Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before June 15, 2022.

• **5679** *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma.*

Compute

$$\lim_{n\to+\infty}(-1)^n \cdot \sin\left(\sum_{k=1}^{3n}\arctan\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11}\right).$$

• **5680** *Proposed by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia..*

Let

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos(2x)}{\log(\tan x)}\,dx \quad \text{and} \quad J=\int_0^{\infty}\frac{\tanh(x)\,\text{sech}(x)}{x}\,dx.$$

(a) Show that the ratio I/J exists, and without explicitly evaluating either of the improper integrals, find its value. (b) Find the value of I.

• **5681** *Proposed by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.*

Let $a>0$. Evaluate

$$\int_0^1\frac{\tan^{-1}x}{x^2-ax-1}\,dx.$$

• **5682** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania.*

Suppose $0 < a \leqslant b$. Prove

$$27\int_a^b\int_a^b\int_a^b (x+y)(y+z)(z+x)dxdydz \leqslant (b-a)^3(a^2+ab+b^2+3)^3.$$

• **5683** *Proposed by Michel Bataille, Rouen, France.*

Let n be a nonnegative integer. Evaluate in closed form

$$\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k.$$

• **5684** *Proposed by Goran Conar, Varaždin, Croatia.*

Let α,β,γ be angles of an acute triangle. Prove that the arithmetic mean of sines of half-angles is bounded between sine of harmonic mean of that half-angles and sine of arithmetic mean of that half-angles. In other words, prove that the following inequalities hold

$$\sin\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)}\right) \leqslant \frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}}{3} \leqslant \frac{1}{2}.$$

When does equality occur?

Solutions

to Formerly Published Problems

• **5661** *Proposed by Kenneth Korbin, New York, NY.*

Given positive acute angles A, B, C with $\sin^2(A+B+C) = 1/10$, find two triples of positive integers (x,y,z), with $x<y<z$, such that $\sin^2 A = 1/x$, $\sin^2 B = 1/y$, $\sin^2 C = 1/z$.

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

Two such triples are $(50,65,325)$ and $(26,170,442)$.

First notice that $\cos^2 A = 1 - 1/x = \dfrac{x-1}{x}$, and similarly, $\cos^2 B = \dfrac{y-1}{y}$ and $\cos^2 C = \dfrac{z-1}{z}$.

From the angle sum formulas for sine and cosine,

$$\begin{aligned}\sin(A+B+C) &= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C\\ &= \frac{1\cdot\sqrt{y-1}\cdot\sqrt{z-1}}{\sqrt{xyz}} + \frac{\sqrt{x-1}\cdot 1\cdot\sqrt{z-1}}{\sqrt{xyz}} + \frac{\sqrt{x-1}\cdot\sqrt{y-1}\cdot 1}{\sqrt{xyz}} - \frac{1}{\sqrt{xyz}}\\ &= \frac{\sqrt{(y-1)(z-1)} + \sqrt{(x-1)(z-1)} + \sqrt{(x-1)(y-1)} - 1}{\sqrt{xyz}}\\ &= \frac{1}{\sqrt{10}},\end{aligned}$$

so that

$$\sqrt{10}\left[\sqrt{(y-1)(z-1)} + \sqrt{(x-1)(z-1)} + \sqrt{(x-1)(y-1)} - 1\right] = \sqrt{xyz}.$$

If $(y-1)(z-1)$, $(z-1)(x-1)$, and $(x-1)(y-1)$ are all perfect squares, then we may write

$$(x,y,z) = (da^2+1, db^2+1, dc^2+1),$$

where a, b, c, and d are positive integers and $a < b < c$. Squaring both sides gives

$$10\left[d(bc+ca+ac)-1\right]^2 = (da^2+1)(db^2+1)(dc^2+1).$$

A search using Mathematica found the following triples (a,b,c), all with $d = 1$:

(a,b,c)	(x,y,z)
(4, 14, 183)	(17, 197, 33490)
(4, 15, 98)	(17, 226, 9605)
(4, 18, 47)	(17, 325, 2210)
(4, 23, 30)	(17, 530, 901)
(5, 9, 73)	(26, 82, 5330)
(5, 13, 21)	(26, 170, 442)
(6, 7, 68)	(37, 50, 4625)
(6, 8, 31)	(37, 65, 962)
(7, 8, 18)	(50, 65, 325)

Of these, the only triples with $x < y < z < 500$ are $(26, 170, 442)$ and $(50, 65, 325)$.

Solution 2 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We apply the identity

$$\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$$

to yield

$$\sqrt{\frac{1}{10}} = \sqrt{\frac{1}{x}\left(1-\frac{1}{y}\right)\left(1-\frac{1}{z}\right)} + \sqrt{\frac{1}{y}\left(1-\frac{1}{x}\right)\left(1-\frac{1}{z}\right)} + \sqrt{\frac{1}{z}\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} - \sqrt{\frac{1}{xyz}},$$

or $xyz = 10[\sqrt{(y-1)(z-1)} + \sqrt{(x-1)(z-1)} + \sqrt{(x-1)(y-1)} - 1]^2$. This in turn inspires us to try values for x, y, and z of the form n^2+1 with n an integer, so we let $x = a^2+1$, $y = b^2+1$, and $z = c^2+1$. Then the previous equation becomes

$$(a^2+1)(b^2+1)(c^2+1) = 10(ab+bc+ca-1)^2.$$

This allows us to find at least nine solutions for (a,b,c), with the corresponding solutions for (x,y,z):

(a,b,c)	(x,y,z)
(4,14,183)	(17,197,33490)
(4,15,98)	(17,226,9605)
(4,18,47)	(17,325,2210)
(4,23,30)	(17,530,901)
(5,9,73)	(26,82,5330)
(5,13,21)	(26,170,442)
(6,7,68)	(37,50,4625)
(6,8,31)	(37,65,962)
(7,8,18)	(50,65,325)

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We deduce from $\sin A = \frac{1}{\sqrt{x}}$, $\sin B = \frac{1}{\sqrt{y}}$, $\sin C = \frac{1}{\sqrt{z}}$ that

$$\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B - \sin A \sin B \sin C =$$

$$= \frac{\sqrt{(x-1)(y-1)} + \sqrt{(y-1)(z-1)} + \sqrt{(z-1)(x-1)} - 1}{\sqrt{xyz}} = \pm\frac{1}{\sqrt{10}}. \qquad (1)$$

An exhaustive computer search reveals that the solutions of (1) with 0<x<y<z⩽1000 are given by (x,y,z)∈{(1,2,5), (17, 530, 901), (26,170,442), (37,65,962), (50,65,325)}. We next show that any integer triple (x,y,z) that solves (1) is a root of a polynomial equation.
When expanding

$$p(s,t,u,v,w) := \prod_{\delta_1,\delta_2,\delta_3,\delta_4\in\{-1,1\}} (s+\delta_1 t+\delta_2 u+\delta_3 v+\delta_4 w)$$

$$= (s+t+u+v+w)(s+t+u+v-w)(s+t+u-v+w)(s+t+u-v-w)$$
$$(s+t-u+v+w)(s+t-u+v-w)(s+t-u-v+w)(s+t-u-v-w)$$
$$(s-t+u+v+w)(s-t+u+v-w)(s-t+u-v+w)(s-t+u-v-w)$$
$$(s-t-u+v+w)(s-t-u+v-w)(s-t-u-v+w)(s-t-u-v-w)$$

we get a polynomial with integer coefficients in s^2, t^2, u^2, v^2, w^2. We put

$$s = \sqrt{\frac{(x-1)(y-1)}{xyz}}, t = \sqrt{\frac{(y-1)(z-1)}{xyz}}, u = \sqrt{\frac{(z-1)(x-1)}{xyz}}, v = -\frac{1}{\sqrt{xyz}},\ w = \frac{1}{\sqrt{10}}.$$

We find with the help of Mathematica that $p(s,t,u,v,w) = 0$ implies either

$$
\begin{aligned}
&2560000 - 5120000x + 2560000x^2 - 5120000y + 10240000xy - 5120000x^2y \\
&+2560000y^2 - 5120000xy^2 + 2240000x^2y^2 + 320000x^3y^2 + 320000x^2y^3 \\
&-320000x^3y^3 + 10000x^4y^4 - 5120000z + 10240000xz - 5120000x^2z \\
&+10240000yz - 22016000xyz + 12544000x^2yz - 768000x^3yz - 5120000y^2z \\
&+12544000xy^2z - 7936000x^2y^2z + 512000x^3y^2z - 768000xy^3z \\
&+512000x^2y^3z + 320000x^3y^3z - 32000x^4y^3z - 32000x^3y^4z \\
&-4000x^4y^4z + 2560000z^2 - 5120000xz^2 + 2240000x^2z^2 + 320000x^3z^2 \\
&-5120000yz^2 + 12544000xyz^2 - 7936000x^2yz^2 + 512000x^3yz^2 \\
&+2240000y^2z^2 - 7936000xy^2z^2 + 6761600x^2y^2z^2 - 1065600x^3y^2z^2 \\
&+45600x^4y^2z^2 + 320000y^3z^2 + 512000xy^3z^2 - 1065600x^2y^3z^2 \\
&+137600x^3y^3z^2 + 2400x^4y^3z^2 + 45600x^2y^4z^2 + 2400x^3y^4z^2 \\
&+600x^4y^4z^2 + 320000x^2z^3 - 320000x^3z^3 - 768000xyz^3 \\
&+512000x^2yz^3 + 320000x^3yz^3 - 32000x^4yz^3 + 320000y^2z^3 \\
&+512000xy^2z^3 - 1065600x^2y^2z^3 + 137600x^3y^2z^3 + 2400x^4y^2z^3 \\
&-320000y^3z^3 + 320000xy^3z^3 + 137600x^2y^3z^3 - 47360x^3y^3z^3 \\
&+480x^4y^3z^3 - 32000xy^4z^3 + 2400x^2y^4z^3 + 480x^3y^4z^3 \\
&-40x^4y^4z^3 + 10000x^4z^4 - 32000x^3yz^4 - 4000x^4yz^4 + 45600x^2y^2z^4 \\
&+2400x^3y^2z^4 + 600x^4y^2z^4 - 32000xy^3z^4 + 2400x^2y^3z^4 \\
&+480x^3y^3z^4 - 40x^4y^3z^4 + 10000y^4z^4 - 4000xy^4z^4 \\
&+600x^2y^4z^4 - 40x^3y^4z^4 + x^4y^4z^4 = 0
\end{aligned}
$$

or

$$
\begin{aligned}
&10000x^4y^4 + 64000x^3y^3z - 32000x^4y^3z - 32000x^3y^4z - 4000x^4y^4z \\
&+160000x^2y^2z^2 - 160000x^3y^2z^2 + 45600x^4y^2z^2 - 160000x^2y^3z^2 \\
&+64000x^3y^3z^2 + 2400x^4y^3z^2 + 45600x^2y^4z^2 + 2400x^3y^4z^2 \\
&+600x^4y^4z^2 + 64000x^3yz^3 - 32000x^4yz^3 - 160000x^2y^2z^3 \\
&+64000x^3y^2z^3 + 2400x^4y^2z^3 + 64000xy^3z^3 + 64000x^2y^3z^3 \\
&-37760x^3y^3z^3 + 480x^4y^3z^3 - 32000xy^4z^3 + 2400x^2y^4z^3 \\
&+480x^3y^4z^3 - 40x^4y^4z^3 + 10000x^4z^4 - 32000x^3yz^4 - 4000x^4yz^4 \\
&+45600x^2y^2z^4 + 2400x^3y^2z^4 + 600x^4y^2z^4 - 32000xy^3z^4 \\
&+2400x^2y^3z^4 + 480x^3y^3z^4 - 40x^4y^3z^4 + 10000y^4z^4 - 4000xy^4z^4 \\
&+600x^2y^4z^4 - 40x^3y^4z^4 + x^4y^4z^4 = 0.
\end{aligned}
$$

Solution 4 by William Chang, University of Southern California, Los Angeles, CA.

