Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before February 1, 2023.

• **5697** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve the following equation in real numbers:

$$\left(x^2-6x+5\right)^5+\left(x^2-9x+14\right)^5-\left(2x^2-15x+19\right)^5=0.$$

• **5698** *Proposed by Florică Anastase, "Alexandru Odobescu" high school, Lehliu-Gară, Călăraşi, Romania.*

Prove

$$\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}dx \geqslant \log(1+\sqrt{2})\log(e\sqrt{2}).$$

• **5699** *Proposed by Narendra Bhandari, Bajura, Nepal.*

Prove

$$\sum_{n=1}^{\infty}\binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{64^n(2n-1)^2}=\frac{\sinh^{-1}(1)}{8\pi}.$$

• **5700** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata," Rome, Italy.*

Evaluate

$$\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x \ln(\cos x)}{1+\sin x \cos x} dx.$$

• **5701** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

Evaluate

$$\int e^{\cos^{-1} t} \cdot \left(\frac{3t^2-1}{\sqrt{t^3-t^5}} \right) dt.$$

• **5702** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

(a) Calculate

$$\lim_{n\to\infty} n \left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2} \mathrm{d}x \right).$$

(b) Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2} \mathrm{d}x \right).$$

Solutions

To Formerly Published Problems

• **5679** *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma.*

Compute

$$\lim_{n\to+\infty} (-1)^n \cdot \sin \left(\sum_{k=1}^{3n} \arctan \frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11} \right).$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Note

$$\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11} = \frac{\sqrt{3}-a_k}{1+\sqrt{3}a_k},$$

where

$$a_k = \frac{\sqrt{3}(2k^2-4)}{k^4+3k^2+1}.$$

Next, for integer $k \geqslant 2$,

$$\begin{aligned}\arctan\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11} &= \frac{\pi}{3} - \arctan\frac{\sqrt{3}(2k^2-4)}{k^4+3k^2+1} \\ &= \frac{\pi}{3} - \arctan\frac{\sqrt{3}(k-1)}{1+(k-1)^2} + \arctan\frac{\sqrt{3}(k+1)}{1+(k+1)^2}.\end{aligned}$$

For $k=1$, the formula is

$$\arctan\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11} = -\frac{2\pi}{3} - \arctan\frac{\sqrt{3}(k-1)}{1+(k-1)^2} + \arctan\frac{\sqrt{3}(k+1)}{1+(k+1)^2}.$$

Thus,

$$\begin{aligned}&\sum_{k=1}^{3n}\arctan\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11} \\ &\quad= (n-1)\pi + \arctan\frac{\sqrt{3}(3n+1)}{1+(3n+1)^2} + \arctan\frac{\sqrt{3}\cdot 3n}{1+(3n)^2} - \arctan\frac{\sqrt{3}}{2},\end{aligned}$$

so

$$\begin{aligned}&\sin\left(\sum_{k=1}^{3n}\arctan\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11}\right) \\ &\quad= (-1)^{n-1}\sin\left(\arctan\frac{\sqrt{3}(3n+1)}{1+(3n+1)^2} + \arctan\frac{\sqrt{3}\cdot 3n}{1+(3n)^2} - \arctan\frac{\sqrt{3}}{2}\right)\end{aligned}$$

and

$$\lim_{n\to\infty}(-1)^n\cdot\sin\left(\sum_{k=1}^{3n}\arctan\frac{\sqrt{3}(k^4+k^2+5)}{k^4+9k^2-11}\right) = \sin\left(\arctan\frac{\sqrt{3}}{2}\right) = \sqrt{\frac{3}{7}}.$$

Solution 2 by Moti Levy, Rehovot, Israel.

$$\begin{aligned}&\sum_{k=1}^{3n}\arctan\left(\sqrt{3}\frac{k^4+k^2+5}{k^4+9k^2-11}\right) \\ &= \sum_{k=1}^{3n}\arctan\left(\sqrt{3}\right) + \sum_{k=1}^{3n}\left(\arctan\left(\sqrt{3}\frac{k^4+k^2+5}{k^4+9k^2-11}\right) - \arctan\left(\sqrt{3}\right)\right) \\ &= \sum_{k=1}^{3n}\arctan\left(\sqrt{3}\right) + \sum_{k=1}^{3n}\arctan\frac{\sqrt{3}\frac{k^4+k^2+5}{k^4+9k^2-11} - \sqrt{3}}{1+3\frac{k^4+k^2+5}{k^4+9k^2-11}} = n\pi + \sum_{k=1}^{3n}\arctan\left(2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)\end{aligned}$$

$$\sin\left(\sum_{k=1}^{3n}\arctan\left(\sqrt{3}\frac{k^4+k^2+5}{k^4+9k^2-11}\right)\right)=\sin(n\pi)\cos\left(\sum_{k=1}^{3n}\arctan 2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)$$
$$+\cos(n\pi)\sin\left(\sum_{k=1}^{3n}\arctan 2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)$$
$$=(-1)^n\sin\left(\sum_{k=1}^{3n}\arctan 2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)$$

$$\lim_{n\to\infty}(-1)^n\sin\left(\sum_{k=1}^{3n}\arctan\left(\sqrt{3}\frac{k^4+k^2+5}{k^4+9k^2-11}\right)\right)=\lim_{n\to\infty}\sin\left(\sum_{k=1}^{3n}\arctan 2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)$$
$$=\sin\left(\sum_{k=1}^{\infty}\arctan\left(2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)\right).$$

Now we have the task of evaluating the infinite sum $\sum_{k=1}^{\infty}\arctan\left(2\sqrt{3}\frac{k^2-2}{k^4+3k^2+1}\right)$.

Let $s=\sigma+i\tau\in\mathbb{C}$. Then

$$\arg\left(1+s\frac{k^2-2}{k^4+2k^2+3}\right)=\arg\left(\frac{k^4+2k^2+3+\sigma\left(k^2-2\right)}{k^4+2k^2+3}+i\frac{\tau\left(k^2-2\right)}{k^4+2k^2+3}\right)\qquad(1)$$
$$=\arctan\left(\frac{\tau\left(k^2-2\right)}{k^4+2k^2+3+\sigma\left(k^2-2\right)}\right).$$

Setting $s=1+2\sqrt{3}i$ in (1),

$$\arg\left(1+\left(1+2\sqrt{3}i\right)\frac{k^2-2}{k^4+2k^2+3}\right)=\arctan\left(\frac{2\sqrt{3}\left(k^2-2\right)}{k^4+3k^2+1}\right)$$

$$\sum_{k=1}^{\infty}\arctan\left(\frac{2\sqrt{3}\left(k^2-2\right)}{k^4+3k^2+1}\right)=\arg\left(\prod_{k=1}^{\infty}\left(1+\left(1+2\sqrt{3}i\right)\frac{k^2-2}{k^4+2k^2+3}\right)\right).$$

The infinite product $\prod_{k=1}^{\infty}\left(1+s\frac{k^2-2}{k^4+2k^2+3}\right)$ will be evaluated using the Summation Theorem.

Theorem

Let $f(z)$ be analytic in $\mathbb{C}$ except for some finite set of isolated singular points. Also let $\left|f(z)\right|<\frac{M}{|z|^k}$ along the rectangular path C_N, where $k>1$ and M is constant independent of N. Then

$$\sum_{n=-\infty}^{\infty}f(n)=-\sum\left\{residues\ \ of\ \ \pi\cot(\pi z)*f(z)\quad at\ \ poles\ of\ \ f\right\}.$$

We apply the theorem on $\log \prod_{k=1}^{\infty}\left(1+s\frac{k^2-2}{k^4+2k^2+3}\right)=\frac{1}{2}\sum_{k=-\infty}^{\infty}\log\left(1+s\frac{k^2-2}{k^4+2k^2+3}\right)$.

