Problems and Solutions   Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

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Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

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Solutions to the problems published in this issue should be submitted before May 15, 2022.

• **5673** *Proposed by Goran Conar, Varaždin, Croatia.*

Let α, β, γ be angles of an arbitrary triangle. Prove the inequality

$$\alpha \cot \alpha + \beta \cot \beta + \gamma \cot \gamma \leqslant \frac{\pi}{\sqrt{3}}.$$

When does equality occur?

• **5674** *Proposed by Kenneth Korbin, New York, NY.*

Find positive rational numbers x and y such that

$$\left[(x+iy)^7 + (x-iy)^7\right]^2 + \left[(y+ix)^7 + (y-ix)^7\right]^2 = 4,$$

where $i^2 = -1$.

• **5675** *Proposed by Nikos Ntorvas, Athens, Greece.*

Suppose $a, b, c, n > 0$ and $a + b + c = 1$. Prove:

$$e^n (a+1)^{nb} (b+1)^{nc} (c+1)^{na} < e^4 (na)^{na} (nb)^{nb} (nc)^{nc}.$$

• **5676** *Proposed by Peter Fulop, Gyomro, Hungary.*

Without using integral identities of the Catalan's constant G, prove

$$\frac{1}{2}\int_0^{\pi/4}\int_0^{\pi/4}\left[\frac{1}{\cos(u+v)}+\frac{1}{\cos(u-v)}\right]dudv=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

• **5677** *Proposed by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.*

Solve the differential equation

$$\frac{dy}{dx}=\tan(x+y)-\cot(x-y).$$

• **5678** *Proposed by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.*

For positive integers m and n define

$$S_m(n)=\sum_{k=1}^{n}\tan^{2m}\left(\frac{k\pi}{2n+1}\right).$$

Express $S_1(n)$, $S_2(n)$ and $S_3(n)$, each as a polynomial in n.

Solutions

• **5655** *Proposed by Kenneth Korbin, New York, NY.*

Given a Heronian triangle $\triangle ABC$ with altitude $\overline{CD}=y-1$, sides $\overline{AC}=y$ and $\overline{BC}=y+1$, find four possible values of y.

Editor's note: A trianlge with integer sides and integer area is called *Heronian*.

Solution 1 by Dionne Bailey, Elsie Campbell. and Charles Diminnie, Angelo State University, San Angelo, TX.

Since $\angle CDB = \angle CDA = 90^\circ$,

$$\begin{aligned}\overline{DB} &= \sqrt{\overline{BC}^2 - \overline{CD}^2}\\ &= \sqrt{(y+1)^2 - (y-1)^2}\\ &= \sqrt{4y}\\ &= 2\sqrt{y}\end{aligned}$$

and

$$\begin{aligned}\overline{AD} &= \sqrt{\overline{AC}^2 - \overline{CD}^2}\\ &= \sqrt{y^2 - (y-1)^2}\\ &= \sqrt{2y-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\overline{AB} &= \overline{DB} + \overline{AD}\\ &= 2\sqrt{y} + \sqrt{2y-1}.\end{aligned}$$

To insure that $\overline{AB}$ is an integer, we must find positive integers x and z for which $y = x^2$ and $2y - 1 = z^2$, i.e., $x^2 = \dfrac{z^2+1}{2}$. After considerable trial and error, we came up with the following assignments for x and z.

x	z	y	$2y-1$	$\overline{AB}$
5	7	25	49	$10 + 7 = 17$
29	41	841	1,681	$58 + 41 = 99$
169	239	28,561	57,121	$338 + 239 = 577$
985	1,393	970,225	1,940,449	$1,970 + 1,393 = 3,363$

.

Since the area X of $\triangle ABC$ is given by

$$X = \frac{1}{2}\overline{(CD)}\left(\overline{AB}\right),$$

we obtain

y	$\overline{CD} = y-1$	$\overline{AB}$	X
25	24	17	204
841	840	99	41,580
28,561	28,560	577	8,239,560
970,225	970,224	3,363	1,631,431,656

.

It follows that these four values of y satisfy the required properties to make $\triangle ABC$ Heronian.

Solution 2 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

Four possible values of y are 25, 841, 28561 and 970225.

Assuming D lies between A and B, then $AD^2 = y^2 - (y-1)^2 = 2y - 1$ and $BD^2 = (y+1)^2 - (y-1)^2 = 4y$. If y and $2y - 1$ are both squares, then $AB = AD + BD = \sqrt{2y-1} + 2\sqrt{y}$ will be an integer. If y is odd, then the area of ΔABC will be $AB \cdot \dfrac{y-1}{2}$, which is also an integer, and the triangle will be Heronian.

Thus, if a and b are integers satisfying $2y - 1 = a^2$ and $y = b^2$, then $a^2 = 2b^2 - 1$, giving Pell's equation $a^2 - 2b^2 = -1$. Solutions for this equation may be determined from the convergents of the continued fraction expansion of $\sqrt{2}$, which is $[1; \overline{2}]$. We set $p_0 = q_0 = 1$, $p_1 = 3$, $q_1 = 2$, and for integers $k \geqslant 2$,

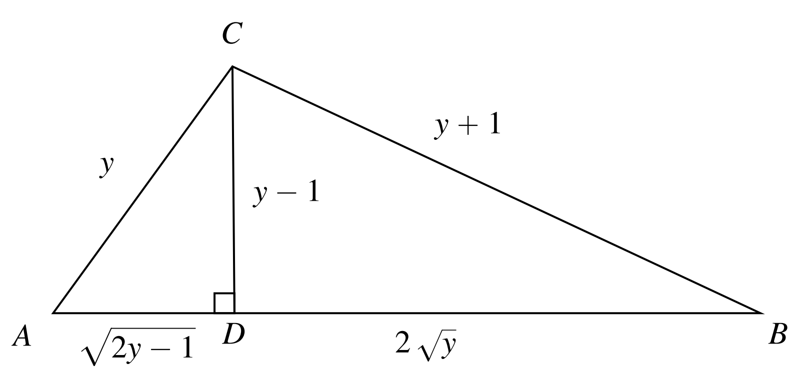
$$p_k = 2p_{k-1} + p_{k-2}, \; q_k = 2q_{k-1} + q_{k-2}.$$

Since the period of the continued fraction expansion of $\sqrt{2}$ is 1, which is odd, then solutions to Pell's equation are given by $a = p_{2k}, b = q_{2k}$, where k is a nonnegative integer. The first few solutions are given in the table below.

k	$a = p_{2k}$	$b = q_{2k}$	$y = b^2 = AC$	BC	AB	AD
0	1	1	1	2	3	0
1	7	5	25	26	17	24
2	41	29	841	842	99	840
3	239	169	28,561	28,562	577	28,560
4	1393	985	970,225	970,226	3363	970,224

Notice that if $a = b = 1$, then ΔABC is a degenerate triangle with $AC = 1$, $BC = 2$, $AB = 3$, and altitude $AD = 0$. For the first four positive values of k, we get the values of y given in the table, and Heronian triangles with side lengths $(25, 26, 17)$, $(841, 842, 99)$, $(28561, 28562, 577)$, and $(970225, 970226, 3363)$. Notice that since all values of y are odd, then in each case the altitude $y - 1$ is even, and the area of ΔABC is an integer.

Solution 3 by Ajay Srinivasan, University of Southern California, Los Angeles, CA.



By $\overline{AB}$ we denote the line segment that joins the points labelled A and B. We are given $\overline{CD} = y - 1$, $\overline{AC} = y$ and that $\overline{BC} = y + 1$. Since $\triangle ABC$ is Heronian, y is obviously an integer. Using the pythagorean theorem, we get the following:

$$\overline{AD} = \sqrt{2y-1}$$
$$\overline{BD} = 2\sqrt{y}$$

This makes $\overline{AB} = \sqrt{2y-1} + 2\sqrt{y}$ for $y \in \mathbb{Z}^+$. Note that $\overline{AB} \in \mathbb{Z}^+$ since the triangle is Heronian. Evidently $\exists x \in \mathbb{Z}^+ : y = x^2$. We also need $2y - 1 = 2x^2 - 1 = z^2$ for some $z \in \mathbb{Z}^+$. Consider the relevant diophantine equation $2x^2 - 1 = z^2$. Since the LHS is an odd number, $z \equiv 1 \pmod 2$, i.e. $\exists k \in \mathbb{Z}_{\geqslant 0} : z = 2k + 1$. The new diophantine is $2x^2 - 1 = 4k^2 + 4k + 1$, and evidently x must be odd (and $y - 1$ must be even). So all positive integral solutions to $2x^2 - 1 = z^2$ yield triangles with both integral area and integral sides.

$$z^2 - 2x^2 = -1 \quad \text{is a negative Pell's Equation}$$

It is soluble (refer to A031396 in OEIS). The fundamental solution for this eqn. is $(1, 1)$, and all positive integral solutions to the equation are generated by $(1, 1)$ in the following manner:

$$\begin{aligned} z_n + \sqrt{2}x_n &= \left(z_1 + \sqrt{2}x_1\right)^n \\ z_n &= 3z_{n-2} + 4x_{n-2} \\ x_n &= 3x_{n-2} + 2z_{n-2} \end{aligned} \tag{1}$$

where $(z_1, x_1) = (1, 1)$ and n is odd. The solutions generated include $(7, 5), (41, 29), (239, 169)$, and $(1393, 985)$. 4 possible values for y include: $5^2, 29^2, 169^2, 985^2$ since $y = 1$ gives a triangle with zero area.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We show that there are infinitely many possible values for y, the smallest of which are 25, 841, 28561, and 970225. Since $\overline{CD} < \overline{AC}$, we note that $\angle A$ cannot be a right angle. Letting $c = \overline{AB}$, we have $c = 2\sqrt{y} + \sqrt{2y-1}$ if $\angle A$ is acute and $c = 2\sqrt{y} - \sqrt{2y-1}$ if $\angle A$ is obtuse. To ensure that c is a positive integer, we first define the sequence $\{x_n\}$ by $x_0 = 1$, $x_1 = 5$, and $x_{n+1} = 6x_n - x_{n-1}$ for $n \geqslant 1$ and then let $y_n = x_n^2$ for each $n \geqslant 0$. We establish that for each positive integer n, y_n is a possible value of y. It is straightforward to verify that for each $n \geqslant 0$, $x_n = [(2 + \sqrt{2})\gamma^n + (2 - \sqrt{2})\delta^n]/4$, where $\gamma = 3 + 2\sqrt{2}$ and $\delta = 3 - 2\sqrt{2}$. Then $y_n = (\gamma^{2n+1} + 2 + \delta^{2n+1})/8$, so $2\sqrt{y_n} = 2x_n$ is a positive integer for each n. Next,

$$\sqrt{2y_n - 1} = \frac{\sqrt{2}}{4}[(2 + \sqrt{2})\gamma^n - (2 - \sqrt{2})\delta^n],$$

so letting $z_n = \sqrt{2y_n - 1}$, we have $z_0 = 1$, $z_1 = 7$, and $z_{n+1} = 6z_n - z_{n-1}$ for $n \geqslant 1$. Hence $c_n = 2\sqrt{y_n} \pm \sqrt{2y_n - 1}$ is a positive integer as required. Finally, since each y_n is odd, the area of ΔABC is a positive integer, given by $(1/2)c_n(y_n - 1)$. *Addenda.* (i) Each value of y_n produces two triangles. The side lengths of these two triangles for the first four values of y_n are: $(25, 26, 17)$ with area = 204 and $(25, 26, 3)$ with area = 36 $(841, 842, 99)$ with area = 41580 and $(841, 842, 17)$ with area = 7140 $(28561, 28562, 577)$ with area = 8239560 and $(28561, 28562, 99)$ with area = 1413720 $(970225, 970226, 3363)$ with area = 1631431656 and $(970225, 970226, 577)$ with area = 279909624

(ii) The values of x_n and z_n are connected with the continued fraction convergents 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, 577/408, 1393/985, ... to $\sqrt{2}$:

$$\{x_n\}_{n=0}^{\infty} = \{1, 5, 29, 169, 985, \dots\} \quad \text{and} \quad \{z_n\}_{n=0}^{\infty} = \{1, 7, 41, 239, 1393, \dots\}.$$

Solution 5 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

By the Pythagorean theorem,

$$\overline{AD} = \sqrt{y^2 - (y-1)^2} = \sqrt{2y-1} \text{ and } \overline{DB} = \sqrt{(y+1)^2 - (y-1)^2} = 2\sqrt{y}.$$

We need $2y - 1$ and y to be perfect squares in order for $\overline{AB} = \overline{AD} + \overline{DB}$ to be an integer as well (cf [2]). This amounts to solving the negative Pell's equation $2n^2 - 1 = m^2$, which has a general solution given by the matrix formula (see Theorem 3.4.1 of [1])

$$\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad k \in \mathbb{N}.$$

The first four values of $m = \overline{AD}$, n, $y = n^2$, $\overline{CD}$, $\overline{DB} = 2n$, $\overline{AB}$, and the area of $\triangle ABC = \frac{1}{2}\overline{AB} \cdot \overline{CD}$ are provided in the following table, with values of y in bold.

k	m	n	y	$\overline{CD}$	$\overline{DB}$	$\overline{AB}$	area of $\triangle ABC$
1	7	5	**25**	24	10	17	204
2	41	29	**841**	840	58	99	41 580
3	239	169	**28 561**	28 560	338	577	8 239 560
4	1393	985	**970 225**	970 224	1970	3363	1 631 431 656

References

[1] Andreescu, Titu and Andrica, Dorin and Cucurezeanu, Ion, An introduction to Diophantine equations, Birkhäuser Verlag, New York, (2010), ISBN 978-0-8176-4548-9.

[2] Besicovitch, A. S., On the linear independence of fractional powers of integers, *J. London Math. Soc.*, **15**, (1940), 3-6.

Solution 6 by David E. Manes, Oneonta, NY.

The four possible values of y that yield a Heronian triangle $\triangle ABC$ are $y = 25, 841, 28561$ and 970225.

Note that the altitude $\overline{CD}$ divides the triangle $\triangle ABC$ into two right triangles $\triangle ACD$ and $\triangle CDB$ with right angles at vertex D. Let $x = \overline{AD}$ and $z = \overline{DB}$. Then $y^2 = x^2+(y-1)^2$ and $(y+1)^2 = z^2+(y-1)^2$. These equations reduce to $x^2 + 1 = 2y$ and $z^2 = 4y$. Therefore, $z^2 = 2(x^2 + 1)$ or $z^2 - 2x^2 = 2$. This equation has the immediate solution $z = 10$ and $x = 7$. These values then define the following Heronian triangle $\triangle ABC$: $\overline{AC} = y = z^2/4 = 25$, $\overline{BC} = y + 1 = 26$ and $\overline{AB} = x + z = 17$. The area of triangle $\triangle ABC$ is $A = (1/2)(\overline{AB})(\overline{CD}) = (1/2)(17)(24) = 204$.