Let $s := A + B + C$ and suppose $s \in [0, \pi/2]$ so that $\sin(s) = \dfrac{1}{\sqrt{10}}$ and $\cos(s) = \dfrac{3}{\sqrt{10}}$. From this it follows that

$$\begin{aligned} z &= \frac{1}{\sin^2(s-(A+B))} &(1)\\ &= \frac{1}{(\sin(s)\cos(A+B)-\cos(s)\sin(A+B))^2} &(2)\\ &= \frac{1}{(\frac{1}{\sqrt{10}}\cos(A+B)-\frac{3}{\sqrt{10}}\sin(A+B))^2} &(3)\end{aligned}$$

By the cosine addition angle formulas we have

$$\begin{aligned} \cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B) &(4)\\ &= \frac{\sqrt{x-1}\sqrt{y-1}-1}{\sqrt{xy}} &(5)\end{aligned}$$

where in the second equality we took the positive square roots because A, B, C are acute. Similarly,

$$\begin{aligned} \sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B) &(6)\\ &= \frac{\sqrt{y-1}+\sqrt{x-1}}{\sqrt{xy}} &(7)\end{aligned}$$

Thus, we have

$$z = \frac{10xy}{(\sqrt{x-1}\sqrt{y-1}-3(\sqrt{x-1}+\sqrt{y-1})-1)^2} \qquad (8)$$

Suppose that $\sqrt{x-1}$ and $\sqrt{y-1}$ are integers, we set $m = \sqrt{x-1}$ and $n = \sqrt{y-1}$ so our expression for z becomes

$$z = \frac{10(m^2+1)(n^2+1)}{(mn-3m-3n-1)^2} \qquad (9)$$

Testing pairs of m, n that make z an integer we have $(m, n) = (4, 8), (5, 7)$. This gives $(x, y, z) = (17, 65, 442), (26, 50, 3250)$. Since for these values of x, y, z we have $A, B, C < \pi/3$, they satisfy the assumption that $s \in [0, \pi/2]$.

To verify that such x, y, z are indeed solutions, we expand $\sin^2(A + B + C)$ as follows

$$\begin{aligned} \sin^2(A+B+C) &= (\sin(A+B)\cos C+\cos(A+B)\sin C)^2 &(10)\\ &= ((\sin A\cos B+\cos A\sin B)\cos C+(\cos A\cos B-\sin A\sin B)\sin C)^2 &(11)\\ &= (\cos B\cos C\sin A+\cos A\cos C\sin B+\cos A\cos B\sin C-\sin A\sin B\sin C)^2 &(12)\\ &= \frac{1}{xyz}(\sqrt{y-1}\sqrt{z-1}+\sqrt{x-1}\sqrt{z-1}+\sqrt{x-1}\sqrt{y-1}-1)^2 &(13)\end{aligned}$$

Plugging both pairs of (x, y, z) into the equation above we get $\sin^2(A+B+C) = \dfrac{1}{10}$ as desired.

Also solved by David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; and the proposer.

• **5662** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral $\displaystyle\int_1^\infty \frac{x \ln x}{(x+1)(x^2+1)}\, dx.$

Solution 1 by Péter Fülöp, Gyömrő, Hungary.

Using the $x = \dfrac{1}{t}$ substution:

$$I = -\int_0^1 \frac{\ln(t)}{(t+1)(t^2+1)} dt$$

Let's extend the integrand by $(1-t)$

$$I = -\int_0^1 \frac{(1-t)\ln(t)}{(1-t^2)(1+t^2)} dt = -\int_0^1 \frac{(1-t)\ln(t)}{(1-t^4)} dt$$

Applying fact $\ln(x) = \dfrac{d(x^a)}{da}_{|a=0}$ and performing the $y = t^4$ substitution we get:

$$I = \frac{d}{da}_{|a=0}\Big[-\frac{1}{4}\int_0^1 \frac{y^{\frac{a-3}{4}}}{(1-y)} dy + \frac{1}{4}\int_0^1 \frac{y^{\frac{a-2}{4}}}{(1-y)} dy\Big]$$

Introducing the β function:

$$I = \frac{d}{da}_{|a=0}\Big[-\frac{1}{4}\beta(\frac{a+1}{4}, 0) + \frac{1}{4}\beta(\frac{a+2}{4}, 0)\Big]$$

Beta function is also given by the series: $\beta_x(a, b) = \displaystyle\sum_{k=0}^{\infty} \frac{(1-b)_k}{k!(k+a)} x^{k+a}$

$$I = \frac{d}{da}_{|a=0}\Big[-\frac{1}{4}\sum_{k=0}^{\infty} \frac{1}{(k+\frac{a+1}{4})} + \frac{1}{4}\sum_{k=0}^{\infty} \frac{1}{(k+\frac{a+2}{4})}\Big]$$

After derivation:

$$I = \frac{1}{16}\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{4})^2} - \frac{1}{16}\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{2})^2} = \frac{1}{16}\Big(\psi^{(1)}(\frac{1}{4}) - \psi^{(1)}(\frac{1}{2})\Big)$$

Where $\psi^{(1)}$ is the trigamma function. It is known that

$$\psi^{(1)}(\frac{1}{2}) = \frac{\pi^2}{2} \qquad \psi^{(1)}(\frac{1}{4}) = \pi^2 + 8G$$

Where G is the Catalan's constant. Now the value of the integral can be calculated.

$$\int_1^{\infty}\frac{x\ln(x)}{(x+1)(x^2+1)}dx = \boxed{\frac{G}{2} + \frac{\pi^2}{32}}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We claim that

$$\int_1^{\infty}\frac{x\ln x}{(x+1)(x^2+1)}dx = \frac{G}{2} + \frac{\pi^2}{32},$$

where G is Catalan's constant (see for instance https://en.wikipedia.org/wiki/Catalan%27s_constant). We perform the change of variables x→1/x and find

$$\int_1^{\infty}\frac{x\ln x}{(x+1)(x^2+1)}dx = -\int_0^1\frac{\ln x}{(x+1)(x^2+1)}dx = \int_0^1\left(\frac{x-1}{2(x^2+1)} - \frac{1}{2(x+1)}\right)\ln x\, dx =$$

$$= \frac{G}{2} + \int_0^1\left(\frac{x}{2(x^2+1)} - \frac{1}{2(x+1)}\right)\ln x\, dx,$$

taking into account that $G = -\int_0^1 \frac{\ln x}{x^2+1}dx$. Finally, using the expansions $\frac{x}{x^2+1} = \sum_{k=0}^{\infty}(-1)^k x^{2k+1}$ and $\frac{1}{x+1} = \sum_{k=0}^{\infty}(-1)^k x^k]$, we find by termwise integration

$$\int_0^1\left(\frac{x}{2(x^2+1)} - \frac{1}{2(x+1)}\right)\ln x\, dx = \frac{1}{2}\sum_{k=0}^{\infty}\int_0^1 (-1)^k\left(x^{2k+1} - x^k\right)\ln x\, dx =$$

$$= -\frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{4(k+1)^2} - \frac{1}{(k+1)^2}\right) = \frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{3}{8}\left(\sum_{k=1}^{\infty}\frac{1}{k^2} - 2\sum_{k=1}^{\infty}\frac{1}{(2k)^2}\right) =$$

$$= \frac{3}{16}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{3}{16}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{32}.$$

Solution 3 by William Chang, University of Southern California, Los Angeles, CA.

We first do a u substitution $u = 1/x$, to get $dx = \frac{-1}{u^2}du$ to get

$$\int_1^\infty \frac{x\ln x}{(x+1)(x^2+1)}dx = -\int_0^1 \frac{\ln u}{(u+1)(u^2+1)}du$$

We now use the partial fraction decomposition $\frac{-1}{(u+1)(u^2+1)} = \frac{\frac{1}{2}u - \frac{1}{2}}{u^2+1} - \frac{\frac{1}{2}}{u+1}$ our integral becomes

$$\int_1^\infty \frac{x\ln x}{(x+1)(x^2+1)}dx = -\frac{1}{2}\int_0^1 \frac{\ln u}{1+u^2}du + \frac{1}{2}\int_0^1 \frac{u\ln u}{1+u^2}du - \frac{1}{2}\int_0^1 \frac{\ln u}{1+u}du$$

Using u substitution $u = x^2$, we get $du = 2xdx$ so that $\int_0^1 \frac{x\ln x}{1+x^2}dx = \frac{1}{2}\int_0^1 \frac{\ln u}{1+u}du$

$$\begin{aligned}
&= \frac{1}{2}C - \frac{3}{4}\int_0^1 \frac{\ln x}{1+x}dx \\
&= \frac{1}{2}C - \frac{3}{4}(-\frac{\pi^2}{12}) \\
&= \frac{1}{2}C + \frac{\pi^2}{16}
\end{aligned}$$

where $C = -\int_0^1 \frac{\ln x}{1+x^2}dx$ is catalan's constant and $[\int_0^1 \frac{\ln x}{1+x}dx = -\frac{\pi^2}{12}$.

Solution 4 by Ajay Srinivasan, University of Southern California, Los Angeles, CA.

Claim: $\int_1^\infty \frac{x\ln x}{(x+1)(x^2+1)}\,dx = \frac{C}{2} - \frac{\pi^2}{32}$ where C is Catalan's constant.

Proof: Notice that the denominator is just $\frac{x^4-1}{x-1}$, so letting I be the given integral:

$$\begin{aligned}
I &:= \int_1^\infty \frac{x\ln x}{(x+1)(x^2+1)}\,dx \\
&= \int_1^\infty \frac{x(x-1)\ln x}{x^4-1}\,dx
\end{aligned}$$

Using the substitution $x = e^t$, with $dx = e^t\,dt$:

$$\begin{aligned}
I &= \int_0^\infty \frac{te^{2t}(e^t-1)}{e^{4t}-1}\,dt \\
&= \int_0^\infty \frac{t(e^{-t}-e^{-2t})}{1-e^{-4t}}\,dt
\end{aligned}$$

Note that $\dfrac{1}{1-e^{-4t}}$ is just the sum of an infinite geometric progression with ratio e^{-4t} for all $t \in]0,\infty[$, so: $\dfrac{1}{1-e^{-4t}} = \sum_{n=0}^{\infty} e^{-4nt}$.

$$\begin{aligned}
I &= \int_0^\infty t(e^{-t}-e^{-2t})\left(\sum_{n=0}^{\infty} e^{-4nt}\right)\,dt \\
&= \int_0^\infty \left(\sum_{n=0}^{\infty} t(e^{-t}-e^{-2t})e^{-4nt}\right)\,dt \\
&= \int_0^\infty \left(\sum_{n=0}^{\infty} t(e^{-(4n+1)t}-e^{-(4n+2)t})\right)\,dt
\end{aligned}$$

Since both the integral of the sum, and the sum of the integral converge; as a consequence of Fubini's theorem, we may interchange the summation and the integral operators.

$$I = \sum_{n=0}^{\infty}\left(\left(\int_0^\infty te^{-(4n+1)t}\,dt\right) - \left(\int_0^\infty te^{-(4n+2)t}\,dt\right)\right)$$

Using the substitutions $u = (4n+1)t$ and $v = (4n+2)t$:

$$\begin{aligned}
I &= \sum_{n=0}^{\infty}\left(\left(\frac{1}{(4n+1)^2}\int_0^\infty ue^{-u}\,du\right) - \left(\frac{1}{(4n+2)^2}\int_0^\infty ue^{-u}\,du\right)\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{\Gamma(2)}{(4n+1)^2} - \frac{\Gamma(2)}{(4n+2)^2}\right)
\end{aligned}$$

We can use $\Gamma(2) = 1$ and

$$\sum_{n=0}^{\infty}\frac{1}{(4n+2)^2} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{32}.$$

Also

$$\sum_{n=0}^{\infty}\frac{1}{(4n+1)^2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n+1)^2} + \frac{1}{(2n+1)^2}\right) = \frac{C}{2} + \frac{\pi^2}{16}$$

where C is Catalan's constant.

$$I = \frac{C}{2} + \frac{\pi^2}{16} - \frac{\pi^2}{32} = \frac{C}{2} - \frac{\pi^2}{32}.$$

Solution 5 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

Denote the value of the integral to be found by I. We show that

$$I = \frac{\pi^2}{32} + \frac{G}{2}.$$

Here $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ denotes Catalan's constant.