The function $\pi\cot(\pi z)\log\left(1+s\frac{z^2-2}{z^4+2z^2+3}\right)$ has one pole of order 1 at $z=0$.

The residue at $z=0$ is

$$\text{residue}\left\{\pi\cot(\pi z)\log\left(1+s\frac{z^2-2}{z^4+2z^2+3}\right),\ \text{at } z=0\right\}=\ln\left(1-\frac{2s}{3}\right).$$

By the Summation Theorem

$$\log\prod_{k=1}^{\infty}\left(1+s\frac{k^2-2}{k^4+2k^2+3}\right)=-\frac{1}{2}\ln\left(1-\frac{2s}{3}\right).$$

$$\prod_{k=1}^{\infty}\left(1+s\frac{k^2-2}{k^4+2k^2+3}\right)=e^{-\frac{1}{2}\ln\left(1-\frac{2z}{3}\right)}=\frac{\sqrt{3}}{\sqrt{3-2s}}.$$

$$\sum_{k=1}^{\infty}\arctan\left(\frac{2\sqrt{3}\left(k^2-2\right)}{k^4+3k^2+1}\right)=\arg\left(\frac{\sqrt{3}}{\sqrt{3-2\left(1+2\sqrt{3}i\right)}}\right)=\arctan\frac{\sqrt{3}}{2}$$

We conclude that the limit is equal to $\sin\left(\arctan\frac{\sqrt{3}}{2}\right)=\sqrt{\frac{3}{7}}$.

Also solved by Albert Stadler, Herrliberg, Switzerland and the proposer.

• **5680** *Proposed by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Saudi Arabia..*

Let

$$I=\int_0^{\frac{\pi}{2}}\frac{\cos(2x)}{\log(\tan x)}\,dx \quad \text{and} \quad J=\int_0^{\infty}\frac{\tanh(x)\,\text{sech}(x)}{x}\,dx.$$

(a) Show that the ratio I/J exists, and without explicitly evaluating either of the improper integrals, find its value. (b) Find the value of I.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

(a) We perform a change of variables: y=tan x, dy=1/cos^{2}x dx and use that

$$\cos(2x)=\cos^2 x-\sin^2 x=\frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x}=\frac{1-y^2}{1+y^2},\ \cos^2 x=\frac{\cos^2 x}{\cos^2 x+\sin^2 x}=\frac{1}{1+y^2}.$$

Then

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(2x)}{\log(\tan x)} dx = \int_0^{\infty} \frac{1-y^2}{(1+y^2)^2} \bullet \frac{1}{\log(y)} dy \underbrace{=}^{\substack{y=e^z \\ dy=e^z dz}} \int_{-\infty}^{\infty} \frac{1-e^{2z}}{(1+e^{2z})^2} \bullet \frac{e^z}{z} dz =$$

$$= -\int_{-\infty}^{\infty} \frac{e^z - e^{-z}}{z\left(e^z + e^{-z}\right)^2} dz = -2\int_0^{\infty} \frac{e^z - e^{-z}}{z\left(e^z + e^{-z}\right)^2} dz = -\int_0^{\infty} \frac{\tanh(x)\,\mathrm{sech}(x)}{x} dx = -J.$$

So I/J=-1.
(b) We will prove that J=4G/π, where G denotes Catalan's constant. With (a) we conclude that I=-4G/π.
Partial integration yields

$$J = \int_{-\infty}^{\infty} \frac{e^z - e^{-z}}{z\left(e^z + e^{-z}\right)^2} dz = -\frac{1}{z}\left(\frac{1}{e^z + e^{-z}} - \frac{1}{2}\right)\Bigg|_{z=-\infty}^{z=\infty} - \int_{-\infty}^{\infty} \frac{1}{z^2}\left(\frac{1}{e^z + e^{-z}} - \frac{1}{2}\right) dz =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{z^2} \frac{\left(e^{\frac{z}{2}} - e^{-\frac{z}{2}}\right)^2}{e^z + e^{-z}} dz = \int_{-\infty}^{\infty} \frac{\sinh^2\left(\frac{z}{2}\right)}{z^2 \cosh z} dz.$$

We evaluate this integral by means of the residue calculus. We note that

$$\frac{\sinh^2\left(\frac{z}{2}\right)}{z^2 \cosh z}$$

is a bounded and analytic function in the region $\mathbb{C} \backslash \cup_{k\mathbb{Z}} \{z \in \mathbb{C}: |z-(k+1/2)i\pi| \leqslant 1/2\}$. Therefore, if we shift the line of integration upwards (and parallel to the real line) we pick up the residues at the poles that are located at z=(k+1/2)iπ, k=0, 1, 2, . . . , while the integral along the shifted line of integration tends to zero. Hence

$$J = \int_{-\infty}^{\infty} \frac{\sinh^2\left(\frac{z}{2}\right)}{z^2 \cosh z} dz = 2\pi i \sum_{k=0}^{\infty} \frac{\sinh^2\left(\frac{z}{2}\right)}{z^2 \sinh z}\Bigg|_{z=\left(k+\frac{1}{2}\right)i\pi} =$$

$$= 2\pi i \sum_{k=0}^{\infty} \frac{-\sin^2\left(\frac{(2k+1)\pi}{4}\right)}{-\left(k+\frac{1}{2}\right)^2 \pi^2 i \sin\left(\left(k+\frac{1}{2}\right)\pi\right)} = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \frac{4G}{\pi}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

(a) With the substitution $x = \tan^{-1} t$,

$$I = \int_0^\infty \frac{1 - t^2}{(1 + t^2)^2 \log t}\, dt.$$

On the other hand, with the substitution $x = \log t$,

$$J = 2 \int_1^\infty \frac{t^2 - 1}{(1 + t^2)^2 \log t}\, dt.$$

Moreover, with the change of variable $t \to 1/t$,

$$\int_1^\infty \frac{t^2 - 1}{(1 + t^2)^2 \log t}\, dt = \int_0^1 \frac{t^2 - 1}{(1 + t^2)^2 \log t}\, dt.$$

Thus,

$$J = \int_0^\infty \frac{t^2 - 1}{(1 + t^2)^2 \log t}\, dt = -I,$$

and the ratio $I/J = -1$.

(b) Let

$$\mathcal{F}(s) = \int_0^\infty \frac{1 - t^s}{(1 + t^2)^2 \log t}\, dt.$$

Then

$$\mathcal{F}'(s) = -\int_0^\infty \frac{t^s}{(1 + t^2)^2}\, dt.$$

With the substitution $t = \tan\theta$,

$$\begin{aligned}
\mathcal{F}'(s) &= -\int_0^{\pi/2} \sin^s \theta \cos^{2-s} \theta \, d\theta \\
&= -\frac{1}{2} B\left(\frac{1+s}{2}, \frac{3-s}{2}\right) = -\frac{1-s}{4} B\left(\frac{1+s}{2}, \frac{1-s}{2}\right) \\
&= -\frac{\pi(1-s)}{4} \cdot \frac{1}{\sin \frac{\pi(1-s)}{2}},
\end{aligned}$$

where $B(x, y)$ is the beta function. It then follows that

$$I = \mathcal{F}(2) = -\frac{\pi}{4} \int_0^2 \frac{1-s}{\sin \frac{\pi(1-s)}{2}}\, ds.$$

The substitutions

$$y = \frac{\pi(1-s)}{2} \quad \text{followed by} \quad u = \tan\frac{y}{2}$$

yields

$$I = -\frac{4}{\pi}\int_0^1 \frac{\tan^{-1} u}{u}\, du = -\frac{4}{\pi}G,$$

where

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

is Catalan's constant.