To find the other values of y, we will use the following result. If the equation $z^2 - dx^2 = c$ is solvable, then it has infinitely many solutions given by: if u, v satisfy $z^2 - dx^2 = c$ and r, s satisfy $z^2 - dx^2 = 1$, then

$$(ur \pm dvs)^2 - d(us \pm vr)^2 = (u^2 - dv^2)(r^2 - ds^2) = c.$$

Therefore, $z = ur \pm dvs$ and $x = us \pm vr$ is a solution of $z^2 - 2x^2 = 1$. Note that if $d = 2$, then $z = 3, x = 2$ is a fundamental solution of $z^2 - 2x^2 = 1$. In the following, we will use the positive values for $ur + dvs$ and $us + vr$. Moreover, for all cases of this result we will use $d = 2 = s$ and $r = 3$.

We begin by using the above result with the first solution of $z^2 - 2x^2 = 2$; namely, $u = 10$ and $v = 7$. Then

$$\begin{aligned}(ur + dvs)^2 - 2(us + vr)^2 &= (10 \cdot 3 + 2 \cdot 7 \cdot 2)^2 - 2(10 \cdot 2 + 7 \cdot 3)^2 \\ &= 58^2 - 2(41)^2 = 2.\end{aligned}$$

Therefore, $u = 58$ and $v = 41$ is a solution of $z^2 - 2x^2 = 2$ and defines the Heronian triangle $\triangle ABC$ such that $\overline{AC} = y = u^2/4 = 58^2/4 = 841$, $\overline{BC} = y + 1 = 842$ and $\overline{AB} = u + v = 58 + 41 = 99$. The area of $\triangle ABC$ is $A = (1/2)(\overline{AB})(\overline{CD}) = (1/2)(99)(840) = 41580$.

Continuing in this manner, let $u = 58$ and $v = 41$. Then

$$\begin{aligned}(ur + 2vs)^2 - 2(us + vr)^2 &= (58 \cdot 3 + 4 \cdot 41)^2 - 2(58 \cdot 2 + 41 \cdot 3)^2 \\ &= 338^2 - 2(239)^2 = 2.\end{aligned}$$

Hence, the values $u = 338, v = 239$ define the Heronian triangle $\triangle ABC$ with $\overline{AC} = u^2/4 = 338^2/4 = 28561$, $\overline{BC} = y + 1 = 28562$ and $\overline{AB} = u + v = 338 + 239 = 577$ with area $A = (1/2)(577)(28560) = 8\,239\,560$.

For the fourth value of y, let $u = 338$ and $v = 239$. Then

$$\begin{aligned}(3u + 4v)^2 - 2(2u + 3v)^2 &= (338 \cdot 3 + 4 \cdot 239)^2 - 2(2 \cdot 338 + 3 \cdot 239)^2 \\ &= (1970)^2 - 2(1393)^2 = 2.\end{aligned}$$

Therefore, $u = 1970$ and $v = 1393$ define the Heronian triangle $\triangle ABC$ with sides $\overline{AC} = y = u^2/4 = 1970^2/4 = 970\,225$, $\overline{BC} = y + 1 = 970\,226$ and $\overline{AB} = u + v = 1970 + 1393 = 3363$. The area of $\triangle ABC$ is $A = (1/2)(3363)(970224) = 1\,631\,431\,656$. Furthermore, continuing this procedure, one can see that there are infinitely many Heronian triangles $\triangle ABC$ with altitude $\overline{CD} = y - 1$ and sides $\overline{AC} = y, \overline{BC} = y + 1$. This completes the solution.

Solution 7 by Michael Brozinsky, Central Islip, NY, and Andrew Bulawa, Brooklyn, NY.

The Pell's equation $x^2 - 2y^2 = -1$ clearly has its fundamental solution $x_1 = 1$, $y_1 = 1$ and thus its general solution x_n, y_n is given by $x_n + y_n\sqrt{2} = (x_1 + y_1\sqrt{2})^n$ for $n = 1, 3, 5, 7, \ldots$. (See, for example, Introduction to Number Theory by James E. Shockley pages 174-178 in the 1967 edition). In particular, we note the following expansions

$$\begin{aligned}
(1 + \sqrt{2})^3 &= 7 + 5\sqrt{2} \\
(1 + \sqrt{2})^5 &= 41 + 29\sqrt{2} \\
(1 + \sqrt{2})^7 &= 239 + 169\sqrt{2} \\
(1 + \sqrt{2})^9 &= 1393 + 985\sqrt{2} \\
(1 + \sqrt{2})^{11} &= 8119 + 5741\sqrt{2}
\end{aligned}$$

Now in the problem at hand we have using the Pythagorean Theorem $DB = \sqrt{(y+1)^2 - (y-1)^2} = 2\sqrt{y}$ and so if angle A is acute $AB = 2\sqrt{y} + \sqrt{y^2 - (y-1)^2} = 2\sqrt{y} + \sqrt{2y - 1}$. Otherwise, i.e., if angle A is not acute, $AB = 2\sqrt{y} - \sqrt{y^2 - (y-1)^2} = 2\sqrt{y} - \sqrt{2y-1}$.

We thus need y and $2y - 1$ to be perfect squares so let $y = k^2$ and $2y - 1 = m^2$ so that $m^2 - 2k^2 = -1$ is in the form of Pell's equation. We thus can make the following table using the results above.

m	k	$y = k^2$	$AB = 2\sqrt{y} + \sqrt{2y-1}$	$AB = 2\sqrt{y} - \sqrt{2y-1}$	Sides $y = AC$, $y+1 = BC$ and side AB
1	1	1	1	1	no triangles since altitude $y - 1 = 0$
7	5	25	17	3	Two triangles 25, 26, 17 and 25, 26, 3
41	29	841	99	17	Two triangles 841, 842, 99 and 841, 842, 17
239	169	28561	577	99	Two triangles 28561, 28562, 577 and 28561, 28562, 99
1393	985	970225	3363	577	Two triangles 970225, 970226, 3363 and 970225, 970226, 577
8119	5741	32959081	19601	3363	Two triangles 32959081, 32959082, 19601 and 32959081, 32959082, 3363

Hence 4 possible values for y are 25, 841, 28561, 970225.

Solution 8 by Michel Bataille, Rouen, France.

Let $x = \overline{AB}$ and let S be the area of the triangle ABC. Then, from well-known formulas, we have

$$2S = x(y-1) \quad \text{and} \quad 16S^2 = (2y+1+x)(2y+1-x)(x-1)(x+1) = (x^2-1)((2y+1)^2-x^2). \quad (1)$$

It follows that y is a solution if and only if y is an integer with $y \geqslant 2$ such that for some integer x satisfying $1 < x < 2y+1$, the equations (1) are compatible.
Here are four solutions: $y_1 = 25$ (with $x_1 = 3$), $y_2 = 841$ (with $x_2 = 17$), $y_3 = 28561$ (with $x_3 = 99$) and $y_4 = 970225$ (with $x_4 = 577$).
For $i = 1,2,3,4$, it is easily checked that the inequalities $1 < x_i < 2y_i + 1$ hold and that the equations $2S_i = x_i(y_i - 1)$ and $16S_i^2 = (x_i^2 - 1)((2y_i + 1)^2 - x_i^2)$ are satisfied with $S_1 = 36$, $S_2 = 7140$, $S_3 = 1413720$, and $S_4 = 279909624$.
These four values of y were found as follows: Expressing that $x = \overline{AB} = \sqrt{(y+1)^2 - (y-1)^2} \pm \sqrt{y^2 - (y-1)^2}$ leads to $(6y - 1 - x^2)^2 = 16y(2y-1)$. This shows that $y(2y-1)$ must be a perfect square, which is achieved by taking $y = k^2, 2y - 1 = \ell^2$ where k, ℓ are positive integers satisfying $\ell^2 - 2k^2 = -1$. From classical results about Fermat-Pell equations, we can take k, ℓ defined by $\ell + k\sqrt{2} = (1 + \sqrt{2})^{2n+1}$ for some positive integer n. Taking successively $n = 1,2,3,4$ we obtain $k_1 = 5, k_2 = 29, k_3 = 169, k_4 = 985$. Thus, $y_1 = 5^2, y_2 = 29^2, y_3 = 169^2, y_4 = 985^2$ are possible solutions. Checking (as pointed above) shows that they are indeed solutions.

Also solved by Daniel Văcaru, Pitești, Romania; Albert Stadler, Herrliberg, Switzerland; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News,

VA; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; and the proposer.

• **5656** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Find $\lim\limits_{n\to\infty} n\left(\dfrac{\pi^2}{4} - a_n^2\right)$ where $a_n = \displaystyle\sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right)$.

Solution 1 by Toyesh Prakash Sharma (Student) Agra College, Agra, India.

As $\displaystyle\sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right) = \sum_{k=1}^{n} \arctan\left(\frac{k-(k-1)}{1+k(k-1)}\right) = \sum_{k=1}^{n}\big(\arctan(k) - \arctan(k-1)\big) =$ $\arctan n$.

Now, applying Stolz-Cesaro result to the given limit. Then,

$$\lim_{n\to\infty} \frac{\left(\frac{\pi^2}{4} - a_n^2\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(\frac{\pi^2}{4} - a_n^2\right) - \left(\frac{\pi^2}{4} - a_{n-1}^2\right)}{\frac{1}{n} - \frac{1}{n-1}} = \lim_{n\to\infty} \frac{a_{n-1}^2 - a_n^2}{\frac{1}{n} - \frac{1}{n-1}}$$

$$= \lim_{n\to\infty} \frac{\left(\tan^{-1} n + \tan^{-1}(n-1)\right)\left(\tan^{-1} n - \tan^{-1}(n-1)\right)}{-\left(\frac{1}{n} - \frac{1}{n-1}\right)} = \pi \bullet \lim_{n\to\infty} \frac{\tan^{-1} n - \tan^{-1}(n-1)}{-\left(\frac{1}{n} - \frac{1}{n-1}\right)}$$

By using L Hospital rule

$$\pi \bullet \lim_{n\to\infty} \frac{\tan^{-1} n - \tan^{-1}(n-1)}{-\left(\frac{1}{n} - \frac{1}{n-1}\right)} = \pi \bullet \lim_{n\to\infty} \frac{\frac{1}{1+n^2} - \frac{1}{1+(1-n)^2}}{\frac{1}{n^2} - \frac{1}{(1-n)^2}} = \pi$$

Solution 2 by Daniel Văcaru, Pitești, Romania.

We write

$$\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan\left(\frac{k-(k-1)}{1+k(k-1)}\right) = \arctan k - \arctan(k-1).$$

It follows that

$$a_n = [k = 1]n\sum \arctan\left(\frac{1}{k^2-k+1}\right) = [k = 1]n\sum\left[\arctan k - \arctan(k-1)\right] = \arctan n.$$

We have

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - a_n^2\right) = \lim_{n\to\infty} n\left[\left(\frac{\pi}{2}\right)^2 - (\arctan n)^2\right] = \lim_{n\to\infty} n\left(\frac{\pi}{2} - \arctan n\right)\left(\arctan n + \frac{\pi}{2}\right),$$

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4}-a_n^2\right)=\lim_{n\to\infty}\left(\arctan n+\frac{\pi}{2}\right)\cdot\lim_{n\to\infty}\frac{\frac{\pi}{2}-\arctan n}{\frac{1}{n}}=\pi\cdot\lim_{n\to\infty}\frac{\frac{\pi}{2}-\arctan n}{\frac{1}{n}}.$$

We use Césaro - Stolz for

$$\lim_{n\to\infty}\frac{\frac{\pi}{2}-\arctan n}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\arctan(n+1)-\arctan n}{\frac{1}{n^2+n}}=\lim_{n\to\infty}\frac{\arctan\frac{1}{n^2+n+1}}{\frac{1}{n^2+n}},$$

$$\lim_{n\to\infty}\frac{\frac{\pi}{2}-\arctan n}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\arctan\frac{1}{n^2+n+1}}{\frac{1}{n^2+n+1}}\cdot\lim_{n\to\infty}\frac{\frac{1}{n^2+n+1}}{\frac{1}{n^2+n}}=1\cdot 1=1.$$

Consequently, our limit is π.

Solution 3 by Ajay Srinivasan, University of Southern California, Los Angeles, CA.