Enforcing a substitution of $x \mapsto \frac{1}{x}$ gives

$$I = -\int_0^1 \frac{\log(x)}{(x+1)(x^2+1)}\,dx.$$

From the partial fraction decomposition of

$$\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{1}{2(x^2+1)} - \frac{x}{2(x^2+1)},$$

the integral may be rewritten as

$$I = \frac{1}{2}\int_0^1 \frac{x\log(x)}{x^2+1}\,dx - \frac{1}{2}\int_0^1 \frac{\log(x)}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{\log(x)}{x^2+1}\,dx.$$

In the first of the integrals appearing to the right of the equality, if a substitution of $x \mapsto \sqrt{x}$ is enforced we see that

$$I = -\frac{3}{8}\int_0^1 \frac{\log(x)}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{\log(x)}{x^2+1}\,dx = -\frac{3}{8}I_1 - \frac{1}{2}I_2.$$

For the first of the integrals, I_1, expanding the denominator in terms of an infinite geometric series, we have

$$I_1 = \int_0^1 \sum_{n=0}^{\infty}(-1)^n x^n \log(x)\,dx. \tag{14}$$

As

$$\int_0^1 \sum_{n=0}^{\infty}\left|(-1)^n x^n \log(x)\right|dx = \int_0^1 \frac{\log(x)}{1-x} < \infty,$$

by Fubini's theorem the summation and integration signs appearing in (14) can be interchanged. Doing so, followed by integrating by parts, we have

$$I_1 = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^n \log(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

after a reindexing of $n \mapsto n-1$ in the sum is made. The value of the series is well known. Here

$$I_1 = -\left(\sum_{n=1}^{\infty}\frac{1}{n^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{2}\cdot\frac{\pi^2}{6} = -\frac{\pi^2}{12}.$$

For the second of the integrals, I_2, expanding the denominator in terms of an infinite geometric series, we have

$$I_2 = \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{2n} \log(x)\, dx. \tag{15}$$

As

$$\int_0^1 \sum_{n=0}^{\infty} \left|(-1)^n x^{2n} \log(x)\right| dx = \int_0^1 \frac{\log(x)}{1-x^2} < \infty,$$

by Fubini's theorem the summation and integration signs appearing in (15) can be interchanged. Doing so, followed by integrating by parts, we have

$$I_2 = \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n} \log(x)\, dx = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -G.$$

So for the value of the integral I, we have

$$I = -\frac{3}{8}\left(-\frac{\pi^2}{12}\right) - \frac{1}{2}(-G) = \frac{\pi^2}{32} + \frac{G}{2},$$

as announced.

Solution 6 by Michel Bataille, Rouen, France.

Let I denote the integral to be calculated. We show that $I = \dfrac{G}{2} + \dfrac{\pi^2}{32}$ where G is the Catalan number.

The change of variables $x = \dfrac{1}{u}$ gives $I = -\displaystyle\int_0^1 \frac{\ln u}{(u+1)(u^2+1)}\, du$ so that $I = \dfrac{1}{2}(J - K - L)$ where J, K, L are the integrals

$$J = \int_0^1 \frac{u \ln u}{u^2+1}\, du, \qquad K = \int_0^1 \frac{\ln u}{u+1}\, du, \qquad L = \int_0^1 \frac{\ln u}{u^2+1}\, du.$$

We calculate J, K, L in turn. Let $\epsilon \in (0,1)$.
(a) Using an integration by parts, we easily obtain

$$\int_\epsilon^1 \frac{u \ln u}{1+u^2}\, du = \frac{(-\ln \epsilon)(\ln(1+\epsilon^2))}{2} - \frac{1}{4}\int_{\epsilon^2}^1 \frac{\ln(1+v)}{v}\, dv.$$

As $\epsilon \to 0^+$, we have $(\ln \epsilon)\ln(1+\epsilon^2) \sim \epsilon^2 \ln \epsilon$, hence taking the limit as $\epsilon \to 0^+$, we obtain

$$\int_0^1 \frac{u \ln u}{1+u^2}\, du = -\frac{1}{4}\int_0^1 \frac{\ln(1+v)}{v}\, dv = -\frac{\pi^2}{48}$$

the last equality because

$$\int_0^1 \frac{\ln(1+v)}{v}\, dv = \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} v^{n-1}}{n}\right) dv = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^1 v^{n-1}\, dv = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

(Note that $\displaystyle\sum_{n=1}^{\infty}\int_0^1\left|\frac{(-1)^{n-1}v^{n-1}}{n}\right|dv=\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$, so we can interchange sum and integral.)

(b) In a similar way, we have

$$\int_{\epsilon}^{1}\frac{\ln u}{u+1}\,du=(\ln\epsilon)(\ln(\epsilon+1))-\int_{\epsilon}^{1}\frac{\ln(1+u)}{u}\,du,$$

hence $K=-\dfrac{\pi^2}{12}$.

(c) Lastly,

$$\int_{\epsilon}^{1}\frac{\ln u}{1+u^2}\,du=(\ln\epsilon)(\arctan\epsilon)-\int_{\epsilon}^{1}\frac{\arctan u}{u}\,du,$$

hence

$$L=-\int_0^1\frac{\arctan u}{u}\,du=-\int_0^1\sum_{n=0}^{\infty}\frac{(-1)^n u^{2n}}{2n+1}=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=-G.$$

We conclude $I=\dfrac{1}{2}\left(-\dfrac{\pi^2}{48}+\dfrac{\pi^2}{12}+G\right)=\dfrac{G}{2}+\dfrac{\pi^2}{32}$.

Solution 7 by Moti Levy, Rehovot, Israel.

Begin with substitution $x=\dfrac{1}{t}$,

$$I:=\int_1^{\infty}\frac{x\ln(x)}{(x+1)(x^2+1)}dx=-\int_0^1\frac{\ln(u)}{(1+u)(1+u^2)}du.$$

Partial fractions expansion of $\dfrac{1}{(1+u)(1+u^2)}=\dfrac{1}{2}\dfrac{1}{1+u}-\dfrac{1}{2}\dfrac{u}{1+u^2}+\dfrac{1}{2}\dfrac{1}{1+u^2}$.

$$I=-\frac{1}{2}\int_0^1\frac{\ln(u)}{1+u}du+\frac{1}{2}\int_0^1\frac{u\ln(u)}{1+u^2}du-\frac{1}{2}\int_0^1\frac{\ln(u)}{1+u^2}du$$

By substitution $u=w^2$, the second integral becomes,

$$\frac{1}{2}\int_0^1\frac{u\ln(u)}{1+u^2}du=\frac{1}{8}\int_0^1\frac{\ln(w)}{1+w}dw.$$

Thus,

$$I=-\frac{3}{8}\int_0^1\frac{\ln(u)}{1+u}du-\frac{1}{2}\int_0^1\frac{\ln(u)}{1+u^2}du.$$

Using the integral representations of the ζ function and the Catalan's constant G,

$$\int_0^1\frac{\ln(u)}{1+u}du=-\frac{1}{2}\zeta(2),$$

$$\int_0^1\frac{\ln(u)}{1+u^2}du=-G,$$

$$I = \frac{1}{32}\pi^2 + \frac{1}{2}G.$$

Solution 8 by Narendra Bhandari, Bajura district, Nepal.

The answer is $\dfrac{G}{2} + \dfrac{\pi^2}{32}$ where G is Catalan's constant, $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^2}$. In the original integral, we substitute $x = \dfrac{1}{y}$ resulting

$$\begin{aligned}\int_1^{\infty} \frac{x\ln(x)}{(1+x)(1+x^2)}dx &= -\int_0^1 \frac{\ln(y)}{(1+y)(1+y^2)}dy = -\frac{1}{2}\int_0^1 \ln(y)\left(\frac{1}{1+y} + \frac{1-y}{1+y^2}\right)dy \\ &= -\frac{1}{2}\int_0^1 \frac{\ln(y)}{1+y}dy - \frac{1}{2}\int_0^1 \frac{\ln(y)}{1+y^2}dy - \frac{1}{2}\int_0^1 \frac{y\ln(y)}{1+y^2}dy \\ &= \frac{\pi^2}{24} + \frac{G}{2} + \frac{\pi^2}{96} = \frac{G}{2} + \frac{\pi^2}{32}\end{aligned}$$

Since

$$\begin{aligned}\int_0^1 \frac{\ln(y)}{1+y}dy &= \sum_{k=0}^{\infty}(-1)^k \int_0^1 y^k \log(y)dy = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} = -\frac{\pi^2}{12} \\ \int_0^1 \frac{\ln(y)}{1+y^2}dy &= \sum_{k=0}^{\infty}(-1)^k \int_0^1 y^{2k} \ln(y)dy = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = -G \\ \int_0^1 \frac{y\ln(y)}{1+y^2}dy &= \sum_{k=0}^{\infty}(-1)^k \int_0^1 y^{2k+1} \ln(y)dy = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2)^2} = -\frac{\pi^2}{48}\end{aligned}$$

In the above solution we used the elementary integral result, $\displaystyle\int_0^1 x^n \ln^m(x)dx = (-1)^m \frac{m!}{(n+1)^{m+1}}$ for all $n, m > -1$ and dominating convergence theorem allows us to interchange the limit of sum and integral.

Also solved by Ankush Kumar Parcha (Student) Indira Gandhi National Open University, New Delhi, India; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania; and the proposer.

• **5663** *Proposed by Ángel Plaza, Universidad de las Palmas de Gran Canaria, Spain.*

Let $n \in \mathbb{N}$ and suppose $|r| < 1$. Evaluate the sum $\displaystyle\sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n} i_j\right) r^{\left(\sum_{j=1}^{n} i_j\right)}$.

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

For $|r| < 1$, we obtain

$$\begin{aligned}
&\sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n} i_j \right) r^{\left(\sum_{j=1}^{n} i_j\right)} \\
= \quad & r \frac{d}{dr} \sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} r^{\left(\sum_{j=1}^{n} i_j\right)} \\
= \quad & r \frac{d}{dr} \left(\sum_{i_n=0}^{\infty} r^{i_n} \cdots \sum_{i_2=0}^{\infty} r^{i_2} \sum_{i_1=0}^{\infty} r^{i_1} \right) = r \frac{d}{dr} (1-r)^{-n} = \frac{nr}{(1-r)^{n+1}}.
\end{aligned}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

Let $S_n(r) := \sum_{i_n=0}^{\infty} \ldots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{0} \left(\sum_{j=1}^{n} i_j \right) r^{\left(\sum_{j=1}^{n} i_j\right)}$. We claim that

$$S_n(r) = \frac{nr}{(1-r)^{n+1}}. \qquad (1)$$

First proof
We have

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \sum_{i=0}^{\infty} i r^i = r \frac{d}{dr} \sum_{i=0}^{\infty} r^i = \frac{r}{(1-r)^2}.$$

Then

$$S_1(r) = \sum_{i_1=0}^{\infty} i_1 r^{i_1} = \frac{r}{(1-r)^2}$$

and

$$S_n(r) = \sum_{i_n=0}^{\infty} \left(\sum_{i_{n-1}=0}^{\infty} \ldots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n-1} i_j + i_n \right) r^{\left(\sum_{j=1}^{n-1} i_j\right)} \right) r^{i_n} =$$

$$= S_{n-1}(r) \sum_{i_n=0}^{\infty} r^{i_n} + \frac{1}{(1-r)^{n-1}} \sum_{i_n=0}^{\infty} i_n r^{i_n} = \frac{1}{1-r} S_{n-1}(r) + \frac{r}{(1-r)^{n+1}}.$$

So (1) follows by induction.
Second proof

$$S_n(r) = \sum_{k=0}^{\infty} k \left(\sum_{\substack{i_1 \geqslant 0, i_2 \geqslant 0, \ldots, i_n \geqslant 0 \\ i_1 + i_2 + \cdots + i_n = k}} 1 \right) r^k = \sum_{k=0}^{\infty} k \binom{k+n-1}{n-1} r^k$$

$$= \sum_{k=0}^{\infty} (k+1) \binom{k+n}{n-1} r^{k+1} = n \sum_{k=0}^{\infty} \binom{-n-1}{k} (-1)^k r^{k+1} = \frac{nr}{(1-r)^{n+1}},$$

which is (1). Here we have used three facts:
(i) The integer solutions of $i_1 + i_2 + \cdots + i_n = k$ with $i_1 \geqslant 0, i_2 \geqslant 0, \ldots, i_n \geqslant 0$ map bijectively to the permutations of k ones and n-1 separators. Therefore

$$\left(\sum_{\substack{i_1 \geqslant 0, i_2 \geqslant 0, \ldots, i_n \geqslant 0 \\ i_1 + i_2 + \cdots + i_n = k}} 1 \right) = \binom{k+n-1}{n-1}.$$