Solution 3 by Moti Levy, Rehovot, Israel.

(a) By change of the integration variable $u = \ln\left(\tan(x)\right)$,

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\cos(2x)}{\ln\left(\tan(x)\right)} dx = \int_{-\infty}^{\infty} \frac{1}{u}\left(\cos\left(2\arctan\left(e^u\right)\right)\right)\frac{e^u}{e^{2u}+1}\, du \\
&= \int_{-\infty}^{\infty} \frac{1}{u}\left(\cos^2\left(\arctan\left(e^u\right)\right) - \sin^2\left(\arctan\left(e^u\right)\right)\right)\frac{e^u}{e^{2u}+1}\, du \\
&= -\int_{-\infty}^{\infty} \frac{1}{u}\left(\frac{e^u - e^{-u}}{e^u + e^{-u}}\right)\frac{1}{e^u + e^{-u}}\, du = -\int_0^{\infty} \frac{1}{u}\left(\frac{e^u - e^{-u}}{e^u + e^{-u}}\right)\frac{2}{e^u + e^{-u}}\, du \\
&= -\int_0^{\infty} \frac{\tanh(u)}{u\cosh(u)}\, du = -J.
\end{aligned}$$

It follows that $\frac{I}{J} = -1$.

(b) Now we evaluate I by complex integration.

The following representation of $\frac{1}{\Gamma(z)}$ as contour integral is well known

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^t}{t^z} dt,$$

where $\mathcal{H}$ is the Hankel contour, which consists of three parts ($\delta \to 0+$):

1) the line segment $[-\infty + i\delta, i\delta]$, denoted by $\mathcal{L}_+$.

2) a small semicircle around the origin with radius δ, denoted by sc.

3) the line segment $[-\infty - i\delta, -i\delta]$, denoted by $\mathcal{L}_-$.

Note that this contour fixes the branch cut of log to lie on the negative real axis.

By change of variable $t \to ty$, we get

$$\frac{y^z}{\Gamma(z+1)} = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{e^{ty}}{t^{z+1}} dt, \tag{2}$$

We replace y in (2) with $\ln\left(\tan\left(y\right)\right)$ and multiply both sides by $\cos\left(2y\right)$ to get

$$\frac{\ln^{z}\left(\tan\left(y\right)\right)\cos\left(2y\right)}{\Gamma\left(z+1\right)} = \frac{1}{2\pi i}\int_{\mathcal{H}}\frac{\tan^{t}\left(y\right)\cos\left(2y\right)}{t^{z+1}}dt \tag{3}$$

Now we integrate both sides of (3) with respect to y from 0 to $\frac{\pi}{2}$ and define $K\left(z\right)$,

$$K\left(z\right) := \int_{0}^{\frac{\pi}{2}}\frac{\ln^{z}\left(\tan\left(y\right)\right)\cos\left(2y\right)}{\Gamma\left(z+1\right)}dy = \frac{1}{2\pi i}\int_{\mathcal{H}}\frac{1}{t^{z+1}}\left(\int_{0}^{\frac{\pi}{2}}\tan^{t}\left(y\right)\cos\left(2y\right)dy\right)dt \tag{4}$$

Entry **3.636.2** from *I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, 7-th edition,* is

$$\int_{0}^{\frac{\pi}{2}}\tan^{t}\left(y\right)\cos\left(2y\right)dy = -\frac{\pi}{2}\sec\left(\frac{\pi t}{2}\right). \tag{5}$$

We substitute (5) in (4) to get

$$K\left(z\right) = -\frac{1}{4\pi i}\int_{\mathcal{H}}\frac{\pi}{t^{z}}\sec\left(\frac{\pi t}{2}\right)dt. \tag{6}$$

$$\operatorname{sech}\left(x\right) = 2\sum_{n=0}^{\infty}\left(-1\right)^{n}e^{-(2n+1)x}$$

and since $\sec\left(x\right) = \operatorname{sech}\left(-ix\right)$

$$\sec\left(x\right) = 2\sum_{n=0}^{\infty}\left(-1\right)^{n}e^{i(2n+1)x}. \tag{7}$$

Replacing (7) in (6),

$$\begin{aligned} K\left(z\right) &:= -\frac{1}{2\pi i}\int_{\mathcal{H}}\frac{\pi}{t^{z}}\sum_{n=0}^{\infty}\left(-1\right)^{n}e^{i(2n+1)\frac{\pi t}{2}}dt \\ &= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\pi\left(-1\right)^{n}\int_{\mathcal{H}}\frac{e^{i(2n+1)\frac{\pi t}{2}}}{t^{z}}dt. \end{aligned}$$

$$\int_{\mathcal{H}}\frac{e^{i(2n+1)\frac{\pi t}{2}}}{t^{z}}dt$$

Now we evaluate $\int_{\mathcal{H}}\frac{e^{i(2n+1)\frac{\pi t}{2}}}{t^{z}}dt.$

On the upper edge of the cut, i.e., on $\mathcal{L}_{+}$,

$$t = ue^{\pi i}\ ,\ \ u > 0,$$

and the integrand is

$$\frac{e^{i(2n+1)\frac{\pi ue^{\pi i}}{2}}}{u^{z}e^{z\pi i}} = \frac{1}{u^{z}}e^{-i\pi z}e^{-i\pi nu}e^{-\frac{1}{2}i\pi u}$$

On the lower edge of the cut, i.e., on $\mathcal{L}_-$,

$$t = ue^{-\pi i} \quad , \quad u > 0$$

and the integrand is

$$\frac{e^{i(2n+1)\frac{\pi u e^{-\pi i}}{2}}}{u^z e^{-z\pi i}} = \frac{1}{u^z e^{-i\pi z}} e^{-i\pi n u} e^{-\frac{1}{2} i\pi u}$$

Summing the contribution of the integrations along the upper edge and lower edge and using the fact the contribution of the circular part sc of $\mathcal{H}$ is zero, we get

$$\frac{1}{u^z e^{-i\pi z}} e^{-i\pi n u} e^{-\frac{1}{2} i\pi u} - \frac{1}{u^z} e^{-i\pi z} e^{-i\pi n u} e^{-\frac{1}{2} i\pi u} = -\frac{2i}{u^z} e^{-i(2n+1)\frac{\pi u}{2}} \sin(\pi z)$$

$$\begin{aligned} K(z) &= -\frac{\sin(\pi z)}{\pi} \sum_{n=0}^{\infty} \pi (-1)^n \int_0^{\infty} \frac{e^{-i(2n+1)\frac{\pi u}{2}}}{u^z} du \\ &= -\frac{\sin(\pi z)\,\Gamma(1-z)}{\pi} \sum_{n=0}^{\infty} \pi (-1)^n \left(i\pi \left(\frac{2n+1}{2} \right) \right)^{z-1} \end{aligned} \tag{8}$$

This is a well known property of the Gamma function,

$$\sin(\pi z)\,\Gamma(1-z) = \frac{\pi}{\Gamma(z)}. \tag{9}$$

Substitution of (9) in (8) gives

$$K(z) = -\frac{\pi}{\Gamma(z)} \sum_{n=0}^{\infty} (-1)^n \left(\pi i \left(\frac{2n+1}{2} \right) \right)^{z-1}$$

or,

$$\Gamma(z)\, K(z) = \pi^z i^{z+1} 2^{1-z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{1-z}}. \tag{10}$$

It follows from (10) and (4) and the identity $\Gamma(z+1) = z\Gamma(z)$ that

$$\int_0^{\frac{\pi}{2}} \ln^z \big(\tan(y)\big) \cos(2y)\, dy = z\pi^z i^{z+1} 2^{1-z} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{1-z}}. \tag{11}$$

Set $z = -1$ in (11) to get the desired result:

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(2y)}{\ln\big(\tan(y)\big)} dy = -\frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = -\frac{4}{\pi}\mathbf{G},$$

where **G** is the Catalan's constant.