It is given that

$$a_n:=\sum_{k=1}^{n}\tan^{-1}\left(\frac{1}{k^2-k+1}\right)$$

Notice $\tan^{-1}\left(\frac{1}{k^2-k+1}\right)=\tan^{-1}\left(\frac{k-(k-1)}{1+k(k-1)}\right)=\tan^{-1}(k)-\tan^{-1}(k-1)$. Thus:

$$a_n=\sum_{k=1}^{n}(\tan^{-1}(k)-\tan^{-1}(k-1))$$

This is a telescoping sum that yields $a_n=\tan^{-1}(n)\ \forall n\in\mathbb{Z}^+$. We now calculate the limit $\lim_{n\to\infty} n\left(\frac{\pi^2}{4}-\arctan^2(n)\right)$. Say $f(x)=x\left(\frac{\pi^2}{4}-\arctan^2(x)\right)$. Observe that

$$f'(x)=\frac{\pi^2}{4}-(\tan^{-1}(x))^2-\frac{2x\tan^{-1}(x)}{1+x^2}>0$$

when $x\in\mathbb{R}^+$. Thus f is monotonic. It now suffices to evaluate $\lim_{x\to\infty} x\left(\frac{\pi^2}{4}-\arctan^2(x)\right)$. A simple calculation using L'Hôpital's Rule yields:

$$\begin{aligned}\lim_{x\to\infty}\frac{\left(\frac{\pi^2}{4}-\arctan^2(x)\right)}{\frac{1}{x}}&=\lim_{x\to\infty}\frac{2x^2\tan^{-1}(x)}{1+x^2}\\&=\lim_{x\to\infty}\frac{2\tan^{-1}(x)}{1+x^{-2}}\\&=2\left(\frac{\pi}{2}\right)=\pi\end{aligned}\qquad(2)$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

By the addition theorem for the arctan function,

$$\arctan\frac{1}{k-1} - \arctan\frac{1}{k} = \arctan\frac{1}{k^2-k+1}.$$

Hence

$$a_n = \arctan 1 + \sum_{k=2}^{n}\left(\arctan\frac{1}{k-1} - \arctan\frac{1}{k}\right) = 2\arctan 1 - \arctan\frac{1}{n} = \frac{\pi}{2} - \arctan\frac{1}{n}$$

and

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - a_n^2\right) = \lim_{n\to\infty} n\left(\frac{\pi}{2} - a_n\right)\left(\frac{\pi}{2} + a_n\right) = \pi\lim_{n\to\infty} n\left(\arctan\frac{1}{n}\right) = \pi.$$

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By induction it may be proved that $\sum_{k=1}^{n}\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan n = \frac{\pi}{2} - \arctan\frac{1}{n}$, so the limit becomes

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - \left(\frac{\pi}{2} - \arctan\frac{1}{n}\right)^2\right) = \lim_{n\to\infty} n\left(\pi\arctan\frac{1}{n} - \arctan^2\frac{1}{n}\right) = \pi.$$

Solution 6 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

For $k > 1$,

$$\arctan\left(\frac{1}{k-1}\right) - \arctan\left(\frac{1}{k}\right) = \arctan\frac{\frac{1}{k-1} - \frac{1}{k}}{1 + \frac{1}{k^2-k}} = \arctan\left(\frac{1}{k^2-k+1}\right),$$

so

$$\begin{aligned} a_n &= \sum_{k=1}^{n}\arctan\left(\frac{1}{k^2-k+1}\right) \\ &= \frac{\pi}{4} + \sum_{k=2}^{n}\left(\arctan\left(\frac{1}{k-1}\right) - \arctan\left(\frac{1}{k}\right)\right) \\ &= \frac{\pi}{2} - \arctan\left(\frac{1}{n}\right). \end{aligned}$$

Therefore,

$$\frac{\pi^2}{4} - a_n^2 = \pi\arctan\left(\frac{1}{n}\right) - \left(\arctan\left(\frac{1}{n}\right)\right)^2.$$

By L'Hôpital's Rule,

$$\lim_{n\to\infty} n \arctan\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\arctan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+1/n^2}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = 1;$$

it then follows that

$$\lim_{n\to\infty} n \left(\arctan\left(\frac{1}{n}\right)\right)^2 = \lim_{n\to\infty} n \arctan\left(\frac{1}{n}\right) \cdot \lim_{n\to\infty} \arctan\left(\frac{1}{n}\right) = 1 \cdot 0 = 0.$$

Finally,

$$\lim_{n\to\infty} n \left(\frac{\pi^2}{4} - a_n^2\right) = \pi.$$

Remark by Solver: This problem is a generalization of Problem 868 from *The Pentagon*, the journal of the Kappa Mu Epsilon mathematics honor society and a special case of Problem 424 from *La Gaceta de la RSME*.

Solution 7 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

Observe that by the angle addition formula for tangent,

$$a_n = \sum_{k=1}^{n} \arctan\left(\frac{k-(k-1)}{1+k(k-1)}\right) = \sum_{k=1}^{n} \left[\arctan(k) - \arctan(k-1)\right] = \arctan(n).$$

The desired limit thus yields a $0 \cdot \infty$ indeterminant form, and so we use L'Hôpital's rule to get

$$\lim_{n\to\infty} n \left(\frac{\pi^2}{4} - a_n^2\right) = \lim_{x\to\infty} \frac{\frac{\pi^2}{4} - (\arctan(x))^2}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{2\arctan(x)}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2x^2 \arctan(x)}{1+x^2} = \pi.$$

Solution 8 by David Huckaby, Angelo State University, San Angelo, TX.

Making use of the identity $\arctan\alpha - \arctan\beta = \arctan\dfrac{\alpha-\beta}{1+\alpha\beta}$, we have

$$\begin{aligned} a_n &= \sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right) = \sum_{k=1}^{n} \arctan\left(\frac{k-(k-1)}{1+k(k-1)}\right) \\ &= \sum_{k=1}^{n} \left[\arctan k - \arctan(k-1)\right] = \arctan n - \arctan 0 = \arctan n. \end{aligned}$$

So

$$\lim_{n\to\infty} n \left(\frac{\pi^2}{4} - a_n^2\right) = \lim_{n\to\infty} n \left[\frac{\pi^2}{4} - (\arctan n)^2\right],$$

which yields the indeterminate form $\infty \cdot 0$.

Rearranging to obtain the indeterminate form $\dfrac{0}{0}$ and then applying L'Hôpital's Rule gives

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - a_n^2\right) = \lim_{n\to\infty} n\left[\frac{\pi^2}{4} - (\arctan n)^2\right] = \lim_{n\to\infty} \frac{\frac{\pi^2}{4} - (\arctan n)^2}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{-2(\arctan n)\cdot \frac{1}{1+n^2}}{-\frac{1}{n^2}} = 2\left(\lim_{n\to\infty} \arctan n\right)\left(\lim_{n\to\infty} \frac{n^2}{1+n^2}\right) = 2\left(\frac{\pi}{2}\right)(1) = \pi.$$

Solution 9 by David E. Manes, Oneonta, NY.

The value of the limit is π.

We begin with the following identity:

$$\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan\left(\frac{k+(1-k)}{1-k(1-k)}\right) = \arctan(k) + \arctan(1-k).$$

Therefore,

$$a_n = \sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right) = \sum_{k=1}^{n}[\arctan(k) + \arctan(1-k)] = \arctan(n)$$

since the sum telescopes using the facts that $\arctan 0 = 0$ and the identity $\arctan(-k) = -\arctan(k)$ for $1 \leqslant k \leqslant n-1$. Hence, $a_n^2 = (\arctan n)^2$. Then

$$\lim_{n\to\infty}\left(\frac{\pi^2}{4} - a_n^2\right) = \frac{\pi^2}{4} - \left(\frac{\pi}{2}\right)^2 = 0.$$

Since $\lim\limits_{n\to\infty} n = \infty$, it follows that the proposed limit is an indeterminate product limit of type $\infty \cdot o$. To evaluate it, define the functions $f(x) = x$ and $g(x) = (\pi^2/4) - (\arctan x)^2$for real numbers x. Note that $\lim\limits_{n\to\infty}\left(\dfrac{1}{f(x)}\right) = 0 = \lim\limits_{n\to\infty} g(x)$. Therefore, using L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{g(x)}{(1/f(x))} = \lim_{n\to\infty} \frac{\frac{d}{dx}\left(\frac{\pi^2}{4} - (\arctan x)^2\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty}\left(\frac{-2\arctan x\left(\frac{1}{1+x^2}\right)}{(-1)(x^{-2})}\right)$$

$$= \lim_{n\to\infty}\left(2\arctan x\left(\frac{x^2}{1+x^2}\right)\right) = 2\left(\frac{\pi}{2}\right)(1) = \pi.$$

Hence,

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - (\arctan n)^2\right) = \lim_{n\to\infty} 2\arctan n\left(\frac{n^2}{1+n^2}\right) = \pi.$$

This completes the solution.

Solution 10 by Michel Bataille, Rouen, France.

Recall that $\arctan(a) + \arctan(b) = \arctan\left(\dfrac{a+b}{1-ab}\right)$ whenever $ab < 1$. In particular for $k > 1$, we have

$$\arctan\left(\frac{1}{k-1}\right) + \arctan\left(-\frac{1}{k}\right) = \arctan\left(\frac{\frac{1}{k-1} - \frac{1}{k}}{1 + \frac{1}{k(k-1)}}\right) = \arctan\left(\frac{1}{k^2-k+1}\right).$$

Since $\arctan(-x) = -\arctan(x)$ and $\arctan(1) = \dfrac{\pi}{4}$, it follows that for $n \geqslant 2$ we have

$$a_n = \frac{\pi}{4} + \sum_{k=2}^{n}\left(\arctan\left(\frac{1}{k-1}\right) - \arctan\left(\frac{1}{k}\right)\right) = \frac{\pi}{4} + \frac{\pi}{4} - \arctan\left(\frac{1}{n}\right) = \frac{\pi}{2} - \arctan\left(\frac{1}{n}\right).$$

We deduce that

$$\begin{aligned} n\left(\frac{\pi^2}{4} - a_n^2\right) = n\left(\frac{\pi}{2} + a_n\right)\left(\frac{\pi}{2} - a_n\right) &= n\left(\pi - \arctan\left(\frac{1}{n}\right)\right)\arctan\left(\frac{1}{n}\right) \\ &= \left(\pi - \arctan\left(\frac{1}{n}\right)\right)\frac{\arctan\left(\frac{1}{n}\right)}{\frac{1}{n}}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \arctan\left(\dfrac{1}{n}\right) = 0$ and $\lim\limits_{x\to 0} \dfrac{\arctan(x)}{x} = 1$, we finally obtain

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - a_n^2\right) = \pi.$$

Solution 11 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

Denote the limit to be found by L. We claim $L = \pi$. From the following well-known identity for the inverse tangent function

$$\arctan(u) - \arctan(v) = \arctan\left(\frac{u-v}{1+uv}\right), \quad uv > -1,$$

we see that

$$\arctan(k) - \arctan(k-1) = \arctan\left(\frac{1}{k^2-k+1}\right),$$

for all $k \in \mathbb{N}$. Thus

$$a_n = \sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right) = \sum_{k=1}^{n}\big[\arctan(k)-\arctan(k-1)\big] = \arctan(n),$$

since the sum telescopes.

We now find an asymptotic expansion for the term $\arctan^2(n)$ as n approaches infinity. From the property

$$\arctan(z) = \frac{\pi}{2} - \arctan\left(\frac{1}{z}\right), \quad z > 0,$$

we can write

$$\begin{aligned}
\arctan^2(n) &= \left(\frac{\pi}{2} - \arctan\left(\frac{1}{n}\right)\right)^2 = \left(\frac{\pi}{2}\right)^2\left(1-\frac{2}{\pi}\arctan\left(\frac{1}{n}\right)\right)^2 \\
&= \left(\frac{\pi}{2}\right)^2\left[1-\frac{2}{\pi}\left\{\frac{1}{n} - O\left(\frac{1}{n^3}\right)\right\}\right]^2 \\
&= \left(\frac{\pi}{2}\right)^2\left[1-\frac{2}{n}\left(\frac{2}{\pi}\right)+\frac{1}{n^2}\left(\frac{2}{\pi}\right)^2 + O\left(\frac{1}{n^3}\right)\right] \\
&= \left(\frac{\pi}{2}\right)^2 - \frac{2}{n}\left(\frac{\pi}{2}\right) + \frac{1}{n^2} + O\left(\frac{1}{n^3}\right).
\end{aligned} \tag{3}$$

Note in the second line the well-known Maclaurin series expansion for $\arctan(x)$ has been used. Returning to the limit we find

$$L = \lim_{n\to\infty} n\left[\frac{2}{n}\left(\frac{\pi}{2}\right) - \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)\right] = \lim_{n\to\infty}\left[\pi - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right] = \pi,$$

as claimed.

Solution 12 by Moti Levy, Rehovot, Israel.

Theorem: Let f be of fixed sign and define

$$h(x) = \frac{f(x+1)-f(x)}{1+f(x+1)f(x)}.$$

Then

$$\sum_{k=1}^{n} \arctan h(k) = \arctan\big(f(n+1)\big) - \arctan\big(f(1)\big).$$

Proof of Theorem: Since $\arctan h(k) = \arctan\big(f(k+1)\big) - \arctan\big(f(k)\big)$, the statement follows by telescoping.

In this case, set $f(x) := x-1$, then $\dfrac{1}{x^2-x+1} = \dfrac{x-(x-1)}{1+x(x-1)}$, hence by the theorem,

$$a_n = \sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right) = \arctan(n).$$

Now we find asymptotic expression for a_n and a_n^2.

$$\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right)$$

$$\arctan\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{3x^3} + O\left(\frac{1}{x^5}\right)$$

$$a_n = \arctan(n) = \frac{\pi}{2} - \frac{1}{n} + \frac{1}{3n^3} + O\left(\frac{1}{n^5}\right)$$

$$a_n^2 = \frac{\pi^2}{4} - \frac{\pi}{n} + O\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4} - a_n^2\right) = \lim_{n\to\infty} n\left(\frac{\pi^2}{4} - \left(\frac{\pi^2}{4} - \frac{\pi}{n} + O\left(\frac{1}{n^2}\right)\right)\right) = \pi.$$

Solution 13 by G. C. Greubel, Newport News, VA.

First note that

$$\begin{aligned}\sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{k^2-k+1}\right) &= \sum_{k=1}^{n} \tan^{-1}\left(\frac{k-(k-1)}{1+k(k-1)}\right) \\ &= \sum_{k=1}^{n} \left(\tan^{-1}(k) - \tan^{-1}(k-1)\right) \\ &= \tan^{-1}(n).\end{aligned}$$

Now, for $x > 0$, $\tan^{-1}(x) = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{1}{x}\right)$ and

$$\begin{aligned}\left(\tan^{-1}(x)\right)^2 &= \frac{\pi^2}{4} - \pi \tan^{-1}\left(\frac{1}{x}\right) + \left(\tan^{-1}\left(\frac{1}{x}\right)\right)^2 \\ &\approx \frac{\pi^2}{4} - \frac{\pi}{x} + \frac{1}{x^2} + \frac{\pi}{3x^3} + O\left(\frac{1}{x^4}\right)\end{aligned}$$

which leads to

$$\frac{\pi^2}{4} - \left(\tan^{-1}(n)\right)^2 \approx \frac{\pi}{n} - \frac{1}{n^2} - \frac{\pi}{n^3} + O\left(\frac{1}{n^4}\right)$$

and

$$n\left(\frac{\pi^2}{4}-\left(\tan^{-1}(n)\right)^2\right)\approx \pi-\frac{1}{n}-\frac{\pi}{3\,n^2}+O\left(\frac{1}{n^3}\right).$$

The limit then follows and yields

$$\lim_{n\to\infty} n\left(\frac{\pi^2}{4}-\left(\tan^{-1}(n)\right)^2\right)=\pi.$$

Also solved by Ankush Kumar Parcha, Indira Gandhi National Open University, New Delhi, India; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Hatef Arshagi, Guilford technical Community College, Jamestown, NC; Arkady Alt, San Jose, CA; Marian Ursărescu-National College "Roman-Vodă", Roman City, Romania; Péter Fülöp, Gyömrő, Hungary; and the proposer.