(ii) $(k+1)\binom{k+n}{n-1} = n\left(\frac{k+1+n}{n}\right)\binom{k+n}{n-1} - n\binom{k+n}{n-1} = n\binom{k+n+1}{n} - n\binom{k+n}{n-1} = n\binom{k+n}{n} =$

$$= n\frac{(k+n)!}{k!n!} = n\frac{(k+n)(k+n-1)\cdots(n+1)}{k!} = n\binom{-n-1}{k}(-1)^k.$$

(iii) The binomial series (see for instance https://en.wikipedia.org/wiki/Binomial_series):

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k,\ a \in \mathbb{C},\ |x| \leqslant 1.$$

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Working from the innermost sum outward,

$$\begin{aligned} \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n} i_j \right) r^{\left(\sum_{j=1}^{n} i_j\right)} &= r^{\left(\sum_{j=2}^{n} i_j\right)} \left(\left(\sum_{j=2}^{n} i_j \right) \sum_{i_1=0}^{\infty} r^{i_1} + \sum_{i_1=0}^{\infty} i_1 r^{i_1} \right) \\ &= r^{\left(\sum_{j=2}^{n} i_j\right)} \left(\frac{\sum_{j=2}^{n} i_j}{1-r} + \frac{r}{(1-r)^2} \right), \end{aligned}$$

$$
\begin{aligned}
\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty}\left(\sum_{j=1}^{n} i_j\right) r^{\left(\sum_{j=1}^{n} i_j\right)} &= \sum_{i_2=0}^{\infty} r^{\left(\sum_{j=2}^{n} i_j\right)}\left(\frac{\sum_{j=2}^{n} i_j}{1-r}+\frac{r}{(1-r)^2}\right)\\
&= r^{\left(\sum_{j=3}^{n} i_j\right)}\left(\frac{\sum_{j=3}^{n} i_j}{1-r}\sum_{i_2=0}^{\infty} r^{i_2}+\frac{1}{1-r}\sum_{i_2=0}^{\infty} i_2 r^{i_2}+\frac{r}{(1-r)^2}\sum_{i_2=0}^{\infty} r^{i_2}\right)\\
&= r^{\left(\sum_{j=3}^{n} i_j\right)}\left(\frac{\sum_{j=3}^{n} i_j}{(1-r)^2}+\frac{2r}{(1-r)^3}\right),\\
\sum_{i_3=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty}\left(\sum_{j=1}^{n} i_j\right) r^{\left(\sum_{j=1}^{n} i_j\right)} &= \sum_{i_3=0}^{\infty} r^{\left(\sum_{j=3}^{n} i_j\right)}\left(\frac{\sum_{j=3}^{n} i_j}{(1-r)^2}+\frac{2r}{(1-r)^3}\right)\\
&= r^{\left(\sum_{j=4}^{n} i_j\right)}\left(\frac{\sum_{j=4}^{n} i_j}{(1-r)^2}\sum_{i_3=0}^{\infty} r^{i_3}+\frac{1}{(1-r)^2}\sum_{i_3=0}^{\infty} i_3 r^{i_3}+\frac{2r}{(1-r)^3}\sum_{i_3=0}^{\infty} r^{i_3}\right)\\
&= r^{\left(\sum_{j=4}^{n} i_j\right)}\left(\frac{\sum_{j=4}^{n} i_j}{(1-r)^3}+\frac{3r}{(1-r)^4}\right),
\end{aligned}
$$

and so on, until

$$
\begin{aligned}
\sum_{i_n=0}^{\infty}\cdots\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty}\left(\sum_{j=1}^{n} i_j\right) r^{\left(\sum_{j=1}^{n} i_j\right)} &= \sum_{i_n=0}^{\infty} r^{i_n}\left(\frac{i_n}{(1-r)^{n-1}}+\frac{(n-1)r}{(1-r)^n}\right)\\
&= \frac{nr}{(1-r)^{n+1}}.
\end{aligned}
$$

Solution 4 by Michel Bataille, Rouen, France.

Let S_n be the sum to be evaluated. We show that $S_n = \dfrac{nr}{(1-r)^{n+1}}$.
The proof is by induction. First, we have

$$
S_1 = \sum_{i_1=0}^{\infty} i_1 r^{i_1} = r\sum_{i_1=1}^{\infty} i_1 r^{i_1-1} = r\frac{d}{dr}\left(\frac{1}{1-r}\right) = \frac{r}{(1-r)^2},
$$

hence the formula holds for $n = 1$.
Second, assume that for some integer $n \geqslant 2$, we have $S_{n-1} = \dfrac{(n-1)r}{(1-r)^n}$. We calculate S_n as follows:

$$
\begin{aligned}
S_n &= \sum_{i_n=0}^{\infty}\cdots\sum_{i_2=0}^{\infty} r^{i_2+\cdots+i_n}\sum_{i_1=0}^{\infty}(i_1+i_2+\cdots+i_n)r^{i_1}\\
&= \sum_{i_n=0}^{\infty}\cdots\sum_{i_2=0}^{\infty} r^{i_2+\cdots+i_n}\left(\sum_{i_1=0}^{\infty} i_1 r^{i_1} + (i_2+\cdots+i_n)\sum_{i_1=0}^{\infty} r^{i_1}\right)\\
&= \frac{r}{(1-r)^2}\sum_{i_n=0}^{\infty}\cdots\sum_{i_2=0}^{\infty} r^{i_2+\cdots+i_n} + \frac{1}{1-r}\cdot S_{n-1}\\
&= \frac{r}{(1-r)^2}\left(\sum_{i_n=0}^{\infty} r^{i_n}\right)\cdots\left(\sum_{i_2=0}^{\infty} r^{i_n}\right) + \frac{S_{n-1}}{1-r}\\
&= \frac{r}{(1-r)^2}\cdot\left(\frac{1}{1-r}\right)^{n-1} + \frac{S_{n-1}}{1-r} = \frac{r}{(1-r)^{n+1}} + \frac{(n-1)r}{(1-r)(1-r)^n} = \frac{nr}{(1-r)^{n+1}}.
\end{aligned}
$$

This completes the induction step and the proof.

Solution 5 by Moti Levy, Rehovot, Israel.

Let $S := \sum_{i_n=0}^{\infty}\cdots\sum_{i_2=0}^{\infty}\sum_{i_1=0}^{\infty}\left(\sum_{j=1}^{n} i_j\right) r^{\left(\sum_{j=1}^{n} i_j\right)}$.

The number of ways to put k unlabeled balls in n distinct bins is $\binom{n+k-1}{k}$, hence

$$S = \sum_{k=1}^{\infty}\binom{n+k-1}{k} k r^k.$$

Let $F(z) := \sum_{k=1}^{\infty}\binom{n+k-1}{k} k z^k$. By changing the order of summation and integration,

$$
\begin{aligned}
\int_0^z \frac{F(t)}{t}dt &= \int_0^z \sum_{k=1}^{\infty}\binom{n+k-1}{k} k t^{k-1} dt = \sum_{k=1}^{\infty}\binom{n+k-1}{k}\int_0^z k t^{k-1} dt\\
&= \sum_{k=0}^{\infty}\binom{n+k-1}{k} z^k = \frac{1}{(1-z)^n},
\end{aligned}
$$

$$F(z) = z\frac{d}{dz}\frac{1}{(1-z)^n} = \frac{nz}{(1-z)^{n+1}}.$$

$$S = F(r) = \frac{nr}{(1-r)^{n+1}}.$$

Solution 6 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

$S_n(r)$. We have

$$S_n(r) = \sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} (i_1 + i_2 + \cdots + i_n)\, r^{i_1+i_2+\cdots+i_n}$$
$$= \sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left\{ i_1 r^{i_1} \cdot r^{i_2+i_3+\cdots+i_n} + (i_2 + i_3 + \cdot + i_n)\, r^{i_1} \cdot r^{i_2+i_3+\cdots+i_n} \right\}.$$

Noting that, for $|r| < 1$, we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{and} \quad \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}.$$

Using these two results the inner most i_1-sum can be found. Thus

$$S_n(r) = \sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \left\{ \frac{r}{(1-r)^2} \cdot r^{i_2+i_3+\cdots+i_n} + \frac{1}{1-r} (i_2 + i_3 + \cdots + i_n)\, r^{i)2+i_3+\cdots+i_n} \right\}$$
$$= \frac{r}{(1-r)^2} \underbrace{\sum_{i_n=0}^{\infty} r^{i_n} \cdots \sum_{i_2=0}^{\infty} r^{i_2}}_{(n-1)\text{ sums}}$$
$$+ \frac{1}{1-r} \sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \left\{ i_2 r^{i_2} \cdot r^{i_3+\cdots+i_n} + (i_3 + \cdots + i_n)\, r^{i_2} \cdot r^{i_3+\cdots+i_n} \right\}$$
$$= \frac{r}{(1-r)^2} \left(\frac{1}{1-r} \right)^{n-1} + \frac{1}{1-r} \sum_{i_n=0}^{\infty} \cdots \sum_{i_3=0}^{\infty} \frac{r}{(1-r)^2} r^{i_3+\cdots+i_n}$$
$$+ \frac{1}{1-r} \sum_{i_n=0}^{\infty} \cdots \sum_{i_3=0}^{\infty} \frac{1}{1-r} (i_3 + \cdots + i_n)\, r^{i_3+\cdots+i_n}$$
$$= \frac{r}{(1-r)^{n+1}} + \frac{r}{(1-r)^3} \underbrace{\sum_{i_n=0}^{\infty} r^{i_n} \cdots \sum_{i_3=0}^{\infty} r^{i_3}}_{(n-2)\text{ sums}}$$
$$+ \frac{1}{(1-r)^2} \sum_{i_n=0}^{\infty} \cdots \sum_{i_3=0}^{\infty} (i_3 + \cdots + i_n)\, r^{i_3+\cdots+i_n}$$
$$= \frac{r}{(1-r)^{n+1}} + \frac{r}{(1-r)^3} \left(\frac{1}{1-r} \right)^{n-2}$$
$$+ \frac{1}{(1-r)^2} \sum_{i_n=0}^{\infty} \cdots \sum_{i_3=0}^{\infty} \left\{ i_3 r^{i_3} \cdot r^{i_4+\cdots+i_n} + (i_4 + \cdots + i_n)\, r^{i_3} \cdot r^{i_4+\cdots+i_n} \right\}.$$

Continuing the process $(n-2)$ more times, we find

$$\begin{aligned}
S_n(r) &= \underbrace{\frac{r}{(1-r)^{n+1}} + \cdots + \frac{r}{(1-r)^{n+1}}}_{(n-1)\text{ terms}} + \frac{1}{(1-r)^{n-1}} \sum_{i_n=0}^{\infty} i_n r^{i_n} \\
&= \frac{(n-1)r}{(1-r)^{n+1}} + \frac{1}{(1-r)^{n-1}} \cdot \frac{r}{(1-r)^2} \\
&= \frac{nr}{(1-r)^{n+1}},
\end{aligned}$$

the desired value for the sum.

Solution 7 by Ajay Srinivasan, University of Southern California, Los Angeles, CA.

Claim: $\displaystyle\sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n} i_j \right) r^{\left(\sum_{j=1}^{n} i_j\right)} = \frac{nr}{(1-r)^{n+1}}$ for all $n \in \mathbb{N}$ and $|r| < 1$.

Proof: The proof is by induction. For simplicity of notation say:

$$S(n,r) := \sum_{i_n=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n} i_j \right) r^{\left(\sum_{j=1}^{n} i_j\right)}$$

for all $n \in \mathbb{N}$ and $|r| < 1$. Let $P(n)$ be the proposition that $S(n,r) = \dfrac{nr}{(1-r)^{n+1}}$ for all $|r| < 1$. The base step is at $n = 1$.

Base Step (n=1): Since $S(1,r) = \displaystyle\sum_{i=1}^{\infty} i r^i = r \sum_{i=1}^{\infty} i r^{i-1} = \frac{r}{(1-r)^2}$ when $|r| < 1$, $P(1)$ holds.