Solution 4 by Narendra Bhandari, Bajura district, Nepal.

Since both the integrals are convergent and so does the ratio of I/J exist and its value is -1 and the value of integral $I = -\frac{4G}{\pi}$ where $G = \sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$ is Catalan's constant.

We start with $\cos(2x) = \cos^2 x - \sin^2 x = \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} = \frac{1-\tan^2 x}{1+\tan^2 x}$. Putting the noted value in I, we tends to have

$$I = \int_0^{\frac{\pi}{2}} \frac{1-\tan^2 x}{1+\tan^2 x}\frac{dx}{\log(\tan x)} = \int_{-\infty}^{\infty} \frac{e^y(1-e^{2y})}{y(1+e^{2y})^2}\,dy = 2\int_0^{\infty} \frac{e^y(1-e^{2y})}{y(1+e^{2y})^2}\,dy$$
$$= -\int_0^{\infty} \frac{\tanh y \operatorname{sech} y}{y}\,dy \overset{\text{x=y}}{=} -\int_0^{\infty} \frac{\tanh x \operatorname{sech} x}{x}\,dx = -J = -\int_0^{\infty} \frac{\sinh x}{x\cosh^2 x}\,dx$$

Hence $I/J = -1$. In course of evaluation, we have made substitution $\log(\tan x) = y$ and we used $\frac{e^x(1-e^{2x})}{(1+e^{2x})} = -\frac{1}{2}\tanh x \operatorname{sech} x$.

Now using the integral representation of Dirichlet Beta function for $\Re(z) > 0$.

$$\beta(z) = \int_0^{\infty} \frac{x^{z-1}e^{-x}}{1+e^{2x}}\,dx = \frac{1}{2\Gamma(z)}\int_0^{\infty} \frac{x^{z-1}}{2\cosh x}\,dx \overset{\text{IBP}}{=} \frac{1}{2\Gamma(1+z)}\int_0^{\infty} \frac{x^z \sinh x}{\cosh^2 x}\,dx$$

Therefore,

$$\int_0^{\infty} \frac{\sinh x}{x\cosh^2 x}\,dx = 2\lim_{x\to -1}\beta(z)\Gamma(1+z) = 2\lim_{z\to -1}\beta(z)\left(\frac{1}{1+z} - \gamma + O(1)\right)$$
$$= \lim_{z\to -1}\frac{\beta(z)}{z+1} + 0 = 2\lim_{z\to -1}\beta'(z) = 2\beta'(-1) = \frac{4}{\pi}\beta(2) = \frac{4G}{\pi},$$

where $\Gamma(z)$ is gamma function and $\beta(2) = \sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. We find the value of $\beta'(-1)$ by differentiating the functional equation of Dirichlet beta function at $z = 2$, namely ,$\beta(1-z) = \left(\frac{\pi}{2}\right)^{-z}\sin\left(\frac{\pi}{2}z\right)\Gamma(z)\beta(z)$. Equating $I = -J = -\frac{4G}{\pi}$.

Alternative way of evaluating integral J. We note that $\int_0^1 \cosh(xy)dy = \frac{\sinh x}{x}$ and using the fact $\int_0^1 \frac{t^{y/2}+t^{-y/2}}{(1+t)^2}\,dt = \frac{\pi}{2}\frac{y}{\sin\left(\frac{\pi y}{2}\right)}$, our integral reduces to

$$\int_0^{\infty} \frac{\sinh x}{x\cosh^2 x}\,dx = \int_0^{\infty}\int_0^1 \frac{\cosh(xy)}{x\cosh^2 x}\,dydx = \int_0^{\infty}\int_0^1 \frac{e^{xy}+e^{-xy}}{(1+e^{-2x})^2}e^{-2x}\,dydx$$
$$= \int_0^1\int_0^1 \frac{t^{y/2}+t^{-y/2}}{(1+t)^2}\,dtdy = \frac{\pi}{2}\int_0^1 \frac{y}{\sin\left(\frac{\pi}{2}y\right)}\,dy = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{y}{\sin(2y)}\,dy$$
$$= \frac{4}{\pi}\int_0^1 \frac{\tan^{-1} y}{y}\,dy = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 y^{2n}\,dy = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \frac{4G}{\pi}.$$

Also solved by Michel Bataille, Rouen, France; Paolo Perfetti, dipartimento di matematica, Universita di "Tor Vergata", Roma, Italy and the proposer.

• **5681** *Proposed by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.*

Let $a > 0$. Evaluate

$$\int_0^1 \frac{\tan^{-1} x}{x^2 - ax - 1}\, dx.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We express the integral in terms of Clausen's function (see for instance https://en.wikipedia.org/wiki/Clausen_func which is defined as

$$Cl_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} = -\int_0^{\theta} \log\left|2\sin\left(\frac{t}{2}\right)\right| dt.$$

We perform a change of variables: x=tan y, dx=1/cos^{2}y dy. Then

$$\int_0^1 \frac{\arctan x}{x^2 - ax - 1} dx = \int_0^{\frac{\pi}{4}} \frac{y}{\left(\tan^2 y - a\tan y - 1\right)\cos^2 y} dy =$$

$$= \int_0^{\frac{\pi}{4}} \frac{y}{\sin^2 y - \cos^2 y - a\sin y\cos y} dy = -\int_0^{\frac{\pi}{4}} \frac{y}{\cos(2y) + \frac{a}{2}\sin(2y)} dy =$$

$$= -\int_0^{\frac{\pi}{4}} \frac{y}{\cos(2y) + \cot(A)\sin(2y)} dy = -\sin(A)\int_0^{\frac{\pi}{4}} \frac{y}{\sin(2y + A)} dy =$$

$$= -\sin(A)\int_0^{\frac{\pi}{4}} \frac{y}{2\sin\left(y + \frac{A}{2}\right)\cos\left(y + \frac{A}{2}\right)} dy, \qquad (1)$$

where A∈]0,π/2[is chosen in such a way that $\cot A = \frac{a}{2}$. Then $\tan A = \frac{2}{a}$, $A = \arctan\left(\frac{2}{a}\right)$,

$$\sin A = \frac{1}{\sqrt{\cot^2 A + 1}} = \frac{1}{\sqrt{1 + \left(\frac{a}{2}\right)^2}},\quad \tan A = \frac{2\tan\left(\frac{A}{2}\right)}{1 - \tan^2\left(\frac{A}{2}\right)},\quad \tan\left(\frac{A}{2}\right) = \sqrt{1 + \left(\frac{a}{2}\right)^2} - \frac{a}{2},$$

$$\tan\left(\frac{\pi}{4}+\frac{A}{2}\right) = \frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}} = \frac{2}{a} + \sqrt{1+\left(\frac{2}{a}\right)^2}.$$