• **5657** *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy.*

Let $\alpha, \beta, \gamma > 0$ and define

$$S_n=\sum_{m=2}^{n}(\ln m)^{\gamma}\prod_{k=2}^{m}\frac{\alpha+k\ln k}{\beta+(k+1)\ln(k+1)}.$$

Determine sufficient and necessary condition(s) governing the parameters α, β and γ so that $\lim\limits_{n\to\infty} S_n$ exists.

Solution 1 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

We claim that the limit exists if and only if $0<\gamma<\beta-\alpha$. To see why, observe that if $\alpha \geqslant \beta$, then

$$\begin{aligned}
S_n &\geqslant \sum_{m=2}^{n}(\ln m)^{\gamma}\prod_{k=2}^{m}\frac{\beta+k\ln k}{\beta+(k+1)\ln(k+1)}\\
&=\sum_{m=2}^{n}(\ln m)^{\gamma}\frac{\beta+2\ln 2}{\beta+(m+1)\ln(m+1)}\\
&>\frac{(\ln 2)^{\gamma}(\beta+2\ln 2)}{\beta+3\ln 3}+\sum_{m=3}^{n}\frac{\beta+2\ln 2}{\beta+(m+1)\ln(m+1)}
\end{aligned}$$

for $n \geqslant 3$. Since $\displaystyle\sum_{m=1}^{\infty}\frac{1}{(m+1)\ln(m+1)}$ is a divergent series, it follows that the sequence $(S_n)_{n\geqslant 2}$ diverges as well due to the Limit Comparison Test.

Meanwhile, if $\alpha < \beta$, then

$$\begin{aligned} S_n &= \sum_{m=2}^{n} (\ln m)^{\gamma} \frac{\alpha + 2\ln 2}{\beta + (m+1)\ln(m+1)} \prod_{k=3}^{m} \frac{\alpha + k\ln k}{\beta + k\ln k} \\ &= \sum_{m=2}^{n} \frac{(\ln m)^{\gamma}(\alpha + 2\ln 2)}{\beta + (m+1)\ln(m+1)} \prod_{k=3}^{m} \left(1 - \frac{\beta-\alpha}{\beta + k\ln k}\right) \\ &= \sum_{m=2}^{n} \frac{(\ln m)^{\gamma}(\alpha + 2\ln 2)}{\beta + (m+1)\ln(m+1)} \prod_{k=3}^{m} \left[e^{-(\beta-\alpha)/(\beta+k\ln k)} - c_k\left(\frac{\beta-\alpha}{\beta + k\ln k}\right)^2\right], \end{aligned}$$

where, by virtue of Taylor's theorem, $\frac{1}{2}\exp(-\frac{\beta-\alpha}{\beta+k\ln k}) \leqslant c_k \leqslant \frac{1}{2}$. We further manipulate S_n to get

$$S_n = \sum_{m=2}^{n} \frac{(\ln m)^{\gamma}(\alpha + 2\ln 2)}{\beta + (m+1)\ln(m+1)} \prod_{k=3}^{m} e^{-(\beta-\alpha)/(\beta+k\ln k)} \left[1 - c_k\left(\frac{\beta-\alpha}{\beta + k\ln k}\right)^2 e^{(\beta-\alpha)/(\beta+k\ln k)}\right]$$

Note that the limit of the product can be reorganized to get

$$\begin{aligned} \prod_{k=3}^{\infty} \left[1 - c_k\left(\frac{\beta-\alpha}{\beta + k\ln k}\right)^2 e^{(\beta-\alpha)/(\beta+k\ln k)}\right] &= \exp\left(\sum_{k=3}^{\infty} \log\left(1 - c_k\left(\frac{\beta-\alpha}{\beta + k\ln k}\right)^2 e^{(\beta-\alpha)/(\beta+k\ln k)}\right)\right) \\ &= \exp\left(\sum_{k=3}^{\infty}\sum_{j=1}^{\infty} \frac{1}{j}\left(c_k\left(\frac{\beta-\alpha}{\beta + k\ln k}\right)^2 e^{(\beta-\alpha)/(\beta+k\ln k)}\right)^j\right). \end{aligned}$$

Since $c_k e^{(\beta-\alpha)/(\beta+k\ln k)} < 1$ and $\ln k > \beta$ for sufficiently large k, we can bound the tail of the double sum inside of the exponential by

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{\beta-\alpha}{\beta(1+k)}\right)^{2j} \leqslant \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{(1-\alpha/\beta)^{2j}}{jk^{2j}} = \sum_{j=1}^{\infty} \frac{(1-\alpha/\beta)^{2j}\zeta(2j)}{j}, \tag{4}$$

which converges by virtue of the Root Test since $\sqrt[j]{(1-\alpha/\beta)^{2j})\zeta(2j)/j}$ can be sandwiched between $(1-\alpha/\beta)^2\sqrt[j]{1/j}$ and $(1-\alpha/\beta)^2\sqrt[j]{\zeta(2)/j}$, both of which tend to $(1-\alpha/\beta)^2 < 1$ as $j \to \infty$. We can thus appeal to the Limit Comparison Test to find that $(S_n)_{n\geqslant 2}$ and

$$\sum_{m=2}^{\infty} \frac{(\ln m)^{\gamma}}{(m+1)\ln(m+1)} \prod_{k=3}^{m} e^{-(\beta-\alpha)/(k\ln k)}$$

either both converge or both diverge. If $\gamma \geqslant \beta - \alpha$, then

$$\begin{aligned} \sum_{m=2}^{\infty} \frac{(\ln m)^{\gamma}}{(m+1)\ln(m+1)} \prod_{k=3}^{m} e^{-(\beta-\alpha)/(k\ln k)} &\geqslant \sum_{m=2}^{\infty} \frac{(\ln m)^{\gamma}}{(m+1)\ln(m+1)} \exp\left(-\frac{\beta-\alpha}{3\ln 3} - \int_3^m \frac{\beta-\alpha}{x\ln x}\,dx\right) \\ &\geqslant \sum_{m=2}^{\infty} \frac{e^{-\frac{\beta-\alpha}{3\ln 3}}(\ln m)^{\gamma}}{(m+1)\ln(m+1)} \exp\big((\beta-\alpha)\ln\ln 3 - (\beta-\alpha)\ln\ln m\big) \\ &= \sum_{m=2}^{\infty} \frac{e^{-\frac{\beta-\alpha}{3\ln 3}}(\ln 3)^{\beta-\alpha}(\ln m)^{\gamma-\beta+\alpha}}{(m+1)\ln(m+1)}, \end{aligned}$$

which can be seen to diverge by comparison to $\sum_{m=1}^{\infty} \frac{1}{(m+1)\ln(m+1)}$. However, if $\gamma < \beta - \alpha$, then

$$\begin{aligned}\sum_{m=2}^{\infty} \frac{(\ln m)^{\gamma}}{(m+1)\ln(m+1)} \prod_{k=3}^{m} e^{-(\beta-\alpha)/(k\ln k)} &\leqslant \sum_{m=2}^{\infty} \frac{(\ln m)^{\gamma}}{(m+1)\ln(m+1)} \exp\left(-\int_3^m \frac{\beta-\alpha}{x\ln x}\,dx\right)\\ &\leqslant \sum_{m=2}^{\infty} \frac{(\ln 3)^{\beta-\alpha}(\ln m)^{\gamma-\beta+\alpha}}{(m+1)\ln(m+1)}\\ &\leqslant \sum_{m=2}^{\infty} \frac{(\ln 3)^{\beta-\alpha}(\ln m)^{\gamma-\beta+\alpha-1}}{m},\end{aligned}$$

which can be seen to converge due to the integral test. Indeed

$$\int_2^{\infty} \frac{(\ln x)^{\gamma-\beta+\alpha-1}}{x}\,dx = -\frac{(\ln 2)^{\gamma-\beta+\alpha}}{(\gamma-\beta+\alpha)} + \lim_{b\to\infty} \frac{(\ln b)^{\gamma-\beta+\alpha}}{(\gamma-\beta+\alpha)} = -\frac{(\ln 2)^{\gamma-\beta+\alpha}}{2(\gamma-\beta+\alpha)},$$

since $\gamma - \beta + \alpha < 0$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\prod_{k=2}^{m} \frac{\alpha + k\ln k}{\beta + (k+1)\ln(k+1)} = \prod_{k=2}^{m} \frac{\alpha + k\ln k}{\beta + k\ln k} \bullet \prod_{k=2}^{m} \frac{\beta + k\ln k}{\beta + (k+1)\ln(k+1)} =$$

$$= \exp\left(\sum_{k=2}^{m} \ln\left(1 + \frac{\alpha}{k\ln k}\right)\right) \exp\left(-\sum_{k=2}^{m} \ln\left(1 + \frac{\beta}{k\ln k}\right)\right) \frac{\beta + 2\ln 2}{\beta + (m+1)\ln(m+1)}.$$

If f(x) is a continuously differentiable function on R_0 then integration by parts yields

$$\int_k^{k+1} f(x)\,dx = \frac{1}{2} f(k+1) + \frac{1}{2} f(k) - \int_k^{k+1} \left(x - [x] - \frac{1}{2}\right) f'(x)\,dx.$$

In particular, if $f(x) = \frac{1}{x\ln x}$, then

$$\ln\ln m \;\; - \ln\ln 2 \;\; = \int_2^m \frac{1}{x\ln x} dx =$$

$$= \frac{1}{2} \sum_{j=2}^{m-1} \left(\frac{1}{j\ln j} + \frac{1}{(j+1)\ln(j+1)}\right) - \int_2^m \left(x - [x] - \frac{1}{2}\right) \frac{d}{dx}\left(\frac{1}{x\ln x}\right) dx$$

and

$$\sum_{j=2}^{m} \frac{1}{j\ln j} = \frac{1}{4\ln 2} + \frac{1}{2m\ln m} + \ln\ln m \;\; - \ln\ln 2 \;\; + \int_2^{\infty} \left(x - [x] - \frac{1}{2}\right) \frac{d}{dx}\left(\frac{1}{x\ln x}\right) dx$$

$$-\int_m^\infty \left(x-[x]-\frac{1}{2}\right)\frac{d}{dx}\left(\frac{1}{x\ln x}\right)dx.$$

We conclude that

$$\sum_{j=2}^m \frac{1}{j\ln j} = \ln\ln m + c_1 + O\left(\frac{1}{m\ln m}\right),$$

and

$$\sum_{k=2}^m \ln\left(1+\frac{\alpha}{k\ln k}\right) = \sum_{k=2}^m \frac{\alpha}{k\ln k} + \sum_{k=2}^\infty \left(\ln\left(1+\frac{\alpha}{k\ln k}\right) - \frac{\alpha}{k\ln k}\right) - \sum_{k=m+1}^\infty \left(\ln\left(1+\frac{\alpha}{k\ln k}\right) - \frac{\alpha}{k\ln k}\right) =$$

$$= \alpha\ln\ln m + c_2(\alpha) + O_\alpha\left(\frac{1}{m\ln m}\right) + \sum_{k=m+1}^\infty O_\alpha\left(\frac{1}{k^2\ln^2 k}\right) = \alpha\ln\ln m + c_2(\alpha) + O_\alpha\left(\frac{1}{m\ln m}\right).$$

Finally

$$\prod_{k=2}^m \frac{\alpha+k\ln k}{\beta+(k+1)\ln(k+1)} =$$

$$= \exp\left((\alpha-\beta)\ln\ln m + c_2(\alpha) - c_2(\beta) + O_{\alpha,\beta}\left(\frac{1}{m\ln m}\right)\right)\frac{\beta+2\ln 2}{\beta+(m+1)\ln(m+1)} =$$

$$= \frac{1}{m}c_3(\alpha,\beta)\ln^{\alpha-\beta-1}(m)\left(1+O_{\alpha,\beta}\left(\frac{1}{m\ln m}\right)\right).$$

Hence the limit of S_n as n tends to infinity exists if and only if $\beta>\alpha+\gamma$, since $\sum_{m=2}^\infty \frac{1}{m\ln^r m}$ converges if and only if $r>1$.

Solution 3 by Michel Bataille, Rouen, France.

We will use the following well-known result: if r, s are real numbers, the series $\sum_{m\geqslant 2}\frac{1}{m^r(\ln m)^s}$ is convergent if and only if ($r>1$) or ($r=1$ and $s>1$).

Let $P_m = \prod_{k=2}^m \frac{\alpha+k\ln k}{\beta+(k+1)\ln(k+1)}$. The problem amounts to determining in which case(s) the series $\sum_{m\geqslant 2}(\ln m)^\gamma P_m$ is convergent.

If $\alpha=\beta$, then $P_m = \frac{\alpha+2\ln 2}{\alpha+(m+1)\ln(m+1)} \sim \frac{\alpha+2\ln 2}{m\ln m}$ as $m\to\infty$ so that $(\ln m)^\gamma P_m \sim \frac{\alpha+2\ln 2}{m(\ln m)^{1-\gamma}}$ and therefore the series $\sum_{m\geqslant 2}(\ln m)^\gamma P_m$ is divergent (since $1-\gamma<1$).