Induction Step ($k \to k+1$): As the induction hypothesis, we assume that $S(k,r) = \dfrac{kr}{(1-r)^{k+1}}$ when $|r| < 1$. Now working with $S(k+1,r)$:

$$S(k+1,r) = \sum_{i_{k+1}=0}^{\infty} \sum_{i_k=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{k+1} i_j \right) r^{\left(\sum_{j=1}^{k+1} i_j\right)}$$

by definition. Then we can say:

$$\begin{aligned}
S(k+1,r) &= \sum_{i_{k+1}=0}^{\infty} \sum_{i_k=0}^{\infty} \cdots \sum_{i_2=0}^{\infty} \sum_{i_1=0}^{\infty} \left(\left(\sum_{j=1}^{k} i_j \right) r^{\left(\sum_{j=1}^{k+1} i_j\right)} + i_{k+1} r^{\left(\sum_{j=1}^{k+1} i_j\right)} \right) \\
&= \sum_{i_{k+1}=0}^{\infty} \left(\frac{kr \cdot r^{i_{k+1}}}{(1-r)^{k+1}} + \frac{i_{k+1} r^{i_{k+1}}}{(1-r)^k} \right) \\
&= \frac{(k+1)r}{(1-r)^{k+2}}
\end{aligned}$$

So for all natural k, $P(k+1)$ holds if $P(k)$ is true. The induction step is done.

Since $P(1)$ holds, and $P(k)$ implies $P(k+1)$ for all natural k; by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$. We are done. ■

Solution 8 by William Chang, University of Southern California, Los Angeles, CA.

Letting S_n denote the given sum, we prove by induction that $S_n = \dfrac{nr}{(1-r)^{n+1}}$. We can compute the base case $S_1 = \sum_{i_1=0}^{\infty} i_1 r^{i_1}$ by noticing $S_1 - rS_1 = \sum_{i_1=1}^{\infty} r^i = \dfrac{r}{1-r}$. Thus, $S_1 = \dfrac{r}{(1-r)^2}$, completing our base case. For the induction step, we relate S_n and S_{n-1} as follows.

$$
\begin{aligned}
S_n &= \sum_{i_n=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n} i_j \right) r^{\sum_{j=1}^{n} i_j} \\
&= \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left[\sum_{i_n=0}^{\infty} \left(i_n + \sum_{j=1}^{n-1} i_j \right) r^{i_n + \sum_{j=1}^{n-1} i_j} \right] \\
&= \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left[\sum_{i_n=0}^{\infty} i_n r^{i_n + \sum_{j=1}^{n-1} i_j} + \sum_{i_n=0}^{\infty} \left(\sum_{j=1}^{n-1} i_j \right) r^{i_n + \sum_{j=1}^{n-1} i_j} \right] \\
&= \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left[r^{\sum_{j=1}^{n-1} i_j} \sum_{i_n=0}^{\infty} i_n r^{i_n} + \left(\sum_{j=1}^{n-1} i_j \right) r^{\sum_{j=1}^{n-1} i_j} \sum_{i_n=0}^{\infty} r^{i_n} \right] \\
&= \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left[r^{\sum_{j=1}^{n-1} i_j} \frac{r}{(1-r)^2} + \left(\sum_{j=1}^{n-1} i_j \right) r^{\sum_{j=1}^{n-1} i_j} \frac{1}{1-r} \right] \\
&= \frac{r}{(1-r)^2} \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} r^{\sum_{j=1}^{n-1} i_j} + \frac{1}{1-r} \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left(\sum_{j=1}^{n-1} i_j \right) r^{\sum_{j=1}^{n-1} i_j} \\
&= \frac{r}{(1-r)^2} \left(\sum_{i=0}^{\infty} r^i \right)^{n-1} + \frac{1}{1-r} S_{n-1} \\
&= \frac{r}{(1-r)^{n+1}} + \frac{S_{n-1}}{1-r}
\end{aligned}
$$

It is now trivial to verify that our formula for S_n satisfies the recursion $S_n = \dfrac{r}{(1-r)^{n+1}} + \dfrac{S_{n-1}}{1-r}$, which completes our inductive step as desired.

Also solved by the proposer.

• **5664** *Proposed by Daniel Sitaru, National Economic College, "Theodor Costescu" Drobeta Turnu-Severin, Mehedinti, Romania.*

Prove that $\forall x, y \in \left(0, \pi/2\right):\ \ \log_{\sin x}\left(\dfrac{\sin 2x}{\sin x+\cos x}\right)+\log_{\cos x}\left(\dfrac{\sin 2x}{\sin x+\cos x}\right)\geqslant 2.$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\log_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)=\frac{\log\left(\frac{2\sin x\cos x}{\sin x+\cos x}\right)}{\log\sin x}+\frac{\log\left(\frac{2\sin x\cos x}{\sin x+\cos x}\right)}{\log\cos x}=$$

$$=2+\frac{\log\left(\frac{2\cos x}{\sin x+\cos x}\right)}{\log\sin x}+\frac{\log\left(\frac{2\sin x}{\sin x+\cos x}\right)}{\log\cos x}.$$

We need to prove that

$$\frac{\log\left(\frac{2\cos x}{\sin x+\cos x}\right)}{\log\sin x}+\frac{\log\left(\frac{2\sin x}{\sin x+\cos x}\right)}{\log\cos x}\geqslant 0$$

which is equivalent to

$$(\log\cos x)\log\left(\frac{2\cos x}{\sin x+\cos x}\right)+(\log\sin x)\log\left(\frac{2\sin x}{\sin x+\cos x}\right)\geqslant 0$$

and to

$$(\log\cos x)^2+(\log\sin x)^2\geqslant\log(\sin x\cos x)\log\left(\frac{\sin x+\cos x}{2}\right)$$

Clearly, by the AM-GM inequality,

$$\sin x\cos x\leqslant\left(\frac{\sin x+\cos x}{2}\right)^2.$$

So it is sufficient to prove that

$$\frac{(\log\cos x)^2+(\log\sin x)^2}{2}\geqslant\left(\log\left(\frac{\sin x+\cos x}{2}\right)\right)^2$$

However this inequality is true by Jensen's inequality, since the function $x\to(\log x)^2$ is convex in the interval (0, e].

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $x \in (0, \pi/2)$, and rewrite

$$\begin{aligned}\log_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)\\ &= \ln\left(\frac{\sin 2x}{\sin x+\cos x}\right)\left(\frac{1}{\ln\sin x}+\frac{1}{\ln\cos x}\right)\\ &= \ln\left(\frac{2}{\frac{1}{\sin x}+\frac{1}{\cos x}}\right)\left(\frac{1}{\ln\sin x}+\frac{1}{\ln\cos x}\right).\end{aligned}$$

Because $\ln\sin x$ and $\ln\cos x$ are both negative, the desired inequality is equivalent to

$$\ln\left(\frac{2}{\frac{1}{\sin x}+\frac{1}{\cos x}}\right)\leqslant\frac{2}{\frac{1}{\ln\sin x}+\frac{1}{\ln\cos x}}.$$

Now, $\ln x$ is an harmonically convex function, so

$$\ln\left(\frac{2}{\frac{1}{\sin x}+\frac{1}{\cos x}}\right)\leqslant\frac{1}{2}\left(\ln\sin x+\ln\cos x\right).$$

Next, by the arithmetic mean - harmonic mean inequality applied to the positive numbers $-\ln\sin x$ and $-\ln\cos x$,

$$\frac{-\ln\sin x-\ln\cos x}{2}\geqslant\frac{2}{-\frac{1}{\ln\sin x}-\frac{1}{\ln\cos x}},$$

so

$$\frac{\ln\sin x+\ln\cos x}{2}\leqslant\frac{2}{\frac{1}{\ln\sin x}+\frac{1}{\ln\cos x}}.$$

Thus,

$$\ln\left(\frac{2}{\frac{1}{\sin x}+\frac{1}{\cos x}}\right)\leqslant\frac{2}{\frac{1}{\ln\sin x}+\frac{1}{\ln\cos x}},$$

as desired.

Solution 3 by Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.

$$\frac{\sin 2x}{\sin x+\cos x}=\frac{2\sin x\cos x}{\sin x+\cos x}=\frac{2}{\frac{1}{\sin x}+\frac{1}{\cos x}}\overset{HM-GM}{\leqslant}\sqrt{\sin x\cos x}$$

$$t\to\log_{\sin x}t;\, t\to\log_{\cos x}t\text{-decreasing, because }\sin x,\cos x\in(0,1),\ \forall x\in(0,\pi/2)$$

$$\log_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)\geqslant$$

$$\log_{\sin x} \sqrt{\sin x \cos x} + \log_{\cos x} \sqrt{\sin x \cos x} =$$
$$\frac{1}{2}\left(\log_{\sin x}(\sin x \cos x) + \log_{\cos x}(\sin x \cos x)\right) =$$
$$\frac{1}{2}\left(2 + \log_{\sin x} \cos x + \log_{\cos x} \sin x\right) \overset{AM-GM}{\geqslant}$$
$$\frac{1}{2}\left(2 + \log_{\sin x} \cos x \cdot \log_{\cos x} \sin x\right) = 2$$

Solution 4 by Michel Bataille, Rouen, France.

Let $f(t) = \dfrac{1}{\ln t}$. The function f is twice differentiable on $(1, \infty)$ with

$$f'(t) = -t^{-1}(\ln t)^{-2}, \qquad f''(t) = t^{-2}(\ln t)^{-3}(2 + \ln t).$$

Since $f''(t) \geqslant 0$, the function f is convex on the interval $(1, \infty)$. It follows that for $a, b > 1$ we have

$$\frac{1}{\ln a} + \frac{1}{\ln b} \geqslant \frac{2}{\ln((a+b)/2)},$$

that is,

$$\left(\ln\left(\frac{a+b}{2}\right)\right)\left(\frac{1}{\ln a} + \frac{1}{\ln b}\right) \geqslant 2$$

(since $\dfrac{a+b}{2} > 1$). Taking $a = \dfrac{1}{\sin x}$ and $b = \dfrac{1}{\cos x}$, we obtain

$$\left(\ln\left(\frac{\sin x + \cos x}{\sin 2x}\right)\right)\left(-\frac{1}{\ln(\sin x)} - \frac{1}{\ln(\cos x)}\right) \geqslant 2$$

or

$$\left(\ln\left(\frac{\sin 2x}{\sin x + \cos x}\right)\right)\left(\frac{1}{\ln(\sin x)} + \frac{1}{\ln(\cos x)}\right) \geqslant 2,$$

which is the required result.

Solution 5 by Moti Levy, Rehovot, Israel.

Lemma 1: $\quad \ln\left(\dfrac{2}{\frac{1}{a} + \frac{1}{b}}\right) \leqslant \dfrac{\ln(a) + \ln(b)}{2}, \quad a, b > 0.$

Proof of lemma 1:

$$\begin{aligned}\frac{\ln(a)+\ln(b)}{2} &\leqslant \ln\left(\frac{a+b}{2}\right) \quad \textit{concavity}\\ \frac{1}{2}\ln(a)+\frac{1}{2}\ln(b) &\leqslant \ln(a+b)-\ln(2)\\ \ln(a)+\ln(b) &\leqslant \frac{1}{2}\ln(a)+\frac{1}{2}\ln(b)+\ln(a+b)-\ln(2)\\ \ln(a)+\ln(b)-\ln(a+b)+\ln(2) &\leqslant \frac{1}{2}\ln(a)+\frac{1}{2}\ln(b)\\ \ln\left(\frac{2ab}{a+b}\right) &\leqslant \frac{\ln(a)+\ln(b)}{2} \Longleftrightarrow \ln\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right) \leqslant \frac{\ln(a)+\ln(b)}{2}.\blacksquare\end{aligned}$$

Lemma 2: $\quad \dfrac{\ln(a)+\ln(b)}{2} \leqslant \dfrac{2}{\frac{1}{\ln(a)}+\frac{1}{\ln(b)}}, \qquad 0<a,b<1.$

Proof of lemma 2:

Since $\ln(a)<0$ and $\ln(a)<0$, the inequality $\dfrac{\ln(a)+\ln(b)}{2} \leqslant \dfrac{2}{\frac{1}{\ln(a)}+\frac{1}{\ln(b)}}$ follows from AM/HM inequality,

$$\frac{A+B}{2} \geqslant \frac{2}{\frac{1}{A}+\frac{1}{B}} \quad A,B>0.\blacksquare$$