We integrate (**??**) by parts and get

$$\int_0^1 \frac{\arctan x}{x^2-ax-1}dx = -\frac{1}{2}\sin(A)\, y\log\left(\tan\left(y+\frac{A}{2}\right)\right)\Bigg|_0^{\frac{\pi}{4}} + \frac{1}{2}\sin(A)\int_0^{\frac{\pi}{4}}\log\left(\tan\left(y+\frac{A}{2}\right)\right)dy.$$

We have

$$\int_0^{\frac{\pi}{4}}\log\left(\tan\left(y+\frac{A}{2}\right)\right)dy = \int_0^{\frac{\pi}{4}}\log\left(\sin\left(y+\frac{A}{2}\right)\right)dy - \int_0^{\frac{\pi}{4}}\log\left(\cos\left(y+\frac{A}{2}\right)\right)dy =$$

$$= \int_0^{\frac{\pi}{4}}\log\left(\sin\left(y+\frac{A}{2}\right)\right)dy - \int_0^{\frac{\pi}{4}}\log\left(\sin\left(y+\frac{\pi}{2}+\frac{A}{2}\right)\right)dy =$$

$$= \int_{\frac{A}{2}}^{\frac{\pi}{4}+\frac{A}{2}}\log(\sin y)\,dy - \int_{\frac{\pi}{2}+\frac{A}{2}}^{\frac{3\pi}{4}+\frac{A}{2}}\log(\sin y)\,dy = \frac{1}{2}\int_A^{\frac{\pi}{2}+A}\log\left(2\sin\left(\frac{y}{2}\right)\right)dy - \frac{1}{2}\int_{\pi+A}^{\frac{3\pi}{2}+A}\log\left(2\sin\left(\frac{y}{2}\right)\right)dy =$$

$$= \frac{1}{2}\left(-Cl_2\left(\frac{\pi}{2}+A\right) + Cl_2(A) + Cl_2\left(\frac{3\pi}{2}+A\right) - Cl_2(\pi+A)\right).$$

So

$$\int_0^1 \frac{\arctan x}{x^2-ax-1}dx =$$

$$= -\frac{\pi}{8}\sin(A)\log\left(\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)\right) + \frac{1}{4}\sin(A)\left(-Cl_2\left(\frac{\pi}{2}+A\right) + Cl_2(A) + Cl_2\left(\frac{3\pi}{2}+A\right) - Cl_2(\pi+A)\right) =$$

$$= -\frac{\pi}{8}\frac{1}{\sqrt{1+\left(\frac{a}{2}\right)^2}}\log\left(\frac{2}{a}+\sqrt{1+\left(\frac{2}{a}\right)^2}\right) + \frac{1}{4}\frac{1}{\sqrt{1+\left(\frac{a}{2}\right)^2}}\left(-Cl_2\left(\frac{\pi}{2}+\arctan\left(\frac{2}{a}\right)\right) + Cl_2\left(\arctan\left(\frac{2}{a}\right.\right.\right.$$

Solution 2 by Paolo Perfetti, dipartimento di matematica, Universita di "Tor Vergata", Roma, Italy.

The roots of the denominator are $\alpha = (a+\sqrt{a^2+4})/2$, $\alpha > 1$ and $\beta = (a-\sqrt{a^2+4})/2$,

$\beta < 0$, $\alpha\beta = -1$. The change of variable $\dfrac{x-\alpha}{x+1/\alpha} = y$, $x = \dfrac{\alpha + \frac{y}{\alpha}}{1-y}$ produces

$$\int_{-\alpha^2}^{\frac{1-\alpha}{1+\frac{1}{\alpha}}} \arctan\left(\frac{\alpha+\frac{y}{\alpha}}{1-y}\right) \frac{\alpha^2(1-y)^2}{y(1+\alpha^2)^2} \frac{\alpha^2+1}{\alpha(1-y)^2} dy =$$

$$= \frac{\alpha}{1+\alpha^2} \int_{-\alpha^2}^{\frac{1-\alpha}{1+\frac{1}{\alpha}}} \frac{\arctan\alpha + \arctan(y/\alpha)}{y} dy =$$

$$= \frac{\alpha\arctan\alpha}{1+\alpha^2} \ln\frac{\alpha-1}{\alpha(\alpha+1)} + \frac{\alpha}{1+\alpha^2} \int_{\frac{\alpha-1}{1+\alpha}}^{\alpha} \frac{\arctan y}{y} dy =$$

$$= \frac{\alpha\arctan\alpha}{1+\alpha^2} \ln\frac{\alpha-1}{\alpha(\alpha+1)} + \frac{\alpha}{1+\alpha^2} \int_0^{\alpha} \frac{\arctan y}{y} dy - \frac{\alpha}{1+\alpha^2} \int_0^{\frac{\alpha-1}{1+\alpha}} \frac{\arctan y}{y} dy$$

$$\frac{\alpha}{1+\alpha^2} = \frac{1}{\sqrt{a^2+4}}, \qquad \frac{\alpha-1}{\alpha(\alpha+1)} = \frac{2a}{(\sqrt{a^2+4}+2)(a+\sqrt{a^2+4})}$$

$$\frac{\alpha-1}{\alpha+1} = \frac{\sqrt{a^2+4}-2}{a}$$

$$\int_0^{\alpha} \frac{\arctan y\, dy}{y} = \int_0^1 \frac{\arctan y\, dy}{y} + \int_1^{\alpha} \frac{\arctan y\, dy}{y} = C + \int_1^{\alpha} \frac{\frac{\pi}{2} - \arctan\frac{1}{y}}{y} dy$$

$C = \sum_{k=0}^{\infty} \dfrac{(-1)^k}{(2k+1)^2}$ is the Catalan constant.

$$\int_1^{\alpha} \frac{1}{y} \arctan\frac{1}{y} dy = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \alpha^{-2k-1} + C$$

Thus

$$\int_0^{\alpha} \frac{\arctan y\, dy}{y} = \frac{\pi}{2} \ln\frac{a+\sqrt{a^2+4}}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \alpha^{-2k-1}$$

Thus the integral is

$$\frac{1}{\sqrt{a^2+4}} \arctan\frac{a+\sqrt{a^2+4}}{2} \ln\frac{2a}{(\sqrt{a^2+4}+2)(a+\sqrt{a^2+4})} +$$
$$+\frac{1}{\sqrt{a^2+4}} \left(\frac{\pi}{2} \ln\frac{a+\sqrt{a^2+4}}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a+\sqrt{a^2+4}}{2}\right)^{-2k-1}\right) +$$
$$-\frac{1}{\sqrt{a^2+4}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{-2+\sqrt{a^2+4}}{a}\right)^{2k+1}$$

Solution 3 by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

One root to the quadratic equation $x^2 - ax - 1 = 0$ where $a > 0$ is

$$\varphi_a = \frac{a + \sqrt{a^2+4}}{2}.$$

Here φ_a is the so-called *metallic ratio*. If $a = 1$ we have the more famous golden ratio, if $a = 2$ the silver ratio, if $a = 3$ the bronze ratio, and so on.