Now, we suppose that $\alpha\neq\beta$ and we set $Q_m = \prod_{k=2}^m \frac{\alpha+k\ln k}{\beta+k\ln k}$. Clearly, we have, as $m\to\infty$

$$P_m = (\beta+2\ln 2)\cdot Q_m\cdot\frac{1}{\beta+(m+1)\ln(m+1)} \sim (\beta+2\ln 2)\cdot\frac{Q_m}{m\ln m}. \tag{1}$$

We have $\dfrac{\alpha + k\ln k}{\beta + k\ln k} = 1 + \dfrac{\alpha-\beta}{\beta + k\ln k}$ and

$$\ln\left(1+\frac{\alpha-\beta}{\beta+k\ln k}\right) - \frac{\alpha-\beta}{\beta+k\ln k} \;\sim\; \frac{-(\alpha-\beta)^2}{2(\beta+k\ln k)^2} \;\sim\; \frac{-(\alpha-\beta)^2}{2k^2(\ln k)^2}$$

as $k \to \infty$. Therefore the series $\displaystyle\sum_{k\geqslant 2}\left(\ln\left(1+\frac{\alpha-\beta}{\beta+k\ln k}\right) - \frac{\alpha-\beta}{\beta+k\ln k}\right)$ is convergent. Let C_1 be its sum. We deduce that as $m \to \infty$

$$\sum_{k=2}^{m}\ln\left(1+\frac{\alpha-\beta}{\beta+k\ln k}\right) = \sum_{k=2}^{m}\frac{\alpha-\beta}{\beta+k\ln k} + C_1 + o(1).$$

Similarly, as $k \to \infty$, we have

$$\frac{1}{\beta+k\ln k} - \frac{1}{k\ln k} = \frac{-\beta}{k\ln k(\beta+k\ln k)} \;\sim\; \frac{-\beta}{k^2(\ln k)^2}$$

and therefore

$$\sum_{k=2}^{m}\frac{1}{\beta+k\ln k} = \sum_{k=2}^{m}\frac{1}{k\ln k} + C_2 + o(1)$$

as $m \to \infty$ for some constant C_2.
Finally, since the function $f(x) = \dfrac{1}{x\ln x}$ is positive, nonincreasing on $[2,\infty)$, the difference

$$\delta_m = \sum_{k=2}^{m}\frac{1}{k\ln k} - \int_2^m \frac{dx}{x\ln x}$$

has a finite limit as $m \to \infty$. [From the inequalities $f(k+1) \leqslant \displaystyle\int_k^{k+1} f(t)\,dt \leqslant f(k)$ $(k = 2,3,\dots,m)$, we deduce that $\delta_m \geqslant 0$ and $\delta_{m+1} - \delta_m = f(m+1) - \displaystyle\int_m^{m+1} f(x)\,dx \leqslant 0$ for $m \geqslant 2$. Thus, (δ_m) is nonincreasing and bounded below, hence convergent.]
It follows that

$$\sum_{k=2}^{m}\frac{1}{k\ln k} = \ln(\ln m) + C_3 + o(1)$$

as $m \to \infty$ for some constant C_3.
Gathering the results provides

$$\sum_{k=2}^{m}\ln\left(1+\frac{\alpha-\beta}{\beta+k\ln k}\right) = (\alpha-\beta)\ln(\ln m) + C_4 + o(1)$$

as $m \to \infty$ for some constant C_4.
By exponentiation, we obtain $Q_m \;\sim\; e^{C_4}(\ln m)^{\alpha-\beta}$ as $m \to \infty$ and using (1),

$$(\ln m)^{\gamma} P_m \;\sim\; \frac{C_5}{m(\ln m)^{1-\gamma-\alpha+\beta}}$$

where C_5 is a positive constant. Thus, $\sum\limits_{m\geqslant 2}(\ln m)^{\gamma}P_m$ is convergent if and only if $\beta > \alpha + \gamma$.

Also solved by the proposer.

• **5658** *Proposed by Titu Zvonaru, Comănești, Romania.*

Let a, b and c be positive with $a+b+c=3$. Prove $\dfrac{1}{5+a^3}+\dfrac{1}{5+b^3}+\dfrac{1}{5+c^3}\leqslant\dfrac{1}{2}$.

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Function $f(x)=\dfrac{1}{5+x^3}$ is concave for $x\in(0,1)$ since $f''(x)=\dfrac{6x(2x^3-5)}{(5+x^3)^3}<0$ for $x\in(0,1)$. Therefore, by Jensen's inequality

$$\frac{1}{5+a^3}+\frac{1}{5+b^3}+\frac{1}{5+c^3}\leqslant 3\cdot\frac{1}{5+(\frac{a+b+c}{3})^3}=\frac{3}{6}=\frac{1}{2}.$$

Solution 2 by Arkady Alt, San Jose, CA.

Let $F\left(a,b,c\right):=\dfrac{1}{5+a^3}+\dfrac{1}{5+b^3}+\dfrac{1}{5+c^3}$

Assuming $a=\max\{a.b,c\}$ (due symmetry of the inequality) we will prove that

$F(a,b,c)\leqslant F(a,p,p)$, where $p:=\dfrac{b+c}{2}$.

Since $a\geqslant p$ then $3=a+b+c\geqslant 3p \iff p\leqslant 1$ and also $bc\leqslant\left(\dfrac{b+c}{2}\right)^2=p^2$.

Then $F(a,p,p)-F(a,b,c)=\left(\dfrac{1}{5+p^3}-\dfrac{1}{5+b^3}\right)+\left(\dfrac{1}{5+p^3}-\dfrac{1}{5+c^3}\right)=$

$\dfrac{(b-p)\left(b^2+bp+p^2\right)}{(s^3+5)\left(b^3+5\right)}+\dfrac{(c-p)\left(c^2+cp+p^2\right)}{(p^3+5)\left(c^3+5\right)}=\dfrac{(b-c)\left(b^2+bp+p^2\right)}{2\left(p^3+5\right)\left(b^3+5\right)}-\dfrac{(b-c)\left(c^2+cp+s^2\right)}{2\left(p^3+5\right)\left(c^3+5\right)}=$

$\dfrac{(b-c)\left(\left(b^2+bp+p^2\right)\left(c^3+5\right)-\left(c^2+cp+p^2\right)\left(b^3+5\right)\right)}{2\left(p^3+5\right)\left(b^3+5\right)\left(c^3+5\right)}=$

$\dfrac{(b-c)^2\left(5\left(b+c\right)+5p-b^2c^2-p^2\left(b^2+bc+c^2\right)-bcp\left(b+c\right)\right)}{2\left(p^3+5\right)\left(b^3+5\right)\left(c^3+5\right)}.$

We have $5\left(b+c\right)+5p-b^2c^2-p^2\left(b^2+bc+c^2\right)-bcp\left(b+c\right)=$

$15p-p^2\left(b+c\right)^2-bcp^2-b^2c^2=15p-4p^4-bcp^2-b^2c^2\geqslant 0$ because

$15p-4p^4-bcp^2-b^2c^2\geqslant 15p-p^4-4p^4-p^4=15p-6p^4=3p\left(5-2p^3\right)\geqslant 6p>0.$

Since $a=3-2p$ then $\dfrac{1}{2}-F(a,p,p)=\dfrac{1}{2}-F(3-2p,p,p)=$

$\dfrac{1}{2}-\dfrac{1}{5+(3-2p)^3}-\dfrac{2}{5+p^3}=\dfrac{(1-p)^2\left(11+10p^3-5p-3p^2-4p^4\right)}{(p^3+5)\left(5+(3-2p)^3\right)}\geqslant 0$

because $11+10p^3-5p-3p^2-4p^4=5\left(1-p\right)+3\left(1-p^2\right)+4p^3\left(1-p\right)+3+6p^3>0$

and $5+(3-2p)^3 \geqslant 5+(3-2\cdot 1)^3 = 6$.

Solution 3 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

We use the method of Lagrange multipliers to maximize

$$f(a,b,c) = \frac{1}{5+a^3} + \frac{1}{5+b^3} + \frac{1}{5+c^3}$$

subject to the constraints that a, b, and c are positive real numbers and $g(a,b,c) = a+b+c = 3$. Extrema can be found where $\nabla f = \lambda \nabla g$, or $f_a = f_b = f_c = \lambda$, since $\nabla g = \langle 1,1,1\rangle$. Thus,

$$\frac{-3a^2}{(5+a^3)^2} = \frac{-3b^2}{(5+b^3)^2} = \frac{-3c^2}{(5+c^3)^2}.$$

Since a, b, and c are all positive, so are $5+a^3$, $5+b^3$, and $5+c^3$; hence

$$\frac{a}{5+a^3} = \frac{b}{5+b^3} = \frac{c}{5+c^3},$$

and

$$(b-a)[ab(a+b)-5] = 0 = (c-b)[bc(b+c)-5] = (a-c)[ca(c+a)-5].$$

Suppose $a \neq b$; then $ab(a+b) = 5$. If $b = c$, then $a = 3-2b$ and $a+b = 3-b$, so that $(3-2b)b(3-b) = 5$; but this equation has no solution for $0 < b < 3$. If $b \neq c$, then $bc(b+c) = 5 = ab(a+b)$, so that $c(b+c) = a(a+b)$, $c(3-a) = a(3-c)$, and $a = c$. But then $b = 3-2a$, $a+b = 3-a$, and $a(3-2a)(3-a) = 5$, which also has no solution for $0 < a < 3$. So, the only extreme value of f occurs where $a = b = c = 1$; namely $f(1,1,1) = \frac{3}{6} = \frac{1}{2}$. Since $f(2,1/2,1/2) = \frac{186}{533} < \frac{1}{2}$, then $f(1,1,1) = \frac{1}{2}$ must be the absolute maximum, and

$$\frac{1}{5+a^3} + \frac{1}{5+b^3} + \frac{1}{5+c^3} \leqslant \frac{1}{2}.$$

Solution 4 by Michael Brozinsky, Central Islip, NY and Andrew Bulawa, Brooklyn NY.

Without loss of generality let $a \leqslant b \leqslant c$ and so $a \leqslant 1$ and $a+b \leqslant 2$ and $c = 3-a-b$. Consider the function $f(x,y) = \frac{1}{5+x^3} + \frac{1}{5+y^3} + \frac{1}{(3-x-y)^3}$ where x and y are non negative and $0 < x \leqslant 1$ and $x \leqslant y$ and $x+y \leqslant 2$. On the line $x+y = k$ where k is constant where $0 < k \leqslant 2$ we have $x \leqslant \frac{k}{2}$ and $f(x,k-x) = \frac{1}{5+x^3} + \frac{1}{5+(k-x)^3} + \frac{1}{5+(3-k)^3}$ and

$$\begin{aligned}\frac{d}{dx}\left(f(x,k-x)\right) &= -\frac{3(k-2x)\left(k^2x - kx^2 - 5\right)\left(k^3x - 3k^2x^2 + 4kx^3 - 2x^4 + 5k\right)}{\left(x^3+5\right)^2\left(k^3 - 3k^2x + 3kx^2 - x^3 + 5\right)^2} \\ &= 3(2x-k)\frac{\left(k^2x - kx^2 - 5\right)\left(k^3x - 3k^2x^2 + 4kx^3 - 2x^4 + 5k\right)}{\left(x^3+5\right)^2\left(k^3 - 3k^2x + 3kx^2 - x^3 + 5\right)^2}\end{aligned} \tag{5}$$

We will show this derivative changes sign from positive to negative at $x = \frac{k}{2}$ and thus the maximum of $f(x, y)$ must be on the line $y = x$ since if $x = \frac{k}{2}$ then $y = \frac{k}{2}$ because $x + y = k$. In (5) the denominator factors are clearly positive (note $5 + (k-x)^3 = k^3 - 3k^2x + 3kx^2 - x^3 + 5$. Now the numerator factor $k^2x - kx^2 - 5$ is always **negative** since the discriminant $k^4 - 20k = k\left(k^3 - 20\right) < 0$ and the leading coefficient $-k < 0$. The numerator factor $k^3x - 3k^2x^2 + 4k\,x^3 - 2x^4 + 5k$ using Vieta's substitution $x = z + \frac{k}{2}$ becomes $\frac{1}{8}k^4 - 2z^4 + 5k$ so that its zeros are

$$z = \frac{\left(k^4 + 40k\right)^{\frac{1}{4}}}{2}, z = \frac{i}{2}\left(k^4 + 40k\right)^{\frac{1}{4}}, z = -\frac{\left(k^4 + 40k\right)^{\frac{1}{4}}}{2}, z = -\frac{i}{2}\left(k^4 + 40k\right)^{\frac{1}{4}}$$

and so the zeros for this numerator factor are

$$x = \frac{\left(k^4 + 40k\right)^{\frac{1}{4}}}{2} + \frac{k}{2}, x = \frac{i\left(k^4 + 40k\right)^{\frac{1}{4}}}{2} + \frac{k}{2}, x = -\frac{\left(k^4 + 40k\right)^{\frac{1}{4}}}{2} + \frac{k}{2}, x = -\frac{i\left(k^4 + 40k\right)^{\frac{1}{4}}}{2} + \frac{k}{2}.$$

Now since $\left(k^4 + 40k\right)^{1/4} > k$ the first real zero $x = \frac{\left(k^4 + 40k\right)^{1/4}}{2} + \frac{k}{2} > k$ and the second (and last) real zero $x = -\frac{\left(k^4 + 40k\right)^{1/4}}{2} + \frac{k}{2} < 0$ and so $k^3x - 3k^2x^2 + 4kx^3 - 2x^4 + 5k$ must be of constant sign when $0 \leqslant x \leqslant \frac{k}{2}$ (by the intermediate value theorem). Choosing $x = 0$ in this factor i.e., $k^3x - 3k^2x^2 + 4k\,x^3 - 2x^4 + 5k$ we get $5k > 0$ and so the constant sign for this factor is **positive**. Finally since $2x - k$ changes sign from negative to positive at $x = \frac{k}{2}$, the derivative in question changes sign from positive to negative at $x = \frac{k}{2}$ and thus the maximum of $f(x, y)$ must be on the line $y = x$.

Hence replacing y by x in $f(x, y)$ we get

$$f(x, x) = \frac{1}{5 + x^3} + \frac{1}{5 + x^3} + \frac{1}{(3 - 2x)^3}$$

If we can show this expression increases with x then the maximum value will be when $x = 1$ and will be $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$. (*)

Now

$$\begin{aligned} \frac{d}{dx}\left(f(x, x)\right) &= -\frac{6\left(15x^6 - 96x^5 + 216x^4 - 226x^3 + 81x^2 - 25\right)}{\left(x^3 + 5\right)^2(-3 + 2x)^4} \\ &= -\frac{6\left(15x^6 - 96x^5 + 216x^4 - 226x^3 + 81x^2 - 25\right)}{\left(x^3 + 5\right)^2(-3 + 2x)^4} \\ &= -\frac{6\left(\left(5x^3 - 12x^2 + 9x + 5\right)\left(3x^3 - 12x^2 + 9x - 5\right)\right)}{\left(x^3 + 5\right)^2(-3 + 2x)^4} \end{aligned} \quad (6)$$

Now the factor $(5x^3 - 12x^2 + 9x + 5) > 0$ since $0 < x \leqslant 1$ because $5x^3 - 12x > -12x$ and so $x\left(5x^2 - 12\right) > -12x$ and thus $x\left(5x^2 - x\right) + 9x + 5 > -12x + 9x + 5 = -3x + 5 > 0$ since $x < \dfrac{5}{3}$, but this is just (reading from left to right) $5x^3 - 12x^2 + 9x + 5 > 0$. Now the factor $\left(3x^3 - 12x^2 + 9x - 5\right) < 0$ since $0 < x \leqslant 1$ because $\dfrac{d}{dx}\left(3x^3 - 12x^2 + 9x - 5\right) = 3\left(3x^2 - 8x + 3\right)$ which can be written as $9(x - r_1)(x - r_2)$ where $r_1 = \dfrac{4 + \sqrt{7}}{3}$ and $r_2 = \dfrac{4 - \sqrt{7}}{3}$. Note $0 < r_2 < 1$ and $r_1 > 1$ and so (by the first derivative test) $3x^3 - 12x^2 + 9x - 5$ has an absolute maximum on $[0, 1]$ when $x = \dfrac{4 - \sqrt{7}}{3}$ or $-\dfrac{65}{9} + \dfrac{14\sqrt{7}}{9}$ which is negative. The denominator in (6) is clearly positive and (noticing the negative sign) in front of (6) we have shown (6) is a positive expression and the goal stated in (*) has been verified. Identifying a with x, b with y, and c with $3 - x - y$ completes the proof.