By lemma 1 and Lemma 2,

$$\ln\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right) \leqslant \frac{2}{\frac{1}{\ln(a)}+\frac{1}{\ln(b)}}, \quad 0<a,b<1,$$

or

$$\frac{\ln\left(\frac{2}{\frac{1}{a}+\frac{1}{b}}\right)}{\frac{2}{\frac{1}{\ln(a)}+\frac{1}{\ln(b)}}} \geqslant 1, \quad 0<a,b<1. \tag{16}$$

$$\begin{aligned}&\log_{\sin(x)}\left(\frac{\sin(2x)}{\sin(x)+\cos(x)}\right)+\log_{\cos(x)}\left(\frac{\sin(2x)}{\sin(x)+\cos(x)}\right)\\ &=\frac{\ln\left(\frac{2}{\frac{1}{\sin(x)}+\frac{1}{\cos(x)}}\right)}{\ln(\sin(x))}+\frac{\ln\left(\frac{2}{\frac{1}{\sin(x)}+\frac{1}{\cos(x)}}\right)}{\ln(\cos(x))}=\ln\left(\frac{2}{\frac{1}{\sin(x)}+\frac{1}{\cos(x)}}\right)\left(\frac{1}{\ln(\sin(x))}+\frac{1}{\ln(\cos(x))}\right)\\ &=2\frac{\ln\left(\frac{2}{\frac{1}{\sin(x)}+\frac{1}{\cos(x)}}\right)}{\frac{2}{\frac{1}{\ln(\sin(x))}+\frac{1}{\ln(\cos(x))}}}.\end{aligned}$$

Since $x \in \left(0, \dfrac{\pi}{2}\right)$ $0 < \sin(x), \cos(x) < 1$, then by (16)

$$2^{\frac{\ln\left(\frac{2}{\frac{1}{\sin(x)}+\frac{1}{\cos(x)}}\right)}{\frac{2}{\frac{1}{\ln(\sin(x))}+\frac{1}{\ln(\cos(x))}}}} \geqslant 2$$

and the inequality is proved.

Solution 6 by Toyesh Prakash Sharma (Student), Agra College, India.

As $\dfrac{\sin 2x}{\sin x + \cos x}$ looks as a constant let it be a and suppose some function $f(t) = \dfrac{1}{\log_a t}$ then, $f''(t) = \dfrac{\log a(\log t + 2)}{t^2 \log^3 t} > 0$ so now applying Jensen's inequality for convex function we have

$$\frac{f(x)+f(y)}{2} \geqslant f\left(\frac{x+y}{2}\right) \Rightarrow \frac{1}{\log_a x} + \frac{1}{\log_a y} \geqslant \frac{2}{\log_a\left(\frac{x+y}{2}\right)}$$

Suppose x and y to be $\sin x$ and $\cos x$ then,

$$\frac{1}{\log_a \sin x} + \frac{1}{\log_a \cos x} \geqslant \frac{2}{\log_a\left(\frac{\sin x + \cos x}{2}\right)}$$

Using basic property of logarithms

$$\log_{\sin x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) + \log_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) \geqslant 2\log_{\left(\frac{\sin x+\cos x}{2}\right)}\left(\frac{2\sin x\cos x}{\sin x + \cos x}\right)$$

$$\log_{\sin x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) + \log_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) \geqslant 2.$$

Solution 7 by William Chang, University of Southern California, Los Angeles, CA.

Letting $u = \sin x$, $v = \cos x$, with $u, v \in (0,1)$, then

$$\begin{aligned}\log_{\sin x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) + \log_{\cos x}\left(\frac{\sin 2x}{\sin x + \cos x}\right) &= \log_u\left(\frac{2uv}{u+v}\right) + \log_u\left(\frac{2uv}{u+v}\right) \\ &= \ln\left(\frac{2uv}{u+v}\right) \cdot \frac{\ln(uv)}{\ln u \cdot \ln v}\end{aligned}$$

By AM-GM, we have $u+v \geqslant 2\sqrt{uv}$, we have that $\ln\left(\frac{2uv}{u+v}\right) \leqslant \ln(\sqrt{uv})$. We now multiply both sides of this inequality by $\frac{\ln(uv)}{\ln u \cdot \ln v} \leqslant 0$ to obtain

$$\begin{aligned}
\log_{\sin x}\left(\frac{\sin 2x}{\sin x+\cos x}\right)+\log_{\cos x}\left(\frac{\sin 2x}{\sin x+\cos x}\right) &\geqslant \ln(\sqrt{uv})\frac{\ln(uv)}{\ln u\cdot \ln v}\\
&= \frac{1}{2}\cdot\frac{(\ln u+\ln v)^2}{\ln u\cdot\ln v}\\
&= \frac{1}{2}\cdot\frac{(|\ln u|+|\ln v|)^2}{|\ln u|\cdot|\ln v|} \qquad \geqslant 2
\end{aligned}$$

where in the last line we applied AM-GM to $|\ln u|$ and $|\ln v|$.

Also solved by the proposer.

• **5665** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

Prove that if $a, b \geqslant 0$, then

$$\int_0^{1/2}\int_0^{1/2}\sqrt{a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2+b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2}\,dxdy \leqslant \frac{\sqrt{9a}+\sqrt{9b}}{64}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We prove more precisely that

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\sqrt{a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2+b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2}\,dxdy \leqslant \frac{1}{22}\sqrt{a+b}.$$

(Note that $\frac{1}{22}\sqrt{a+b} \leqslant \frac{3}{64}\left(\sqrt{a}+\sqrt{b}\right)$.)

By the inequality of Cauchy-Schwarz,

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\sqrt{a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2+b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2}\,dxdy \leqslant$$

$$\leqslant\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}dxdy\right)^{\frac{1}{2}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2+b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2\right)dxdy\right)^{\frac{1}{2}} = \frac{1}{2}C\sqrt{a+b},$$

where

$$C = \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[\ln \left(\frac{1+x}{1+xy} \right) \right]^2 dxdy \right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[\ln \left(\frac{1+y}{1+xy} \right) \right]^2 dxdy \right)^{\frac{1}{2}}.$$

C evaluates numerically to $0.0909060678120890295051151598171 < \dfrac{1}{11} = 0.\overline{09}$.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

With $0 \leqslant x, y \leqslant 1/2$,

$$\frac{1+x}{1+xy} \geqslant 1 \quad \text{and} \quad \frac{1+y}{1+xy} \geqslant 1,$$

so

$$\ln \left(\frac{1+x}{1+xy} \right) \geqslant 0, \quad \ln \left(\frac{1+y}{1+xy} \right) \geqslant 0,$$

and

$$\sqrt{a \left[\ln \left(\frac{1+x}{1+xy} \right) \right]^2 + b \left[\ln \left(\frac{1+y}{1+xy} \right) \right]^2} \leqslant \sqrt{a} \ln \left(\frac{1+x}{1+xy} \right) + \sqrt{b} \ln \left(\frac{1+y}{1+xy} \right).$$

Moreover,

$$\begin{aligned} \ln \left(\frac{1+x}{1+xy} \right) &= \ln(1+x) - \ln(1+xy) \\ &= x - xy - \sum_{n=1}^{\infty} \left(\frac{x^{2n} - x^{2n}y^{2n}}{2n} - \frac{x^{2n+1} - x^{2n+1}y^{2n+1}}{2n+1} \right). \end{aligned}$$

For each integer $n \geqslant 1$,

$$\begin{aligned} & x^{2n} - x^{2n}y^{2n} - (x^{2n+1} - x^{2n+1}y^{2n+1}) \\ & \quad = x^{2n}(1-y) \left(1 + y + y^2 + \cdots + y^{2n-1} - x(1 + y + y^2 + \cdots + y^{2n}) \right). \end{aligned}$$

With $0 \leqslant x, y \leqslant 1/2$,

$$1 + y + y^2 + \cdots + y^{2n-1} \geqslant 1 \quad \text{and} \quad x(1 + y + y^2 + \cdots + y^{2n}) \leqslant 2(1/2) = 1,$$

so $x^{2n} - x^{2n}y^{2n} - (x^{2n+1} - x^{2n+1}y^{2n+1}) \geqslant 0$. Thus,

$$\frac{x^{2n} - x^{2n}y^{2n}}{2n} - \frac{x^{2n+1} - x^{2n+1}y^{2n+1}}{2n+1} \geqslant \frac{x^{2n} - x^{2n}y^{2n} - (x^{2n+1} - x^{2n+1}y^{2n+1})}{2n+1} \geqslant 0,$$

and

$$\ln \left(\frac{1+x}{1+xy} \right) \leqslant x - xy.$$

Similarly,

$$\ln\left(\frac{1+y}{1+xy}\right) \leqslant y - xy.$$

Next,

$$\int_0^{1/2}\int_0^{1/2}(x-xy)\,dx\,dy = \int_0^{1/2}\int_0^{1/2}(y-xy)\,dx\,dy = \frac{3}{64}.$$

Finally,

$$\begin{aligned}
&\int_0^{1/2}\int_0^{1/2}\sqrt{a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2 + b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2}\,dx\,dy \\
\leqslant\ & \sqrt{a}\int_0^{1/2}\int_0^{1/2}\ln\left(\frac{1+x}{1+xy}\right)\,dx\,dy + \sqrt{b}\int_0^{1/2}\int_0^{1/2}\ln\left(\frac{1+y}{1+xy}\right)\,dx\,dy \\
\leqslant\ & \sqrt{a}\int_0^{1/2}\int_0^{1/2}(x-xy)\,dx\,dy + \sqrt{b}\int_0^{1/2}\int_0^{1/2}(y-xy)\,dx\,dy \\
=\ & \frac{3}{64}(\sqrt{a}+\sqrt{b}) = \frac{\sqrt{9a}+\sqrt{9b}}{64}.
\end{aligned}$$

Solution 3 by Michel Bataille, Rouen, France.

For $x, y \in [0, 1/2]$, we have $\frac{1+x}{1+xy} - 1 = \frac{x(1-y)}{1+xy} \geqslant 0$, hence

$$1 \leqslant \frac{1+x}{1+xy} = 1 + \frac{x(1-y)}{1+xy}.$$

Recalling that $0 \leqslant \ln(1+u) \leqslant u$ for $u \geqslant 0$, we deduce that

$$0 \leqslant \ln\left(\frac{1+x}{1+xy}\right) \leqslant \frac{x(1-y)}{1+xy}.$$

It follows that the integrand $f(x,y) = \sqrt{a\left[\ln\left(\frac{1+x}{1+xy}\right)\right]^2 + b\left[\ln\left(\frac{1+y}{1+xy}\right)\right]^2}$ satisfies

$$f(x,y) \leqslant \frac{\sqrt{ax^2(1-y)^2 + by^2(1-x)^2}}{1+xy} \leqslant \sqrt{ax^2(1-y)^2 + by^2(1-x)^2}.$$

But for $u, v \geqslant 0$ we have $\sqrt{u+v} \leqslant \sqrt{u} + \sqrt{v}$, therefore $f(x,y) \leqslant \sqrt{a}x(1-y) + \sqrt{b}y(1-x)$ and

$$\int_0^{1/2}\int_0^{1/2} f(x,y)\,dxdy \leqslant \sqrt{a}\int_0^{1/2}\int_0^{1/2} x(1-y)\,dxdy + \sqrt{b}\int_0^{1/2}\int_0^{1/2} x(1-y)\,dxdy.$$

Now, we calculate

$$\int_0^{1/2}\int_0^{1/2} y(1-x)\,dxdy = \int_0^{1/2}\int_0^{1/2} x(1-y)\,dxdy = \int_0^{1/2}\left(\int_0^{1/2}(x-xy)\,dy\right)\,dx$$

$$= \int_0^{1/2}\left[xy-\frac{xy^2}{2}\right]_0^{1/2}\,dx = \int_0^{1/2}\left(\frac{x}{2}-\frac{x}{8}\right)\,dx = \frac{3}{64}$$

and finally

$$\int_0^{1/2}\int_0^{1/2} f(x,y)\,dxdy \leqslant \frac{3}{64}(\sqrt{a}+\sqrt{b}) = \frac{\sqrt{9a}+\sqrt{9b}}{64}.$$

Solution 4 by Moti Levy, Rehovot, Israel.