Denote the value of the integral to be found by $I(a)$. Enforce a substitution of

$$x = \frac{\varphi_a t - \frac{1}{\varphi_a}}{1+t}.$$

Under such a substitution one has

$$\begin{aligned} x^2 - ax - 1 &= \frac{-(a^2+4)t}{(t+1)^2}, \text{ and} \\ dx &= \frac{\varphi_a + \frac{1}{\varphi_a}}{(1+t)^2}\,dt = \frac{\sqrt{a^2+4}}{(1+t)^2}\,dt, \end{aligned}$$

while for the limits of integration we have

$$(0,1) \mapsto \left(\frac{1}{\varphi_a^2}, \frac{1+\frac{1}{\varphi_a}}{\varphi_a - 1}\right).$$

Also

$$\arctan(x) = \arctan\left(\frac{\varphi_a t - \frac{1}{\varphi_a}}{1+t}\right) = \arctan(\varphi_a t) - \arctan\left(\frac{1}{\varphi_a}\right),$$

where the following difference property for the inverse tangnet function of

$$\arctan(u) - \arctan(v) = \arctan\left(\frac{u-v}{1+uv}\right), \quad uv > -1,$$

has been used.

So under the given substitution, the integral becomes

$$\begin{aligned} I(a) &= \frac{1}{\sqrt{a^2+4}} \arctan\left(\frac{1}{\varphi_a}\right) \int_{\frac{1}{\varphi_a^2}}^{\frac{1+\frac{1}{\varphi_a}}{\varphi_a-1}} \frac{dt}{t} - \frac{1}{\sqrt{a^2+4}} \int_{\frac{1}{\varphi_a^2}}^{\frac{1+\frac{1}{\varphi_a}}{\varphi_a-1}} \frac{\arctan(\varphi_a t)}{t}\,dt \\ &= \frac{1}{\sqrt{a^2+4}} \arctan\left(\frac{1}{\varphi_a}\right) I_1 - \frac{1}{\sqrt{a^2+4}} I_2. \end{aligned}$$

The first of the integrals is elementary. Here

$$I_1 = \int_{\frac{1}{\varphi_a^2}}^{\frac{1+\frac{1}{\varphi_a}}{\varphi_a-1}} \frac{dt}{t} = \log(t)\Big|_{\frac{1}{\varphi_a^2}}^{\frac{1+\frac{1}{\varphi_a}}{\varphi_a-1}} = \log\left(\frac{1+\frac{1}{\varphi_a}}{\varphi_a-1}\right) - \log\left(\frac{1}{\varphi_a^2}\right)$$
$$= \log\left(\frac{\varphi_a(\varphi_a+1)}{\varphi_a-1}\right).$$

And for the second of the integrals, let $x = \varphi_a t$. Then

$$I_2 = \int_{\frac{1}{\varphi_a}}^{\frac{\varphi_a+1}{\varphi_a-1}} \frac{\arctan(x)}{x}\,dx = \mathrm{Ti}_2\left(\frac{\varphi_a+1}{\varphi_a-1}\right) - \mathrm{Ti}_2\left(\frac{1}{\varphi_a}\right).$$

Here $\mathrm{Ti}_2(x)$ denotes the inverse tangent integral which is defined by

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\arctan(t)}{t}\,dt.$$

So the value for the integral is

$$I(a) = \frac{1}{\sqrt{a^2+4}}\left\{\arctan\left(\frac{1}{\varphi_a}\right)\log\left(\frac{\varphi_a(\varphi_a+1)}{\varphi_a-1}\right) + \mathrm{Ti}_2\left(\frac{1}{\varphi_a}\right) - \mathrm{Ti}_2\left(\frac{\varphi_a+1}{\varphi_a-1}\right)\right\},$$

where $\varphi_a = \left(a + \sqrt{a^2+4}\right)/2$, $a > 0$.

Also solved by the proposer.

• **5682** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania.*

Suppose $0 < a \leqslant b$. Prove

$$27\int_a^b\int_a^b\int_a^b (x+y)(y+z)(z+x)dxdydz \leqslant (b-a)^3(a^2+ab+b^2+3)^3.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

The integral evaluates to $\frac{27}{4}(b-a)^3\left(5a^3+11a^2b+11ab^2+5b^3\right)$. So we need to prove that

$$\frac{27}{4}\left(5a^3+11a^2b+11ab^2+5b^3\right) \leqslant \left(a^2+ab+b^2+3\right)^3.$$

We claim that this inequality holds true for *all* a, b $\in \mathbb{R}$. Clearly, $a^2 + ab + b^2 \geqslant 0$ for all a, b $\in \mathbb{R}$. Hence, by the AM-GM inequality,

$$a^2 + ab + b^2 + 3 \geqslant 2\sqrt{3\left(a^2 + ab + b^2\right)}.$$

So it is sufficient to prove that

$$\left|\frac{27}{4}\left(5a^3 + 11a^2b + 11ab^2 + 5b^3\right)\right| \leqslant \left(2\sqrt{3\left(a^2 + ab + b^2\right)}\right)^3$$

which is equivalent to

$$12^3\left(a^2 + ab + b^2\right)^3 - \frac{27^2}{4^2}\left(5a^3 + 11a^2b + 11ab^2 + 5b^3\right)^2 = \frac{27}{16}(a-b)^2\left(349a^4 + 800a^3b + 1158a^2b^2 + 800ab^3\right.$$

However, the last inequality holds true, since

$$349x^4 + 800x^3 + 1158x^2 + 800x + 349 =$$

$$= 349x^2\left(x + \frac{400}{349}\right)^2 + \frac{244142}{349}\left(x + \frac{139600}{244142}\right)^2 + \frac{14682779}{122071} > 0.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

By the arithmetic mean - geometric mean inequality,

$$27(x+y)(y+z)(z+x) \leqslant 8(x+y+z)^3,$$

so

$$\begin{aligned}
27\int_a^b\int_a^b\int_a^b (x+y)(y+z)(z+x)\,dx\,dy\,dz &\leqslant 8\int_a^b\int_a^b\int_a^b (x+y+z)^3\,dx\,dy\,dz \\
&= 36(b-a)^3(a+b)(a^2+ab+b^2) \\
&\leqslant (b-a)^3\left(\frac{2}{3}(a^2+ab+b^2+3) + (a+b)\right)^3,
\end{aligned}$$

where the last inequality follows from the arithmetic mean - geometric mean inequality applied to 6, $3(a+b)$, and $2(a^2+ab+b^2)$. Thus, it suffices to establish that

$$\frac{2}{3}(a^2+ab+b^2+3) + (a+b) \leqslant a^2+ab+b^2+3,$$

or

$$3(a+b) \leqslant a^2+ab+b^2+3.$$

But this is equivalent to

$$\frac{3}{4}(b-1)^2+\frac{1}{4}(2a+b-3)^2 \geqslant 0.$$

Solution 3 by Michel Bataille, Rouen, France.