Note the factorization of $15x^6 - 96x^5 + 216x^4 - 226x^3 + 81x^2 - 25$ was obtained via Maple.

Solution 5 by Ajay Srinivasan, University of Southern California, Los Angeles, CA.

This is a constrained optimization problem. Define the constraint function as $g(x, y, z) = x + y + z$, and the objective function as $f(x, y, z) = \dfrac{1}{5 + x^3} + \dfrac{1}{5 + y^3} + \dfrac{1}{5 + z^3}$. It suffices to show that the maximum value of f under the constraint $g(x, y, z) = 3$ is $\dfrac{1}{2}$. The Lagrange multiplier for this scenario is:

$$\mathcal{L}(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z) = \frac{1}{5 + x^3} + \frac{1}{5 + y^3} + \frac{1}{5 + z^3} - \lambda x - \lambda y - \lambda z$$

Evidently,

$$\nabla \mathcal{L} = \left\langle \frac{-3x^2}{(5 + x^3)^2} - \lambda, \frac{-3y^2}{(5 + y^3)^2} - \lambda, \frac{-3z^2}{(5 + z^3)^2} - \lambda \right\rangle$$

$\nabla \mathcal{L} = 0$ implies that

$$\frac{-3x^2}{(5 + x^3)^2} = \frac{-3y^2}{(5 + y^3)^2} = \frac{-3z^2}{(5 + z^3)^2}$$

One possible solution to this is $x = y = z$. Using $x = y = z$ in $g(x, y, z) = 3$ yields that $(x, y, z, \lambda) = (1, 1, 1, \frac{-1}{12})$ is a solution for the Lagrange multiplier problem $\mathcal{L}(x, y, z, \lambda) = 0$ and $g(x, y, z) = 3$. The Bordered Hessian Matrix for $\mathcal{L}$ is:

$$\mathbf{H_4} = \begin{pmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} & \mathcal{L}_{\lambda z} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} & \mathcal{L}_{xz} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} & \mathcal{L}_{yz} \\ \mathcal{L}_{z\lambda} & \mathcal{L}_{zx} & \mathcal{L}_{zy} & \mathcal{L}_{zz} \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & \mathcal{L}_{xx} & 0 & 0 \\ -1 & 0 & \mathcal{L}_{yy} & 0 \\ -1 & 0 & 0 & \mathcal{L}_{zz} \end{pmatrix}$$

Using the values of double-derivatives:

$$\mathbf{H_4} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & \frac{-1}{12} & 0 & 0 \\ -1 & 0 & \frac{-1}{12} & 0 \\ -1 & 0 & 0 & \frac{-1}{12} \end{pmatrix}$$

$$\det(\mathbf{H_4}) = \begin{vmatrix} 0 & -1 & -1 & -1 \\ -1 & \frac{-1}{12} & 0 & 0 \\ -1 & 0 & \frac{-1}{12} & 0 \\ -1 & 0 & 0 & \frac{-1}{12} \end{vmatrix} = -\frac{1}{48} < 0$$

$$\det(\mathbf{H_3}) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & \frac{-1}{12} & 0 \\ -1 & 0 & \frac{-1}{12} \end{vmatrix} = \frac{1}{6} > 0$$

By the second derivative test for constrained optimization problems, $(1, 1, 1)$ is a local maximum for f under the constraint $g(x, y, z) = 3$. Since $f(1, 1, 1) = \frac{1}{2}$, $\frac{1}{2}$ is the local maximal value for f.

Solution 6 by David E. Manes, Oneonta, NY.

The proposed inequality is equivalent to the following:

$$\left(\frac{1}{5 + a^3} - \frac{1}{6}\right) + \left(\frac{1}{5 + b^3} - \frac{1}{6}\right) + \left(\frac{1}{5 + c^3} - \frac{1}{6}\right) \leqslant 0.$$

Therefore,

$$\frac{1 - a^3}{6(5 + a^3)} + \frac{1 - b^3}{6(5 + b^3)} + \frac{1 - c^3}{6(5 + c^3)} \leqslant 0.$$

Factoring the numerators, one obtains

$$\frac{(1 - a)(a^2 + a + 1)}{6(5 + a^3)} + \frac{(1 - b)(b^2 + b + 1)}{6(5 + b^3)} + \frac{(1 - c)(c^2 + c + 1)}{6(5 + c^3)} \leqslant 0.$$

Multiplying by -6 yields

$$(a - 1)\left(\frac{a^2 + a + 1}{5 + a^3}\right) + (b - 1)\left(\frac{b^2 + b + 1}{5 + b^3}\right) + (c - 1)\left(\frac{c^2 + c + 1}{5 + c^3}\right) \geqslant 0.$$

Without loss of generality, we may assume $a \leqslant b \leqslant c$. This implies $a-1 \leqslant b-1 \leqslant c-1$ and $\dfrac{a^2+a+1}{5+a^3} \leqslant \dfrac{b^2+b+1}{5+b^3} \leqslant \dfrac{c^2+c+1}{5+c^3}$. By Chebyshev's inequality, we get

$$\sum_{cyclic}(a-1)\left(\frac{a^2+a+1}{5+a^3}\right) \geqslant \frac{1}{3}(a-1+b-1+c-1)\sum_{cyclic}\left(\frac{a^2+a+1}{5+a^3}\right) = 0$$

since $a+b+c=3$. Equality occurs if and only if $a=b=c=1$. This completes the solution.

Solution 7 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Solution (a):

The function $f(x) = 1/(5+x^3)$ is concave on $(0,1]$: $f''(x) = 6x(2x^3-5)/(x^3+5)^3 < 0$. Therefore, we can apply Jensen's inequality to see that

$$\sum_{cyclic}\frac{1}{5+a^3} \leqslant 3\cdot\frac{1}{5+\left(\frac{a+b+c}{3}\right)^3} = \frac{3}{6} = \frac{1}{2}.$$

Equality holds if and only if $a=b=c=1$.

Solution (b):

First the AGM inequality gives us $5+a^3 = 1+1+1+1+1+a^3 \geqslant 6\sqrt[6]{a^3} = 6\sqrt{a}$, so that $1/(5+a^3) \leqslant 1/(6\sqrt{a})$. Then the power mean inequality yields

$$\left(\frac{a^{-1/2}+b^{-1/2}+c^{-1/2}}{3}\right)^{-2} \leqslant \frac{a+b+c}{3},$$

which implies

$$\sum_{cyclic}\frac{1}{5+a^3} \leqslant \sum_{cyclic}\frac{1}{6\sqrt{a}} \leqslant \frac{3}{6}\left(\frac{a+b+c}{3}\right)^{-1/2} = \frac{1}{2}.$$

Equality holds if and only if $a=b=c=1$.

Solution (c):

The function $f(x) = 1/(5+x^3)$ is concave on $(0,1]$: $f''(x) = 6x(2x^3-5)/(x^3+5)^3 < 0$. Therefore, the function lies on or below the tangent line drawn to the point $(1, 1/6)$ on the curve $y = f(x)$:

$$\frac{1}{5+x^3} \leqslant \frac{1}{2}(x-1)+\frac{1}{6}.$$

Therefore,

$$\sum_{cyclic}\frac{1}{5+a^3} \leqslant \sum_{cyclic}\left(\frac{1}{2}(a-1)+\frac{1}{6}\right) = \frac{1}{2}(a+b+c-3)+\frac{3}{6} = \frac{1}{2}.$$

Equality holds if and only if $a = b = c = 1$.

Solution (d):

The AGM inequality yields $5 + a^3 = 1 + 1 + 1 + 1 + 1 + a^3 \geqslant 6\sqrt{a}$, so that

$$\sum_{cyclic} \frac{1}{5 + a^3} \leqslant \frac{1}{6} \sum_{cyclic} \frac{1}{\sqrt{a}}.$$

But the power mean inequality gives us

$$\left(\frac{a^{-1/2} + b^{-1/2} + c^{-1/2}}{3} \right)^{-2} \leqslant \frac{a + b + c}{3} = 1, \text{ or } \quad \sum_{cyclic} \frac{1}{\sqrt{a}} \leqslant 3,$$

which completes the proof of the given inequality.

Solution 8 by Moti Levy, Rehovot, Israel.

Using the identity $\frac{1}{5 + a^3} = \frac{1}{5} - \frac{1}{5} \frac{a^4}{a(a^3 + 5)}$, we rewrite the original inequality as follows,

$$\frac{3}{5} - \frac{1}{5} \left(\frac{a^4}{a(a^3 + 5)} + \frac{b^4}{b(b^3 + 5)} + \frac{c^4}{c(c^3 + 5)} \right) \leqslant \frac{1}{2},$$

or,

$$\frac{a^4}{a(a^3 + 5)} + \frac{b^4}{b(b^3 + 5)} + \frac{c^4}{c(c^3 + 5)} \geqslant \frac{1}{2}. \tag{7}$$

By Titu's lemma

$$\frac{a^4}{a(a^3 + 5)} + \frac{b^4}{b(b^3 + 5)} + \frac{c^4}{c(c^3 + 5)} \geqslant \frac{\left(a^2 + b^2 + c^2\right)^2}{a(a^3 + 5) + b(b^3 + 5) + c(c^3 + 5)} = \frac{\left(a^2 + b^2 + c^2\right)^2}{a^4 + b^4 + c^4 + 5(a + b + c)}.$$

Hence it is enough to show that

$$\frac{\left(a^2 + b^2 + c^2\right)^2}{a^4 + b^4 + c^4 + 5(a + b + c)} \geqslant \frac{1}{2},$$

or that

$$2\left(a^2 + b^2 + c^2\right)^2 - \left(a^4 + b^4 + c^4 + 5(a + b + c)\right) \geqslant 0 \tag{8}$$

Now we use the p, q, r notation, namely $p = a + b + c$, $q = ab + bc + ca$, $r = abc$.

$$a^2 + b^2 + c^2 = p^2 - 2q$$
$$a^4 + b^4 + c^4 = p^4 - 4p^2q + 2q^2 + 4pr$$

Inequality (8) becomes

$$2\left(p^2 - 2q\right)^2 - \left(p^4 - 4p^2q + 2q^2 + 4pr + 5p\right) \geqslant 0.$$

or

$$p^4 - 4p^2q - 4pr - 5p + 6q^2 \geqslant 0. \tag{9}$$

setting $p = 3$ in (9), we get

$$q^2 - 6q - 2r + 11 \geqslant 0 \tag{10}$$

By AM-GM inequality, $\sqrt[3]{abc} \leqslant \dfrac{a+b+c}{3}$

$$r \leqslant 1, \tag{11}$$

and $\dfrac{ab+bc+ca}{3} \geqslant \left(\sqrt[3]{abc}\right)^2$

$$q \geqslant 3r^{\frac{2}{3}} \geqslant 3r. \tag{12}$$

By the known inequality

$$q \leqslant \frac{p^3 + 9r}{4p}$$

and (11), we get

$$q \leqslant 3.$$

Using (12) in (10) we get

$$q^2 - 6q - 2r + 11 \geqslant q^2 - 6q - 2\frac{q}{3} + 11 = \frac{1}{3}(3-q)(11-3q) \geqslant 0, \quad for \quad 0 \leqslant q \leqslant 3,$$

and this completes the proof.

Solution 9 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

Equality occurs when $a = b = c = 1$. As $a, b, c > 0$ such that $a + b + c = 3$, we see that $0 < a, b, c < 3$. Now consider the function

$$f(x) = \frac{1}{5 + x^3}, \quad 0 < x < 3.$$

Here

$$f'(x) = -\frac{3x^2}{(5+x^3)^2}, \quad \text{and} \quad f''(x) = -\frac{6x(5-2x^3)}{(5+x^3)^3}.$$

As $f''(x) < 0$ for $x > \sqrt[3]{\dfrac{5}{2}}$ $(= 1.35\ldots)$ then f is clearly concave for $x > 2$. Also, on finding the equation of the tangent to the curve f at the point $x = 1$ we have

$$y(x) = -\frac{x}{12} + \frac{1}{4}.$$

This tangent will be above, or touches, the curve f for

$$0 < x < \frac{1 + \sqrt{13}}{2} \ (= 2.30\ldots).$$

So clearly $y(x) \geqslant f(x)$ for $0 < x \leqslant 2$.

For $0 < a, b, c \leqslant 2$ we have:

$$\frac{1}{5+a^3} \leqslant -\frac{a}{12} + \frac{1}{4}$$
$$\frac{1}{5+b^3} \leqslant -\frac{b}{12} + \frac{1}{4}$$
$$\frac{1}{5+c^3} \leqslant -\frac{c}{12} + \frac{1}{4}$$

Thus

$$\sum_{cyc} \frac{1}{5+a^3} \leqslant \sum_{cyc} \left(-\frac{a}{12} + \frac{1}{4} \right) = -\frac{1}{12}(a+b+c) + \frac{3}{4}$$
$$= -\frac{3}{12} + \frac{3}{4} = \frac{1}{2}.$$

And, for $2 < a, b, c < 3$ the conditions for Jensen's inequality as applied to the function f are satisfied. Letting $f(a) = 1/(5+a^3)$, on applying this inequality we see that

$$\sum_{cyc} f(a) \leqslant 3f\left(\frac{1}{3} \sum_{cyc} a \right) = 3f\left(\frac{a+b+c}{3} \right) = 3f(1) = 3 \cdot \frac{1}{6} = \frac{1}{2}.$$

So from the above two results, on combining them, for $0 < a, b, c < 3$ such that $a + b + c = 3$ we conclude that

$$\sum_{cyc} \frac{1}{5+a^3} \leqslant \frac{1}{2},$$

as required to show.

Also solved by G. C. Greubel, Newport News, VA; Hatef Arshagi, Guilford technical Community College, Jamestown, NC; Albert Stadler, Herrliberg, Switzerland; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; Toyesh Prakash Sharma, St. C.F. Andrews School, Agra, India; Goran Conar, Varaždin, Croatia; and the proposer.