Using the inequality $\sqrt{a+b} \leqslant \sqrt{a}+\sqrt{b}$, for $a,b \geqslant 0$,

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\sqrt{\left(\sqrt{a}\ln\left(\frac{1+x}{1+xy}\right)\right)^2+\left(\sqrt{b}\ln\left(\frac{1+y}{1+xy}\right)\right)^2}dxdy$$

$$\leqslant \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(\sqrt{a}\left(\ln\left(\frac{1+x}{1+xy}\right)\right)+\sqrt{b}\ln\left(\frac{1+y}{1+xy}\right)\right)dxdy$$

$$= \left(\sqrt{a}+\sqrt{b}\right)\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\ln\left(\frac{1+x}{1+xy}\right)dxdy$$

Using the inequality, $\ln(1+x) \leqslant x,\ \ for\ \ x \geqslant 0$,

$$\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\ln\left(\frac{1+x}{1+xy}\right)dxdy = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\ln\left(1+x\frac{1-y}{1+xy}\right)dxdy$$

$$\leqslant \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x\frac{1-y}{1+xy}dxdy \leqslant \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} x(1-y)\,dxdy = \frac{3}{64}.$$

Also solved by the proposer.

• **5666** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $H_n = \sum_{k=1}^{n} 1/k$, which is known as the nth harmonic number. Calculate

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\left(\zeta(2)-\sum_{k=1}^{n+1}\frac{1}{k^2}\right).$$

Editor's note: Here ζ denotes the Euler-Riemann zeta function, defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Solution 1 by Moti Levy, Rehovot, Israel.

$$S := \sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) = \sum_{n=1}^{\infty} \frac{H_n}{n+1} \sum_{k=n+2}^{\infty} \frac{1}{k^2} \tag{17}$$

The integral representation of $1/k^2$ is

$$\frac{1}{k^2} = -\int_0^1 x^{k-1} \ln(x)\, dx, \tag{18}$$

and for $1/k^3$,

$$\frac{1}{k^3} = \frac{1}{2} \int_0^1 x^{k-1} \ln^2(x)\, dx. \tag{19}$$

Plugging (18) into (17) and changing the order of summation and integration, we get

$$\begin{aligned} S &= -\sum_{n=1}^{\infty} \frac{H_n}{n+1} \sum_{k=n+2}^{\infty} \int_0^1 x^{k-1} \ln(x)\, dx \\ &= -\int_0^1 \sum_{n=1}^{\infty} \frac{H_n}{n+1} x \ln(x) \sum_{k=n+2}^{\infty} x^{k-2} dx \\ &= -\int_0^1 \sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1} \frac{\ln(x)}{1-x} dx. \end{aligned} \tag{20}$$

To find $\sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1}$, we define

$$F(x) := \sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1}$$

then

$$F'(x) := \sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x}.$$

$$F(x) = -\int_0^x \frac{\ln(1-t)}{1-t} dt = \frac{1}{2} \ln^2(1-x). \tag{21}$$

Now we substitute (21) in (20) and get the required sum as a logarithmic integral:

$$S = -\frac{1}{2} \int_0^1 \frac{\ln^2(1-x) \ln(x)}{1-x} dx \tag{22}$$

$$\begin{aligned} \int_0^1 \frac{\ln^2(1-x) \ln(x)}{1-x} dx &= \int_0^1 \frac{\ln^2(x) \ln(1-x)}{x} dx = -\int_0^1 \frac{\ln^2(x)}{x} \sum_{n=1}^{\infty} \frac{x^n}{n} dx \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \ln^2(x)\, dx = -\sum_{n=1}^{\infty} \frac{1}{n} \frac{2}{n^3} = -2\zeta(4) = -\frac{\pi^4}{45}. \end{aligned}$$

We conclude that

$$S = \zeta(4) = \frac{\pi^4}{90}.$$

Solution 2 by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia.

Denote the value of the sum by S. We show that $S = \dfrac{\pi^4}{90}$.

Write the sum as

$$S = \sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - H_{n+1}^{(2)} \right).$$

Here $H_n^{(2)} = \sum_{k=1}^{n} \frac{1}{k^2}$ denotes the nth generalised harmonic number of order two. Let

$$a_n = \frac{H_n}{n+1} \quad \text{and} \quad b_n = \zeta(2) - H_{n+1}^{(2)}.$$

To find the sum, summation by parts will be used. In view of this, we shall have a need for the following sum

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{H_k}{k+1}.$$

To find this sum, summation by parts will be used. Let

$$a_k = \frac{1}{k+1} \quad \text{and} \quad b_k = H_k.$$

Now

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=2}^{n+1} \frac{1}{k} = \sum_{k=1}^{n+1} \frac{1}{k} - 1 = H_{n+1} - 1.$$

And

$$b_{k+1} - b_k = H_{k+1} - H_k = \left(H_k + \frac{1}{k+1} \right) - H_k = \frac{1}{k+1},$$

where the recurrence relation for the harmonic numbers has been used. Applying summation by parts, namely

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k \left(b_{k+1} - b_k \right),$$

to the sum, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{H_k}{k+1} &= (H_{n+1}-1)\,H_{n+1} - \sum_{k=1}^{n} \frac{H_{k+1}-1}{k+1} \\
&= (H_{n+1}-1)\,H_{n+1} - \sum_{k=1}^{n} \frac{H_k + \frac{1}{k+1} - 1}{k+1} \\
&= (H_{n+1}-1)\,H_{n+1} - \sum_{k=1}^{n} \frac{H_k}{k+1} - \sum_{k=1}^{n} \frac{1}{(k+1)^2} + \sum_{k=1}^{n} \frac{1}{k+1} \\
&= (H_{n+1}-1)\,H_{n+1} - \sum_{k=1}^{n} \frac{H_k}{k+1} - \sum_{k=1}^{n+1} \frac{1}{k^2} + \sum_{k=1}^{n+1} \frac{1}{k} \\
\Rightarrow 2\sum_{k=1}^{n} \frac{H_k}{k+1} &= (H_{n+1}-1)H_{n+1} - H_{n+1}^{(2)} + H_{n+1} \\
\Rightarrow \sum_{k=1}^{n} \frac{H_k}{k+1} &= \frac{1}{2}\left(H_{n+1}^2 - H_{n+1}^{(2)}\right).
\end{aligned}$$

Returning to our sum, we have

$$A_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{H_k}{k+1} = \frac{1}{2}\left(H_{n+1}^2 - H_{n+1}^{(2)}\right),$$

and

$$\begin{aligned}
b_{n+1} - b_n &= \zeta(2) - H_{n+2}^{(2)} - \left(\zeta(2) - H_{n+1}^{(2)}\right) = H_{n+1}^{(2)} - H_{n+2}^{(2)} \\
&= H_{n+1}^{(2)} - \left(H_{n+1}^{(2)} + \frac{1}{(n+2)^2}\right) = -\frac{1}{(n+2)^2},
\end{aligned}$$

where the recurrence relation for the nth generalised harmonic numbers of order two has been used. Applying summation by parts, namely

$$\sum_{n=1}^{\infty} a_n b_n = \lim_{n\to\infty} A_n b_{n+1} - \sum_{n=1}^{\infty} A_n\,(b_{n+1} - b_n),$$

as $\lim_{n\to\infty} A_n b_{n+1} = 0$ (see the Appendix for a proof of this), we have

$$S = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_{n+1}^2 - H_{n+1}^{(2)}}{(n+2)^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n+1)^2}, \tag{23}$$

after a reindexing of $n \mapsto n-1$ has been made. Now, it is known that [**?**, p. 355], [**?**, Eq. (2.11), p. 62]

$$\sum_{n=1}^{\infty} \left(H_n^2 - H_n^{(2)}\right) x^n = \frac{\log^2(1-x)}{1-x}, \quad |x| < 1.$$

Replacing x with t in this identity before integrating both sides with respect to t from 0 to x, we have

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n+1} x^{n+1} = \int_0^x \frac{\log^2(1-x)}{1-x}\, dx,$$

or

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{n+1} x^{n} = -\frac{1}{3}\frac{\log^3(1-x)}{x}.$$

Integrating both sides of this equation with respect to x from 0 to 1 we obtain

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n+1)^2} = -\frac{1}{3}\int_0^1 \frac{\log^3(1-x)}{x}\, dx. \tag{24}$$

For the integral on the right that has appeared, substituting $x = 1 - e^{-u}$ we have

$$\int_0^1 \frac{\log^3(1-x)}{x}\, dx = -\int_0^\infty \frac{u^3 e^{-u}}{1-e^{-u}}\, du = -\sum_{n=0}^{\infty} \int_0^\infty u^3 e^{-(n+1)u}\, du.$$

Here the interchange made between the summation and the integration is permissible due to Tonelli's theorem since all terms involved are positive. Reindexing the sum by $n \mapsto n-1$ gives

$$\int_0^1 \frac{\log^3(1-x)}{x}\, dx = -\sum_{n=1}^{\infty} \int_0^\infty u^3 e^{-nu}\, du.$$

Enforcing a substitution of $u \mapsto \dfrac{u}{n}$ produces

$$\begin{aligned}\int_0^1 \frac{\log^3(1-x)}{x}\, dx &= -\sum_{n=1}^{\infty} \frac{1}{n^4} \cdot \int_0^\infty u^3 e^{-u}\, du = -\zeta(4)\Gamma(4)\\ &= -\frac{\pi^4}{90} \cdot 3! = -\frac{\pi^4}{15}.\end{aligned}$$

So (24) becomes

$$\sum_{n=1}^{\infty} \frac{H_n^2 - H_n^{(2)}}{(n+1)^2} = -\frac{1}{3}\left(-\frac{\pi^4}{15}\right) = \frac{\pi^4}{45},$$

and from (23) we have for our desired sum

$$S = \frac{1}{2} \cdot \frac{\pi^4}{45} = \frac{\pi^4}{90},$$

as claimed.

Appendix

In this appendix we show

$$\lim_{n\to\infty} A_n b_{n+1} = \lim_{n\to\infty} \left(H_{n+1}^2 - H_{n+1}^{(2)}\right) \cdot \left(\zeta(2) - H_{n+2}^{(2)}\right) = 0.$$

The asymptotic expansions for H_n and $H_n^{(2)}$ are [**?**, Eq. (5.11.2), p. 140; Eq. (5.15.8), p. 144]

$$H_n = \log(n) + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

and

$$H_n^{(2)} = \zeta(2) - \frac{1}{n} + \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right),$$

respectively. Here γ is the Euler–Mascheroni constant. Squaring the result for H_n gives

$$H_n^2 = \log^2(n) + 2\gamma\log(n) + \gamma^2 + \frac{\log(n)}{n} + \frac{\gamma}{n} + O\left(\frac{\log^2(n)}{n^2}\right).$$

Asymptotically for large n we have

$$\begin{aligned}\left(H_{n+1}^2 - H_{n+1}^{(2)}\right)\cdot\left(\zeta(2) - H_{n+2}^{(2)}\right) \sim \left(H_n^2 - H_n^{(2)}\right)\left(\zeta(2) - H_n^{(2)}\right)\\ = \frac{\log^2(n) + 2\gamma\log(n) - \zeta(2) + \gamma^2}{n}\\ + O\left(\frac{\log^2(n)}{n^2}\right),\end{aligned}$$

thus

$$\lim_{n\to\infty} A_n b_{n+1} = \lim_{n\to\infty}\left[\frac{\log^2(n) + 2\gamma\log(n) - \zeta(2) + \gamma^2}{n} + O\left(\frac{\log^2(n)}{n^2}\right)\right] = 0,$$

as desired.

