{m+n}.$$

Also, from the integral representation for the harmonic numbers of

$$\frac{H_k}{k} = -\int_0^1 x^{k-1}\log(1-x)\,dx,$$

replacing the positive integer k with the positive integer $m+n$ one has

$$\frac{H_{m+n}}{m+n} = -\int_0^1 x^{m+n-1}\log(1-x)\,dx.$$

The sum S may therefore be rewritten as

$$\begin{aligned} S &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn^2}\cdot\frac{1}{m+n}\cdot\frac{H_{m+n}}{m+n} \\ &= -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn^2}\int_0^1 t^{m+n-1}\,dt\int_0^1 x^{m+n-1}\log(1-x)\,dx, \end{aligned}$$

or, after interchanging the summation and integration signs which is permissible due to the unsigned nature of all terms involved, as

$$S = -\int_0^1\int_0^1 \frac{\log(1-x)}{xt}\sum_{m=1}^{\infty}\frac{(xt)^m}{m}\sum_{n=1}^{\infty}\frac{(xt)^n}{n^2}\,dt\,dx. \tag{31}$$

Recalling

$$-\log(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}, \quad -1 \leqslant z < 1,$$

and

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}, \quad |z| \leqslant 1,$$

where $\mathrm{Li}_2(z)$ denotes the dilogarithm, the sum in (31) can be expressed as

$$S = \int_0^1\int_0^1 \frac{\log(1-x)\log(1-xt)\,\mathrm{Li}_2(xt)}{xt}\,dt\,dx.$$

Enforcing a substitution of $u = xt$, $du = x\,dt$ produces

$$S = \int_0^1 \frac{\log(1-x)}{x} \int_0^x \frac{\log(1-u)\,\mathrm{Li}_2(u)}{u}\,du\,dx.$$

Denoting the inner u-integral by $I(x)$, integrating by parts we have

$$I(x) = -\mathrm{Li}_2(u)\cdot \mathrm{Li}_2(u)\Big|_0^x - \int_0^x \frac{\log(1-u)\,\mathrm{Li}_2(u)}{u}\,du = -\mathrm{Li}_2^2(x) - I(x),$$

or

$$I(x) = -\frac{1}{2}\,\mathrm{Li}_2^2(x).$$

Returning to the sum S one has

$$S = -\frac{1}{2}\int_0^1 \frac{\log(1-x)\,\mathrm{Li}_2^2(x)}{x}\,dx.$$

Integrating by parts again produces

$$S = -\frac{1}{2}\left[-\mathrm{Li}_2(x)\cdot \mathrm{Li}_2^2(x)\Big|_0^1 - 2\int_0^1 \frac{\log(1-x)\,\mathrm{Li}_2^2(x)}{x}\,dx\right] = \frac{1}{2}\,\mathrm{Li}_2^3(1) - 2S.$$

Thus

$$3S = \frac{1}{2}\,\mathrm{Li}_2^3(1) \quad \text{or} \quad S = \frac{1}{6}\,\mathrm{Li}_2^3(1).$$

As

$$\mathrm{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

where the result for the famous Basel problem has been recalled, we have

$$S = \frac{1}{6}\left(\frac{\pi^2}{6}\right)^3 = \frac{\pi^6}{1296},$$

as claimed.

Also solved by Narendra Bhandari, Bajura district, Nepal; and the proposers.

Late Acknowledgement: Joe L. Howard, from Portales, N.M., submitted via regular mail (snail mail) his solution to Problem #5638. Due to restricted accessibility of received physical mail, Joe's solution regrettably did not reach the editor on time for proper timely acknowledgment and publication. In the transition period from the former editor to the current one, some solutions did not reach the current editor on time for timely acknowledgement. These are Problems #5633, #5634, #5636 from Michel Bataille, Rouen, France; and Problems #5634, #5635, #5636 from Albert Stadler, Herrilberg, Switzerland. This editor, in a continuing tradition as practiced by the former editor, thanks all contributors, including the latter three even if their solutions regrettably did not get published.

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Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributers. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following recommendations. As you peruse below, you may construe that the recommendations amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
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Examples:

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#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

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3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

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For all your proposed problems, please adopt for all documents the following FILENAME format:

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Examples:

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6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

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Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