• **5659** *Proposed by Narendra Bhandari, National Academy of Science and Technology, Pokhara University, Nepal.*

Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}} \right) = \frac{\pi^4}{32} - 2G^2$$

where G is Catalan's constant defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \text{ and } H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = 2 \sum_{k=0}^{n} \frac{(-1)^k}{k+n+1}$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

The integral representation of the harmonic numbers H_n, namely

$$H_n = \int_0^1 \frac{1-t^n}{1-t}dt,$$

permits to extend the definition of the harmonic numbers to non-integral indices. We find

$$H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = \int_0^1 \frac{1-t^{\frac{n}{2}}}{1-t}dt - \int_0^1 \frac{1-t^{\frac{n-1}{2}}}{1-t}dt = \int_0^1 \frac{t^{\frac{n-1}{2}}-t^{\frac{n}{2}}}{1-t}dt = \int_0^1 \frac{u^{n-1}-u^n}{1-u^2}2udu,$$

$$H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = 2\int_0^1 \frac{u^n}{1+u}du = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k+n+1}.$$

Thus

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) = 2\int_0^1 \frac{1}{1+u}\sum_{n=0}^{\infty}\frac{(-1)^n u^n}{(2n+1)^3}du \overbrace{=}^{u=v^2} 4\int_0^1 \frac{1}{1+v^2}\sum_{n=0}^{\infty}\frac{(-1)^n v^{2n+1}}{(2n+1)^3}dv =$$

$$= 4\arctan v \sum_{n=0}^{\infty}\frac{(-1)^n v^{2n+1}}{(2n+1)^3}\Bigg|_0^1 - 4\int_0^1 \arctan v \sum_{n=0}^{\infty}\frac{(-1)^n v^{2n}}{(2n+1)^2}dv =$$

$$= \pi\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} - 4\int_0^1 \frac{\arctan v}{v}\int_0^v \frac{\arctan w}{w}dwdv,$$

where we have used Taylor's expansion of the arctan function for $|x|<1$, namely

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}.$$

Clearly,

$$4\int_0^1 \frac{\arctan v}{v}\int_0^v \frac{\arctan w}{w}dwdv = 2\int_0^1 \frac{d}{dv}\left(\int_0^v \frac{\arctan w}{w}dw\right)^2 dv = 2\left(\int_0^1 \frac{\arctan w}{w}dw\right)^2 = 2G^2,$$

while

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\int_0^1\left(\ln^2 x\right)x^{2n}dx = \frac{1}{2}\int_0^1 \frac{\ln^2 x}{1+x^2}dx = \frac{1}{2}\int_1^{\infty}\frac{\ln^2 x}{1+x^2}dx = \frac{1}{4}\int_0^{\infty}\frac{\ln^2 x}{1+x^2}dx =$$

$$= \frac{1}{4}\frac{d^2}{du^2}\int_0^{\infty}\frac{x^{u-1}}{1+x^2}dx\Bigg|_{u=1}.$$

It is known that

$$\int_0^{\infty}\frac{x^{u-1}}{1+x^v}dx = \frac{\pi}{v\sin\left(\frac{\pi u}{v}\right)},\ Re\,(v) > Re\,(u) > 0.$$

(see for instance, I.S. Gradshteyn / I.M. Ryzhik, Table of Integrals, Series, and Products, corrected and enlarged edition, Academic Press, 1980, 3.241, formula 2).
So

$$\frac{1}{4}\frac{d^2}{du^2}\int_0^\infty \frac{x^{u-1}}{1+x^2}dx\bigg|_{u=1} = \frac{1}{4}\frac{d^2}{du^2}\frac{\pi}{2\sin\left(\frac{\pi u}{2}\right)}\Bigg|_{u=1} = \frac{\pi^3}{64}\frac{3+\cos(\pi u)}{\sin^3\left(\frac{\pi u}{2}\right)}\Bigg|_{u=1} = \frac{\pi^3}{32},$$

and the sum of the problem statement equals

$$\frac{\pi^4}{32} - 2G^2.$$

Solution 2 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

Note the H_x with $x > -1$ that appear in the question are the analytic continuation of the nth harmonic numbers defined by $\sum_{k=1}^{n}\frac{1}{k}$. Denote the sum to be proved by S. For $n \in \mathbb{Z}_{\geqslant 0}$ observe that

$$H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = 2\int_0^1 \frac{x^n}{1+x}\,dx,$$

which is result (17) proved in the Appendix, and

$$\int_0^1 x^{2n}\log(x)\,dx = \frac{-1}{(2n+1)^2}.$$

The series given by S can thus be rewritten as

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{1}{(2n+1)^2}\cdot\left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right) \\ &= -2\int_0^1\int_0^1 \frac{\log(t)}{1+x}\sum_{n=0}^{\infty}\frac{(-1)^n(t\sqrt{x})^{2n}}{2n+1}\,dx\,dt \\ &= -2\int_0^1\int_0^1 \frac{\log(t)\arctan(t\sqrt{x})}{t\sqrt{x}(1+x)}\,dx\,dt. \end{aligned}$$

Here the interchange that has been made between the integration signs and the summation is permissible due to Fubini's theorem while the well-known Maclaurin series expansion for $\arctan(x)$ has been used. Enforcing a substitution of $x \mapsto x^2$ in the inner integral followed by an integration by parts yields

$$S = -4\int_0^1\int_0^1 \frac{\log(t)\arctan(xt)}{t(1+x^2)}\,dx\,dt = 2\int_0^1\int_0^1 \frac{x\log^2(t)}{(1+x^2)(1+x^2t^2)}\,dx\,dt.$$

From the partial fraction decomposition of

$$\frac{x}{(1+x^2)(1+x^2t^2)} = \frac{x}{(1-t^2)(1+x^2)} - \frac{xt^2}{(1-t^2)(1+x^2t^2)},$$

we see the inner x-integral is elementary. Thus

$$\begin{aligned} S &= \int_0^1 \log^2(t) \left[\frac{\log(1+x^2) - \log(1+x^2t^2)}{1-t^2} \right]_{x=0}^{x=1} dt \\ &= \int_0^1 \frac{\log^2(t) \log\left(\frac{2}{1+t^2}\right)}{1-t^2} \, dt \\ &= \log(2) \int_0^1 \frac{\log^2(x)}{1-x^2} \, dx - \int_0^1 \frac{\log^2(x) \log(1+x^2)}{1-x^2} \, dx \\ &= \frac{7}{4} \zeta(3) \log(2) - I. \end{aligned} \tag{13}$$

Here the dummy variable has been reverted back to x while the result found for the first of the integrals to the right of the equality comes from setting $n = 2$ in (16).

We now turn our attention to the evaluation of the integral I. Define on the interval $x \in [0, 1]$ the function

$$R(x) = \int_0^x \frac{\log^2(t)}{1-t^2} \, dt.$$

Setting $t \mapsto xt$ produces

$$R(x) = \int_0^1 \frac{x \log^2(xt)}{1-t^2x^2} \, dt.$$

Note that $R(0) = 0$ and $R(1) = \displaystyle\int_0^1 \frac{\log^2(t)}{1-t^2} \, dt = \frac{7}{4}\zeta(3)$, a result which comes from setting $n = 2$ in (16). Integrating by parts we have

$$\begin{aligned} I &= \Big[R(x) \log(1+x^2) \Big]_0^1 - 2 \int_0^1 \frac{xR(x)}{1+x^2} \, dx \\ &= \frac{7}{4} \zeta(3) \log(2) - 2 \int_0^1 \int_0^1 \frac{x^2 \log^2(xt)}{(1-t^2x^2)(1+x^2)} \, dt \, dx. \end{aligned}$$

The partial fraction decomposition of

$$\frac{x^2}{(1+x^2)(1-x^2t^2)} = \frac{1}{(1+t^2)(1-x^2t^2)} - \frac{1}{(1+t^2)(1+x^2)},$$

allows one to rewrite the integral for I as

$$\begin{aligned} I = \frac{7}{4} \zeta(3) \log(2) &+ 2 \int_0^1 \int_0^1 \frac{\log^2(xt)}{(1+t^2)(1+x^2)} \, dt \, dx \\ &- 2 \int_0^1 \int_0^1 \frac{\log^2(xt)}{(1+t^2)(1-x^2t^2)} \, dt \, dx. \end{aligned}$$

Since

$$\log^2(xt) = \log^2(x) + 2 \log(x) \log(t) + \log^2(t),$$

using this to rewrite the $\log^2(xt)$ term appearing in the integrand of the first of the double integrals in I yields

$$I = \frac{7}{4}\zeta(3)\log(2) + 4\int_0^1\int_0^1 \frac{\log^2(x)}{(1+t^2)(1+x^2)}\,dt\,dx + 4\left(\int_0^1 \frac{\log(x)}{1+x^2}\,dx\right)^2$$
$$- 2\int_0^1\int_0^1 \frac{\log^2(xt)}{(1+t^2)(1-x^2t^2)}\,dt\,dx.$$

Here we have exploited those symmetries which appeared in the integrands. For the first of the double integrals we have

$$\int_0^1\int_0^1 \frac{\log^2(x)}{(1+t^2)(1+x^2)}\,dt\,dx = \frac{\pi}{4}\int_0^1 \frac{\log^2(x)}{1+x^2}\,dx = \frac{\pi}{4}\cdot\frac{\pi^3}{16},$$

where result (15) given in the Appendix has been used. The second integral corresponds to result (14) given in the Appendix. Thus

$$I = \frac{7}{4}\zeta(3)\log(2) + 4\cdot\frac{\pi}{4}\cdot\frac{\pi^3}{16} + 4(-G)^2 - \int_0^1 \frac{2}{t(1+t^2)}\left(\int_0^1 \frac{t\log^2(xt)}{1-x^2t^2}\,dx\right)dt$$
$$= \frac{7}{4}\zeta(3)\log(2) + \frac{\pi^4}{16} + 4G^2 - \int_0^1 \frac{2R(t)}{t(1+t^2)}\,dt.$$

For the remaining integral, noting that $R'(t) = \dfrac{\log^2(t)}{1-t^2}$, integrating by parts gives

$$\int_0^1 \frac{2R(t)}{t(1+t^2)}\,dt = \Big[R(t)\left(2\log(t) - \log(1+t^2)\right)\Big]_0^1$$
$$- \int_0^1 \left(2\log(t) - \log(1+t^2)\right)\frac{\log^2(t)}{1-t^2}\,dt$$
$$= -\frac{7}{4}\zeta(3)\log(2) - 2\int_0^1 \frac{\log^3(t)}{1-t^2}\,dt + \int_0^1 \frac{\log^2(t)\log(1+t^2)}{1-t^2}\,dt$$
$$= -\frac{7}{4}\zeta(3)\log(2) - 2\left(\frac{-\pi^4}{16}\right) + I = -\frac{7}{4}\zeta(3)\log(2) + \frac{\pi^4}{8} + I,$$

where we have used $\int_0^1 \frac{\log^3(t)}{1-t^2}\,dt = -\frac{\pi^4}{16}$ and comes from setting $n = 3$ in (16) given in the Appendix. So

$$I = \frac{7}{4}\zeta(3)\log(2) + \frac{\pi^4}{16} + 4G^2 - \left(-\frac{7}{4}\zeta(3)\log(2) + \frac{\pi^4}{8} + I\right)$$
$$= \frac{7}{2}\zeta(3)\log(2) - \frac{\pi^4}{16} + 4G^2 - I,$$

or

$$I = \frac{7}{4}\zeta(3)\log(2) - \frac{\pi^4}{32} + 2G^2.$$

Substituting this result into (13) yields

$$S = \frac{7}{4}\zeta(3)\log(2) - \left(\frac{7}{4}\zeta(3)\log(2) - \frac{\pi^4}{32} + 2G^2\right) = \frac{\pi^4}{32} - 2G^2,$$

as required to prove.

Appendix

In this appendix a number of integrals that we shall have a need for are given.

A well-known integral representation for the Catalan constant G is [1, Entry 4.231.12, p. 539]

$$\int_0^1 \frac{\log(x)}{1+x^2}\, dx = -G. \tag{14}$$

And when the logarithm term appearing in the integrand is squared one has [1, Entry 4.261.6, p. 546]

$$\int_0^1 \frac{\log^2(x)}{1+x^2}\, dx = \frac{\pi^3}{16}. \tag{15}$$

Lemma 1 *If $n \in \mathbb{N}$ then*

$$\int_0^1 \frac{\log^n(x)}{1-x^2}\, dx = (-1)^n n! \left(1 - \frac{1}{2^{n+1}}\right)\zeta(n+1), \tag{16}$$

where $\zeta(x)$ denotes the Riemann zeta function.

Proof. Calling the integral to be evaluated ℓ_n, then

$$\ell_n = \int_0^1 \log^n(x) \sum_{k=0}^{\infty} x^{2k}\, dx = \sum_{k=0}^{\infty} \int_0^1 x^{2k} \log^n(x)\, dx.$$

The interchange made between the order of the summation and integration is permissible since for fixed $n \in \mathbb{N}$ all terms involved are unsigned (all negative if n is odd, all positive if n is even). Integrating by parts n times gives

$$\ell_n = (-1)^n n! \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n} = (-1)^n n! \lambda(n+1).$$

Here $\lambda(x)$ denotes the Dirichlet lambda function. As this function is known to be related to the Riemann zeta function by

$$\lambda(n) = \left(1 - \frac{1}{2^n}\right)\zeta(n),$$

the desired result then follows and completes the proof.

Lemma 2 *If $n > -1$ then the following identity holds:*

$$\int_0^1 \frac{x^n}{1+x}\,dx = \frac{1}{2}\left(H_{\frac{n}{2}} - H_{\frac{n-1}{2}}\right). \tag{17}$$

Here H_z, $z > -1$, is the analytic continuation of the nth harmonic numbers defined by $\sum_{k=1}^{n} \frac{1}{k}$.

Proof. Recalling the following integral representation for the harmonic numbers of

$$H_z = \int_0^1 \frac{1-x^z}{1-x}\,dx,$$

we see that

$$H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = \int_0^1 \frac{1-x^{\frac{n}{2}}}{1-x}\,dx - \int_0^1 \frac{1-x^{\frac{n-1}{2}}}{1-x}\,dx = \int_0^1 \frac{x^{\frac{n-1}{2}} - x^{\frac{n}{2}}}{1-x}\,dx.$$

Enforcing a substitution of $x \mapsto x^2$ produces

$$H_{\frac{n}{2}} - H_{\frac{n-1}{2}} = 2\int_0^1 \frac{x^n(1-x)}{1-x^2}\,dx = 2\int_0^1 \frac{x^n}{1+x}\,dx,$$

and completes the proof.