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

Recalling $\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2}$ and separate the $n+1$ term we get the following:

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}\Big(\underbrace{\sum_{k=1}^{\infty}\frac{1}{k^2} - \sum_{k=1}^{n}\frac{1}{k^2}}_{\sum_{k=1}^{\infty}\frac{1}{(k+n)^2}} - \frac{1}{(n+1)^2}\Big)$$

The sum is equal to:

$$\underbrace{\sum_{n=1}^{\infty}\frac{H_n}{n+1}\sum_{k=1}^{\infty}\frac{1}{(k+n)^2}}_{S_1} - \underbrace{\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}}_{S_2}$$

The following equivalences will be used for the determination of S_1 and S_2:

$$\frac{1}{z} = \int_0^\infty e^{-zx}dx,\ \frac{1}{z^2} = \int_0^\infty xe^{-zx}dx \text{ in } S_1, \text{ then } \frac{2}{z^3} = \int_0^\infty x^2e^{-zx}dx \text{ in } S_2.$$

Applying them:

$$S_1 = \sum_{n=1}^{\infty} H_n \int_0^\infty e^{-(n+1)y}dy \sum_{k=1}^{\infty} \int_0^\infty xe^{-(k+n)x}dx$$

$$S_2 = \sum_{n=1}^{\infty} \int_0^\infty H_n x^2 e^{-(n+1)x}dx$$

1. Integral form of S_1

The order of summations and integrations can be exchanged then the sums can be evaluated:

$$S_1 = \int_0^\infty \int_0^\infty \sum_{n=1}^{\infty} H_n e^{-y(n+1)} \sum_{k=1}^{\infty} xe^{-(k+n)x}dydx = \int_0^\infty xe^{-y} \int_0^\infty \underbrace{\sum_{n=1}^{\infty} H_n e^{-(x+y)(n)}}_{-\frac{\ln(1-e^{-(x+y)})}{1-e^{-(x+y)}}} \underbrace{\sum_{k=1}^{\infty} e^{-kx}}_{\frac{e^{-x}}{1-e^{-x}}} dydx$$

$$S_1 = -\int_0^\infty \int_0^\infty x\frac{e^{-(x+y)}}{1-e^{-x}} \frac{\ln(1-e^{-(x+y)})}{1-e^{-(x+y)}}dydx$$

Performing the following substitutions: $e^{-x} = t$ and $e^{-y} = u$ we get:

$$S_1 = \int_0^1 \int_0^1 \frac{\ln(t)}{(1-t)} \frac{\ln(1-ut)}{1-ut}dudt$$

Using the $ut = z$ substitution:

$$S_1 = \int_0^1 \frac{\ln(t)}{(1-t)t} \underbrace{\int_0^t \frac{\ln(1-z)}{1-z}dz}_{-\frac{1}{2}\ln^2(1-t)}\,dt = -\frac{1}{2}\int_0^1 \frac{\ln(t)\ln^2(1-t)}{(1-t)t}dt$$

After partial fraction decomposition:

$$S_1 = -\frac{1}{2}\int_0^1 \frac{\ln(t)\ln^2(1-t)}{1-t}dt - \frac{1}{2}\int_0^1 \frac{\ln(t)\ln^2(1-t)}{t}dt$$

and exchange $t \to (1-t)$ we get:

$$S_1 = -\frac{1}{2}\int_0^1 \frac{\ln(1-t)\ln^2(t)}{t}dt - \frac{1}{2}\int_0^1 \frac{\ln^2(t)\ln(1-t)}{1-t}dt$$

2. Integral form of S_2

S_2 can be calculated similar way as was used in section 1.

$$S_2 = -\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3} = -\frac{1}{2}\sum_{n=1}^{\infty}\int_0^{\infty} x^2 H_n e^{-(n+1)x}dx$$

Swap the order of the sumation and integration we get:

$$S_2 = -\frac{1}{2}\int_0^{\infty} x^2 e^{-x} \underbrace{\sum_{n=1}^{\infty} H_n e^{-nx}}_{\frac{-\ln(1-e^{-x})}{1-e^{-x}}} dx$$

Performing the following substitution: $e^{-x} = t$:

$$S_2 = \frac{1}{2}\int_0^1 \frac{\ln^2(t)\ln(1-t)}{1-t}dt$$

3. Determination of S

Based on the results of the section 1-2, S can be calculated:

$$S = S_1 + S_2 = -\frac{1}{2}\int_0^1 \frac{\ln(1-t)\ln^2(t)}{t}dt$$

Applying the facts: $\frac{d(x^a)}{da}_{|a=0} = \ln(x)$ and $\ln(1-x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}(-x)^{k+1}$

$$S = -\frac{1}{2}\frac{d^2}{da,b}\Big(\int_0^1 t^{a+b}\sum_{k=0}^{\infty}\frac{(-1)^k(-t)^k}{k+1}\Big)_{|a=b=0}dt$$

Performing the integration, then the derivations according to a and b:

$$S = -\frac{1}{2}\frac{d^2}{da,b}\Big(\sum_{k=1}^{\infty}\frac{1}{k(k+a+b)}\Big)_{|a=b=0} = \sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$\text{The result is } \sum_{n=1}^{\infty} \frac{H_n}{n+1}\Big(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2}\Big) = \zeta(4)$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

We claim that

$$S := \sum_{n=1}^{\infty} \frac{H_n}{n+1} \left((2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) = (4) = \frac{\pi^4}{90}.$$

It is well known that

$$H_n = \int_0^1 \frac{1-x^n}{1-x} dx$$

and

$$(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=n+2}^{\infty} \frac{1}{k^2} = -\sum_{k=n+2}^{\infty} \int_0^1 x^{k-1} \log x \, dx = -\int_0^1 \frac{x^{n+1}}{1-x} \log x \, dx.$$

Hence

$$S = -\sum_{n=1}^{\infty} \frac{1}{n+1} \int_0^1 \frac{1-u^n}{1-u} du \int_0^1 \frac{v^{n+1}}{1-v} \log v \, dv =$$

$$= \int_0^1 \int_0^1 \frac{1}{1-u} \frac{1}{1-v} \log v \left(\log(1-v) - \frac{1}{u} \log(1-uv) \right) du dv.$$

We first integrate with respect to u and find

$$\int_0^1 \frac{1}{1-u} \left(\log(1-v) - \frac{1}{u} \log(1-uv) \right) du = \int_0^1 \left(\frac{1}{1-u} + \frac{1}{u} \right) \left(u \log(1-v) - \log(1-uv) \right) du =$$

$$= \int_0^1 \frac{1}{u} \Big(u \log(1-v) - \log(1-uv) + (1-u) \log(1-v) - \log\big(1-(1-u)v\big) \Big) du =$$

$$= \int_0^1 \frac{1}{u} \left(-\log(1-uv) - \log\left(\frac{1-(1-u)v}{1-v} \right) \right) du = Li_2(v) + Li_2\left(-\frac{v}{1-v} \right) = -\frac{\big(\log(1-v)\big)^2}{2},$$

according a classical dilogarithm identity (e.g., https://en.wikipedia.org/wiki/Spence%27s_function). Therefore

$$S = -\frac{1}{2} \int_0^1 \frac{1}{1-v} (\log v) \big(\log(1-v)\big)^2 dv.$$

We perform the variable change v→1-v and then integrate by parts. We obtain

$$S = -\frac{1}{2} \int_0^1 \frac{1}{v} \big(\log(1-v)\big) (\log v)^2 dv = -\frac{1}{6} \int_0^1 \frac{1}{1-v} (\log v)^3 dv.$$

Finally, we use the expansion $\frac{1}{1-v} = \sum_{k=0}^{\infty} v^k$ and integrate termwise to get

$$S = -\frac{1}{6} \sum_{k=0}^{\infty} \int_0^1 v^k (\log v)^3 dv = \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = (4) = \frac{\pi^4}{90}.$$

Solution 5 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

With
$$\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{k=n+2}^{\infty} \frac{1}{k^2} = -\sum_{k=n+2}^{\infty} \int_0^1 x^{k-1} \ln x \, dx = -\int_0^1 \frac{x^{n+1}}{1-x} \ln x \, dx,$$
it follows that
$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) = -\int_0^1 \left(\sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1} \right) \frac{\ln x}{1-x} \, dx.$$

The generating function for the harmonic numbers is
$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x};$$
upon integration, this yields
$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1} = \frac{1}{2} \ln^2(1-x).$$

Thus,
$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) = -\frac{1}{2} \int_0^1 \frac{\ln^2(1-x) \ln x}{1-x} \, dx = -\frac{1}{2} \int_0^1 \frac{\ln^2 x \ln(1-x)}{x} \, dx.$$

With integration by parts, this becomes
$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) &= -\frac{1}{6} \ln^3 x \ln(1-x) \bigg|_0^1 - \frac{1}{6} \int_0^1 \frac{\ln^3 x}{1-x} \, dx \\ &= -\frac{1}{6} \int_0^1 \frac{\ln^3 x}{1-x} \, dx. \end{aligned}$$

Finally,
$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) = -\frac{1}{6} \sum_{k=0}^{\infty} \int_0^1 x^k \ln^3 x \, dx = \sum_{k=0}^{\infty} \frac{1}{(k+1)^4} = \zeta(4).$$

Solution 6 by Michel Bataille, Rouen, France.

Let S denote the sum to be evaluated. We show that $S = \dfrac{\pi^4}{90}$.

The following known result will be needed: if r, s are nonnegative integers, then $\int_0^1 x^r (\ln x)^s = \frac{(-1)^s s!}{(r+1)^{s+1}}$ (this is readily proved by induction on s with the help of an integration by parts).
We have

$$\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} = \sum_{j=1}^{\infty} \frac{1}{(n+1+j)^2} = -\sum_{j=1}^{\infty} \int_0^1 x^{n+j}(\ln x)\, dx$$

$$= -\int_0^1 (x^n \ln x) \left(\sum_{j=1}^{\infty} x^j \right) dx = -\int_0^1 \frac{x^{n+1} \ln x}{1-x}\, dx$$

(we can interchange sum and integral since $x^{n+j}(\ln x)$ has a constant sign on $(0,1]$).
We deduce that

$$S = -\sum_{n=1}^{\infty} \int_0^1 \frac{H_n}{n+1} \cdot \frac{x^{n+1} \ln x}{1-x}\, dx = -\int_0^1 \left(\frac{\ln x}{1-x} \sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1} \right) dx.$$

But, for $|x| < 1$, it is readily checked that $(1-x)\sum_{n=1}^{\infty} H_n x^n = -\ln(1-x)$ so that $\sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1} = \frac{(\ln(1-x))^2}{2}$. This leads to

$$S = -\frac{1}{2}\int_0^1 \frac{(\ln x)(\ln(1-x))^2}{(1-x)}\, dx = -\frac{1}{2}\int_0^1 \frac{(\ln(1-x))(\ln x)^2}{x}\, dx = \frac{1}{2}\int_0^1 \frac{(\ln x)^2}{x} \cdot \left(\sum_{n=1}^{\infty} \frac{x^n}{n} \right) dx$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1}(\ln x)^2\, dx = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{2(-1)^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Solution 7 by Narendra Bhandari, Bajura district, Nepal.

The answer is $\zeta(4) = \frac{\pi^4}{90}$. We note that

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \left(\sum_{k=1}^{n+1} + \sum_{k=n+2}^{\infty} \right) \frac{1}{k^2} \tag{25}$$

Using the result from (1),the original sum can be written as

$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} \left(\zeta(2) - \sum_{k=1}^{n+1} \frac{1}{k^2} \right) = \sum_{n=1}^{\infty} \frac{H_n}{n+1} \sum_{k=n+2}^{\infty} \frac{1}{k^2} = \sum_{n=1}^{\infty} \frac{H_n}{n+1} \sum_{k=1}^{\infty} \frac{1}{(k+n+1)^2}$$

$$= -\sum_{n=1}^{\infty} \frac{H_n}{n+1} \sum_{k=1}^{\infty} \int_0^1 x^{n+k} \log(x) dx = -\int_0^1 \log(x) \left\{ \sum_{n=1}^{\infty} \frac{H_n}{n+1} x^n \sum_{k=1}^{\infty} x^k \right\} dx$$

$$= -\frac{1}{2}\int_0^1 \frac{\log(x)\log^2(1-x)}{1-x} dx = \zeta(4)$$

Since

$$\int_0^1 \frac{\log(x)\log^2(1-x)}{1-x}dx \overset{\text{IBP}}{=} \int_0^1 \frac{\log^3(1-x)}{x}dx - 2\int_0^1 \frac{\log^2(1-x)\log(x)}{1-x}dx \Rightarrow$$
$$\int_0^1 \frac{\log(x)\log^2(1-x)}{1-x}dx = \frac{1}{3}\int_0^1 \frac{\log^3(x)}{1-x}dx = -2\sum_{m=1}^{\infty}\frac{1}{m^4} = -2\zeta(4)$$

The dominating convergence theorem allows us to interchange the limit of the sums and integral wherever needed and we utilize the following results in the above solution.

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}x^{n+1} = \frac{\log^2(1-x)}{2} \text{ and } \int_0^1 y^n \log^m(y)dy = (-1)^m\frac{m!}{(n+1)^{m+1}}$$

for $x \in [-1,1)$ and $n, m > -1$.

Also solved by the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.

- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