Result (17) is not new. It can be found, for example, in [2, p. 156, Eq. (3.98)].

References

[1] Gradshteyn I. S. and Ryzhik I. M. (2015). *Table of Integrals, Series, and Products*, 8th edition (Academic Press: Amsterdam).

[2] Olaikhan A. S. (2021). *An Introduction to the Harmonic Series and Logarithmic Integrals: For High School Students up to Researchers* (Ali Shadhar Olaikhan: Phoenix, AZ).

Solution 3 by Moti Levy, Rehovot, Israel.

We begin with the following partial fractions expansion:

$$\begin{aligned}\frac{1}{(2n+1)^3(k+n+1)} &= -\frac{1}{(2k+1)^3(k+n+1)} + \frac{2}{(2k+1)(2n+1)^3} \\ &\quad - \frac{2}{(2k+1)^2(2n+1)^2} + \frac{2}{(2k+1)^3(2n+1)},\end{aligned} \tag{18}$$

then apply (18) to expand the sum into four terms,

$$S := 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\frac{(-1)^k}{k+n+1} = -2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(k+n+1)} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}$$
$$-4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}.$$

$$S = -S + 8\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} - 4\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$$
$$S = 4\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\right) - 2\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\right)^2 \tag{19}$$

The Dirichlet beta function is defined as $\beta(s) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$, $\operatorname{Re}(s) > 0$.

$$S = 4\beta(3)\beta(1) - 2\beta^2(1) \tag{20}$$

The first values of Dirichlet beta function are tabulated:

$$\beta(1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}, \tag{21}$$
$$\beta(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = G, \tag{22}$$
$$\beta(3) = \sum_{k=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}. \tag{23}$$

The required result follows after plugging (21), (22), and (23) into (20).

Also solved by G. C. Greubel, Newport News, VA; and the proposer.

• **5660** *Proposed by José Díaz-Barrero, Barcelona Tech, Barcelona, Spain.*

Suppose the sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ of real numbers satisfy the equation

$$x_n^2 + x_{n-1}^2 + y_n^2 + y_{n-1}^2 = (y_n x_{n-1} - x_n y_{n-1}) + \sqrt{3}\,(x_n x_{n-1} + y_n y_{n-1}).$$

Show that the sequences $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ are periodic and find their period.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Put $c_n := \sqrt{x_n^2 + y_n^2} \geqslant 0$. There are unique numbers $\varphi_n \in [0, 2\pi[$ such that

$$x_n = c_n \cos\varphi_n\ ,\ y_n = c_n \sin\varphi_n\ .$$

The equation of the problem statement can then be rewritten as

$$c_{n-1}^2 + c_n^2 =$$

$$= c_{n-1}c_n\left(\sin\varphi_n \cos\varphi_{n-1} \ - \cos\varphi_n \sin\varphi_n\ \right) + c_{n-1}c_n\sqrt{3}\left(\cos\varphi_n \cos\varphi_{n-1} \ - \sin\varphi_n \sin\varphi_n\ \right) =$$

$$= c_{n-1}c_n\left(\sin\left(\varphi_n - \varphi_{n-1}\right)\ +\ \sqrt{3}\cos\left(\varphi_n - \varphi_{n-1}\right)\right) =$$

$$= 2c_{n-1}c_n\sin\left(\varphi_n - \varphi_{n-1} + \frac{\pi}{3}\right).$$

By the AM-GM inequality,

$$c_{n-1}^2 + c_n^2 \geqslant 2c_{n-1}c_n.$$

Hence $\sin\left(\varphi_n - \varphi_{n-1} + \frac{\pi}{3}\right)\ \geqslant 1$ for all n, implying that

$$\varphi_n - \varphi_{n-1} + \frac{\pi}{3} \equiv \frac{\pi}{2}\,(mod\ 2\pi)$$

or equivalently,

$$\varphi_n \equiv \varphi_{n-1} + \frac{\pi}{6}\,(mod\ 2\pi)\,.$$

Furthermore $c_{n-1}^2 + c_n^2 = 2c_{n-1}c_n$ implies $c_n = c_{n-1}$ for all n. We conclude that the sequences $(x_n)_{n0}$ and $(y_n)_{n0}$ are periodic with period 12.

Solution 2 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

If $x_k = y_k = 0$ for some integer $k \geqslant 0$, then both $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ trivially reduce to the zero sequence, so suppose instead that x_n and y_n are never simultaneously zero. For each integer $n \geqslant 0$, let (r_n, θ_n) be the unique polar coordinate representation of (x_n, y_n) in which $r_n > 0$ and $0 \leqslant \theta_n < 2\pi$. Then

$$x_n^2 + x_{n-1}^2 + y_n^2 + y_{n-1}^2 = r_n^2 + r_{n-1}^2,$$

whereas

$$(y_n x_{n-1} - x_n y_{n-1}) = r_n r_{n-1}(\sin\theta_n \cos\theta_{n-1} - \cos\theta_n \sin\theta_{n-1}) = r_n r_{n-1}\sin(\theta_n - \theta_{n-1})$$

and

$$\sqrt{3}(x_n x_{n-1} + y_n y_{n-1}) = \sqrt{3} r_n r_{n-1}(\cos\theta_n \cos\theta_{n-1} + \sin\theta_n \sin\theta_{n-1}) = \sqrt{3} r_n r_{n-1}\cos(\theta_n - \theta_{n-1}).$$

Due to the harmonic addition formula

$$a\cos\theta + b\sin\theta = \operatorname{sgn}(a)\sqrt{a^2+b^2}\cos\left(\theta - \tan^{-1}\left(\frac{b}{a}\right)\right),$$

the original recurrence can be transformed to

$$r_n^2 + r_{n-1}^2 - 2r_n r_{n-1}\cos\left(\theta_n - \theta_{n-1} - \frac{\pi}{6}\right) = 0.$$

Appealing to the law of cosines, we then need $\theta_n - \theta_{n-1} = \pi/6 \pmod{2\pi}$, otherwise there would be a nondegenerate triangle with sides r_n, r_{n-1}, and 0. Consequently,

$$(r_n - r_{n-1})^2 = r_n^2 + r_{n-1}^2 - 2r_n r_{n-1} = 0.$$

Thus $r_n = r_{n-1}$. It follows that $(r_n)_{n\geqslant 0}$ is a constant sequence, and so (x_n, y_n) is attained by iteratively rotating the point (x_0, y_0) counterclockwise about the origin by $\pi/6$ radians n times. Hence, $(x_n)_{n\geqslant 0}$ and $(y_n)_{n\geqslant 0}$ are periodic sequences with period $2\pi/(\pi/6) = 12$.

Solution 3 by Michael Brozinsky, Central Islip, NY and Andrew Bulawa, Brooklyn, NY.

If we view the given equation as quadratic in both x_n and then in y_n the quadratic formula gives

$$x_n = \frac{\sqrt{3}\,x_{n-1}}{2} - \frac{y_{n-1}}{2} + \frac{\sqrt{-x_{n-1}^2 - 2\sqrt{3}\,x_{n-1}y_{n-1} + 4\sqrt{3}\,y_{n-1}y_n + 4y_nx_{n-1} - 4y_n^2 - 3y_{n-1}^2}}{2}$$

and

$$y_n = \frac{\sqrt{3}\,y_{n-1}}{2} + \frac{x_{n-1}}{2} + \frac{\sqrt{-y_{n-1}^2 + 4\sqrt{3}\,x_{n-1}x_n + 2\sqrt{3}\,x_{n-1}y_{n-1} - 4x_n^2 - 4x_ny_{n-1} - 3x_{n-1}^2}}{2}$$

Now since both radicands are non positive because

$$-x_{n-1}^2 - 2\sqrt{3}\,x_{n-1}y_{n-1} + 4\sqrt{3}\,y_{n-1}y_n + 4y_nx_{n-1} - 4y_n^2 - 3y_{n-1}^2 = -\left(x_{n-1} + \sqrt{3}\,y_{n-1} - 2y_n\right)^2$$

and

$$-y_{n-1}^2 + 4\sqrt{3}\,x_{n-1}x_n + 2\sqrt{3}\,x_{n-1}y_{n-1} - 4x_n^2 - 4x_ny_{n-1} - 3x_{n-1}^2 = -\left(\sqrt{3}\,x_{n-1} - 2x_n - y_{n-1}\right)^2$$

both radicands must be 0 since the sequences are of real numbers.

Hence we have the recursion formulas $x_n = \dfrac{\sqrt{3}x_{n-1}}{2} - \dfrac{y_{n-1}}{2}$ and $y_n = \dfrac{\sqrt{3}y_{n-1}}{2} + \dfrac{x_{n-1}}{2}$ and in particular, $x_1 = \dfrac{\sqrt{3}x_0}{2} - \dfrac{y_0}{2}$ and $y_1 = \dfrac{\sqrt{3}y_0}{2} - \dfrac{x_0}{2}$; $x_2 = \dfrac{x_0}{2} - \dfrac{\sqrt{3}y_0}{2}$ and $y_2 = \dfrac{y_0}{2} - \dfrac{\sqrt{3}x_0}{2}$; $x_3 = -y_0$ and $y_3 = x_0$. Since three iterations thus end up interchanging x_0 and y_0 but change the sign of y_0, it follows that three more iterations will give $x_6 = -x_0$ and $y_6 = -y_0$ and 6 more will give $x_{12} = x_0$ and $y_{12} = y_0$ so that the sequences are periodic and repeat very 12 elements.

Solution 4 by Michel Bataille, Rouen, France.

Let $U_n = \left(x_n + \dfrac{1}{2}\,y_{n-1} - \dfrac{\sqrt{3}}{2}\,x_{n-1}\right)^2 + \left(y_n - \dfrac{1}{2}\,x_{n-1} - \dfrac{\sqrt{3}}{2}\,y_{n-1}\right)^2$.

Expanding and using the hypothesis gives $U_n = 0$. It follows that the sequences (x_n) and (y_n) satisfy the recursions

$$x_n = \frac{\sqrt{3}}{2}\,x_{n-1} - \frac{1}{2}\,y_{n-1}, \qquad y_n = \frac{1}{2}\,x_{n-1} + \frac{\sqrt{3}}{2}\,y_{n-1},$$

that is, $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$ where $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

The characteristic polynomial of A is $\lambda^2 - \sqrt{3}\lambda + 1 = (\lambda - e^{i\pi/6})(\lambda - e^{-i\pi/6})$, hence $A = PDP^{-1}$ for some invertible matrix P where

$$D = \begin{pmatrix} e^{i\pi/6} & 0 \\ 0 & e^{-i\pi/6} \end{pmatrix}.$$

It follows that for nonnegative integers n, k we have

$$\begin{pmatrix} x_{n+k} \\ y_{n+k} \end{pmatrix} = A^k \begin{pmatrix} x_n \\ y_n \end{pmatrix} = PD^kP^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Since $D^{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we deduce that $x_{n+12} = x_n,\ y_{n+12} = y_n$ for all n and we conclude that (x_n) and (y_n) are periodic with period 12.

Solution 5 by Moti Levy, Rehovot, Israel.

Let $(u_n)_{n\geqslant 0}$ be a sequence of vectors $u_n := \begin{bmatrix} x_n \\ y_n \end{bmatrix}$.

Using vector notation,

$$x_n^2 + x_{n-1}^2 + y_n^2 + y_{n-1}^2 = u_n^T u_n + u_{n-1}^T u_{n-1}. \tag{24}$$

$$y_n x_{n-1} - x_n y_{n-1} = (Au_n)^T u_{n-1} = u_n^T A^T u_{n-1}, \tag{25}$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$$x_n x_{n-1} + y_n y_{n-1} = u_n^T u_{n-1}. \tag{26}$$

Then the equation in the problem statement becomes

$$\begin{aligned} u_n^T u_n + u_{n-1}^T u_{n-1} &= u_n A^T u_{n-1} + \sqrt{3} u_n^T u_{n-1} \\ &= 2u_n^T L u_{n-1}, \end{aligned} \tag{27}$$

where $L := \frac{1}{2}A^T + \frac{\sqrt{3}}{2}I = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. One can check that L is unitary matrix (that is $L^{-1} = L^T$).

We rewrite (27) to get

$$u_n^T u_n + u_{n-1}^T u_{n-1} = u_n^T L u_{n-1} + u_n^T L u_{n-1},$$

or

$$u_n^T (u_n - Lu_{n-1}) = \left(u_n^T L - u_{n-1}^T\right) u_{n-1}. \tag{28}$$

We manipulate the left and side of (28) to get,

$$\begin{aligned}\left(u_n^T L - u_{n-1}^T\right) u_{n-1} &= u_{n-1}^T \left(L^T u_n - u_{n-1}\right) = u_{n-1}^T L^T L \left(L^T u_n - u_{n-1}\right) \\ &= u_{n-1}^T L^T \left(L L^T u_n - L u_{n-1}\right) = \left(L u_{n-1}\right)^T \left(u_n - L u_{n-1}\right). \qquad (29)\end{aligned}$$

By plugging (29) into (28) we will show that $u_n = Lu_{n-1}$, as follows

$$\begin{aligned} u_n^T \left(u_n - L u_{n-1}\right) &= \left(L u_{n-1}\right)^T \left(u_n - L u_{n-1}\right) \\ \left(u_n^T - \left(L u_{n-1}\right)^T\right)\left(u_n - L u_{n-1}\right) &= 0 \\ \left(u_n - L u_{n-1}\right)^T \left(u_n - L u_{n-1}\right) &= 0 \\ &\Rightarrow u_n = L u_{n-1}. \end{aligned}$$

Now to show that the sequence is periodic, we note that

$$u_n = L^n u_{0,}$$

and find that the eigenvalues of L are $\{e^{\frac{\pi i}{6}}, e^{-\frac{\pi i}{6}}\}$. Hence

$$L = P^{-1} \begin{bmatrix} e^{\frac{\pi i}{6}} & 0 \\ 0 & e^{-\frac{\pi i}{6}} \end{bmatrix} P,$$

and

$$L^n = P^{-1} \begin{bmatrix} e^{\frac{n\pi i}{6}} & 0 \\ 0 & e^{-\frac{n\pi i}{6}} \end{bmatrix} P.$$

The smallest integer for which $L^p = I$ is $p = 12$.

We conclude that sequence is periodic and its period is 12.

Also solved by by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; and the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