We have

$$(x+y)(y+z)(z+x) = (x^2y+xy^2)+(y^2z+yz^2)+(z^2x+zx^2)+2xyz$$

and

$$\int_a^b\int_a^b\int_a^b xyz\,dxdydz = \left(\int_a^b x\,dx\right)\left(\int_a^b y\,dy\right)\left(\int_a^b z\,dz\right) = \frac{(b^2-a^2)^3}{8}$$

$$\begin{aligned}\int_a^b\int_a^b\int_a^b (x^2y+xy^2)\,dxdydz &= (b-a)\int_a^b\int_a^b (x^2y+xy^2)\,dxdy\\ &= (b-a)\int_a^b\left(y\cdot\frac{b^3-a^3}{3}+y^2\cdot\frac{b^2-a^2}{2}\right)dy\\ &= (b-a)\cdot\frac{(b^3-a^3)(b^2-a^2)}{3}.\end{aligned}$$

Since

$$\int_a^b\int_a^b\int_a^b (y^2z+yz^2)\,dxdydz = \int_a^b\int_a^b\int_a^b (z^2x+zx^2)\,dxdydz = \int_a^b\int_a^b\int_a^b (x^2y+xy^2)\,dxdydz$$

a simple calculation gives

$$27\int_a^b\int_a^b\int_a^b (x+y)(y+z)(z+x)\,dxdydz = \frac{27}{4}(b-a)^3(a+b)(5a^2+6ab+5b^2).$$

Therefore we have to show that

$$\frac{(a+b)(5a^2+6ab+5b^2)}{4} \leqslant \left(\frac{a^2+ab+b^2+3}{3}\right)^3. \qquad (1)$$

Let t be defined by $b=ta$. Then, $t\geqslant 1$ and (1) writes as

$$\frac{5t^3+11t^2+11t+5}{4} \leqslant \left(\frac{1}{a}+a\cdot\frac{t^2+t+1}{3}\right)^3.$$

Since $\frac{1}{a}+a\cdot\frac{t^2+t+1}{3} \geqslant 2\cdot\sqrt{\frac{t^2+t+1}{3}}$, it is sufficient to prove that

$$5t^3+11t^2+11t+5 \leqslant 32\left(\frac{t^2+t+1}{3}\right)^{3/2}.$$

It is easily obtained that this can be rewritten as

$$\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)\left(5\left(t+\frac{1}{t}\right)+6\right)\leqslant\frac{32}{3\sqrt{3}}\left(t+\frac{1}{t}+1\right)^{3/2}. \qquad (2)$$

We set $X=\sqrt{t}+\frac{1}{\sqrt{t}}$ so that $X\geqslant 2$ and $t+\frac{1}{t}=X^2-2$. Expressing (2) with X leads to the equivalent inequality $f(X^2)\geqslant 0$ where

$$f(u)=349u^3-1992u^2+2640u-1024.$$

Fot $u\geqslant 4$, we have $f''(u)>0$, hence $f'(u)\geqslant f'(4)>0$ so that f is increasing on $[4,\infty)$. It follows that $f(X^2)\geqslant f(4)=0$, so we are done.

Solution 4 by Paolo Perfetti, dipartimento di matematica, Universita di "Tor Vergata", Roma, Italy.

$(x+y)(y+z)(z+x)=2xyz+(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)$

$$27\int_a^b\int_a^b\int_a^b(x+y)(y+z)(z+x)dy\,dy\,dz=$$

$$=27\left(2\int_a^b xdx\int_a^b ydy\int_a^b zdz+6\int_a^b x^2dx\int_a^b ydy\int_a^b dz\right)=$$

$$=\frac{27}{4}(b^2-a^2)^3+27(b^3-a^3)(b^2-a^2)(b-a)$$

The inequality becomes

$$\frac{1}{4}(b^2-a^2)^3+(b^3-a^3)(b^2-a^2)(b-a)\leqslant\frac{1}{27}(b-a)^3(a^2+ab+b^2+3)^3$$

that is (we can suppose $a\neq b$ because for $a=b$ the equality holds true)

$$\frac{1}{27}(a^2+ab+b^2+3)^3-\frac{1}{4}(b+a)^3-(a^2+b^2+ab)(b+a)\geqslant 0$$

Now define the two variables $a+b=2u$, $ab=v^2$. The Agm yields $a^2+b^2+2ab\geqslant 4ab$ namely $4u^2\geqslant 4v^2$, $u\geqslant v$. In term of (u,v) the inequality reads as

$$\frac{1}{27}(4u^2-v^2+3)^3-2u^3-2(4u^2-v^2)u\geqslant 0$$

$$\frac{-v^6}{27}+v^4(\frac{1}{3}+\frac{4}{9}u^2)+v^2(2u-\frac{8}{3}u^2-1-\frac{16}{9}u^4)+\frac{16}{u}^4+4u^2+\frac{64}{27}u^6+1-10u^3\geqslant 0$$

The further change of variables $v=\sqrt{V}$, $u=\sqrt{U}$ sets the inequality as

$$\frac{-1}{27}V^3+V^2(\frac{1}{3}+\frac{4}{9}U)+V(2\sqrt{U}-\frac{8}{3}U-1-\frac{16}{9}U^2)+\frac{16}{3}U^2+4U+\frac{64}{27}U^3+1-10U^{\frac{3}{2}}\geqslant 0$$

Let $F(V)$ the above function.

$$F'(V) = \frac{-V^2}{9} + \frac{2V}{3} + \frac{8}{9}UV + 2\sqrt{U} - \frac{8U}{3} - 1 - \frac{16}{9}U^2 \underbrace{\leqslant}_{V\leqslant U}$$
$$\leqslant \frac{-V^2}{9} + \frac{2U}{3} + \frac{8}{9}U^2 + 2\sqrt{U} - \frac{8U}{3} - 1 - \frac{16}{9}U^2 \leqslant$$
$$\leqslant -2U + 2\sqrt{U} - 1 - \frac{8U^2}{9} = -\frac{8U^2}{9} - (\sqrt{U}-1)^2 - U \leqslant 0$$

$F(V)$ decreases thus $F(V) \geqslant F(U)$ and

$$F(U) = (1 - \sqrt{U})^2(U^2 + 6U + 2U^{\frac{3}{2}} + 1 + 2\sqrt{U}) \geqslant 0.$$

Solution 5 by Toyesh Prakash Sharma (Student) Agra College, India.

Applying the concept of symmetry, we can rewrite the left-hand side of the above inequality as

$$27\int_a^b\int_a^b\int_a^b (x+y)(y+z)(z+x)\,\mathrm{dxdydz} = 27\left(\int_a^b\int_a^b\int_a^b (x+y)\,\mathrm{dxdydz}\right)^3$$

$$= 27\left(2\int_a^b\int_a^b\int_a^b x\mathrm{dxdydz}\right)^3 = 27\left(2\int_a^b \mathrm{xdx}\left(\int_a^b \mathrm{dy}\right)^2\right)^3 = 27\left(2\left[\frac{x^2}{2}\right]_a^b\left([y]_a^b\right)^2\right)^3$$

$$= 27\left(\left(b^2-a^2\right)(b-a)^2\right)^3 = 27\,(b-a)^6\left(b^2-a^2\right)^3$$

So now we have to establish that

$$27\,(b-a)^6\left(b^2-a^2\right)^3 \leqslant (b-a)^3\left(b^2+ab+a^2+3\right)^3$$

$$\Rightarrow 27\,(b-a)^3\left(b^2-a^2\right)^3 \leqslant \left(b^2+ab+a^2+3\right)^3$$

$$\Rightarrow 27\left(b^3+a^3-a^2b-ab^2\right)^3 \leqslant \left(b^2+ab+a^2+3\right)^3$$

$$\Rightarrow 3\left(b^3+a^3-a^2b-ab^2\right) \leqslant b^2+ab+a^2+3$$

$$\Rightarrow 0 \leqslant \left(b^2+ab+a^2+3a^2b+3ab^2+3\right) - 3\left(b^3+a^3\right)$$

And its correct for all $0 < a \leqslant b$ and we can also very it easily.

Also solved by the proposer.

• **5683** *Proposed by Michel Bataille, Rouen, France.*

Let n be a nonnegative integer. Evaluate in closed form

$$\sum_{k\geqslant 0} \binom{n+1}{2k+1} 5^k.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We have

$$\begin{aligned}\sum_{k\geqslant 0} \binom{2n+1}{2k+1} 5^k &= \sum_{k\geqslant 0} \binom{2n+1}{k} \frac{1-(-1)^k}{2} 5^{(k-1)/2} = \frac{1}{\sqrt{5}} \sum_{k\geqslant 0} \binom{2n+1}{k} \frac{\sqrt{5}^k - \left(-\sqrt{5}\right)^k}{2} \\ &= \frac{\left(1+\sqrt{5}\right)^{2n+1} - \left(1-\sqrt{5}\right)^{2n+1}}{2\sqrt{5}}.\end{aligned}$$

Remark: In virtue of Binet's formula for the Fibonacci numbers F_n the latter expression is equal to $2^{2n} F_{2n+1}$.

Solution 2 by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

From the binomial theorem we have

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$

Here n is a non-negative integer. Observing the convention that $\binom{n}{k} = 0$ whenever $k > n$ or $k < 0$, we may write this as

$$\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n,$$

and is valid for all real x. A reindexing of the sum by $n \mapsto n+1$, produces

$$\sum_{k=0}^{\infty} \binom{n+1}{k} x^k = (1+x)^{n+1}.$$

Replacing x with $-x$ one obtains

$$\sum_{k=0}^{\infty} \binom{n+1}{k} (-1)^k x^k = (1-x)^{n+1}.$$

Finding the difference between these two sums yields

$$2 \sum_{\substack{k=0 \\ k \in \text{odd}}} \binom{n+1}{k} x^k = (1+x)^{n+1} - (1-x)^{n+1}.$$

On shifting the index by $k \mapsto 2k+1$ we find

$$\sum_{k=0}^{\infty} \binom{n+1}{2k+1} x^{2k+1} = \frac{(1+x)^{n+1} - (1-x)^{n+1}}{2}.$$

Setting $x = \sqrt{5}$ on finds

$$\sum_{k=0}^{\infty} \binom{n+1}{2k+1} 5^k = \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2\sqrt{5}} \tag{12}$$

$$= \frac{2^n}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]. \tag{13}$$

Noting that

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1}{\varphi} = \frac{-(1-\sqrt{5})}{2},$$

where φ denotes the golden ratio, the expression in (13) may be rewritten as

$$\sum_{k=0}^{\infty} \binom{n+1}{2k+1} 5^k = \frac{2^n}{\sqrt{5}} \left(\varphi^{n+1} - \frac{(-1)^{n+1}}{\varphi^{n+1}} \right). \tag{14}$$

The expression in (14) may be written in terms of the Fibonacci numbers F_n. Recall the nth Fibonacci number F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$ with $F_0 = 0$ and $F_1 = 1$. Binet's formula for the Fibonacci numbers F_n is

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \frac{(-1)^n}{\varphi^n} \right).$$

So in terms of the Fibonacci numbers, (14) can be expressed as

$$\sum_{k=0}^{\infty} \binom{n+1}{2k+1} 5^k = 2^n F_{n+1},$$

and is the desired closed-form expression for the sum.

Solution 3 by Perfetti Paolo, Universita di “Tor Vergata", Roma, Italy.

It is easy to prove by the resides method that

$$\frac{1}{2i\pi}\oint_C \frac{(1+z)^n}{z^{k+1}}dz = \frac{1}{k!}\frac{d^k}{dz^k}(1+z)^n\Big|_{z=0} = \binom{n}{k}$$

C is a circumference of any radius in the complex space around the origin run counterclockwise. Based on that we can write

$$\sum_{k=0}^{+\infty}\binom{n+1}{2k+1}5^k = \sum_{k=0}^{+\infty}5^k\frac{1}{2i\pi}\oint_C \frac{(1+z)^{n+1}}{z^{2k+2}}dz$$

Note that the binomial coefficient is different from zero as long as $2k+1 \leqslant n+1$ and in fact for $k > n/2$ the integral is equal to zero because the residue is zero. To see it let's write

$$\frac{(1+z)^{n+1}}{z^{2k+2}} = \sum_{j=0}^{n+1}\binom{n+1}{j}\frac{1}{z^{2k+2-j}}, \quad 2k+2-j \geqslant 2k+2-n-1 > n+2-n-1 = 1$$

thus the residue is zero.

Let's take the radius of C greater than $\sqrt{5}$

$$\sum_{k=0}^{+\infty}5^k\frac{1}{2i\pi}\oint_C \frac{(1+z)^{n+1}}{z^{2k+2}}dz = \frac{1}{2i\pi}\oint_C \frac{(1+z)^{n+1}}{z^2}\sum_{k=0}^{+\infty}\frac{5^k}{z^{2k}}dz =$$

$$= \frac{1}{2i\pi}\oint_C \frac{(1+z)^{n+1}}{z^2}\frac{z^2}{z^2-5}dz = \frac{1}{2i\pi}\oint_C \frac{(1+z)^{n+1}}{z^2-5}dz =$$

$$= \text{Res}\frac{(1+z)^{n+1}}{z^2-5}\Big|_{z=\sqrt{5}} + \text{Res}\frac{(1+z)^{n+1}}{z^2-5}\Big|_{z=-\sqrt{5}} = \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2\sqrt{5}} = 2^n F_{n+1}$$

where F_{n+1} is the $(n+1)$–th Finonacci–number. The exchange between the integral and the series in the first line is allowed by standard theorem on complex integrals.

Solution 4 by Moti Levy, Rehovot, Israel.

We propose two solutions, using Gaussian hypergeometric function identity, and using Egorychev method:

First Solution (*Gaussian hypergeometric function identity*)

A very good reference to the application of hypergeometric functions to the evaluation of binomial identities is the wonderful book of Graham, Knuth and Patashnik, "Concrete Mathematics".

The first step is to express the binomial series as a hypergeometric function and then the second step is trying to use the classical hypergeometric theorems and identities.

The following lemma is a direct consequence of the definition of the Gaussian hypergeometric function:

Lemma: Let $(\alpha_k)_{k\geqslant 0}$ be a sequence which satisfies the following conditions:

$$\alpha_0 = 1,$$
$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{1}{k+1}\frac{(k+a)(k+b)}{(k+c)}z.$$

Then

$$\sum_{k=0}^{\infty}\alpha_k = {}_2F_1\left(a,b;c\middle|\,z\right), \tag{15}$$

where ${}_2F_1\left(a,b;c\middle|\,z\right)$ is the Gaussian hypergeometric function.■

Let us define

$$\alpha_k := \frac{\binom{n+1}{2k+1}}{n+1},$$

then one can check that

$$\alpha_0 = 1,$$
$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{\binom{n+1}{2k+3}}{\binom{n+1}{2k+1}}z = \frac{1}{k+1}\frac{\left(k+\frac{1-n}{2}\right)\left(k-\frac{n}{2}\right)}{\left(k+\frac{3}{2}\right)}z.$$

It follows from the lemma that

$$\frac{1}{n+1}\sum_{k=0}^{\infty}\binom{n+1}{2k+1}z^k = {}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{3}{2}\middle|\,z\right).$$

Now we use a well-known identity, entry **15.1.10** from M. Abramowitz & I. A. Stegun. *Handbook of Mathematical Functions*. Washington, D.C. National Bureau of Standards, 1964.

$${}_2F_1\left(a+\frac{1}{2},a;\frac{3}{2}\middle|\,z^2\right) = \frac{1}{2(1-2a)z}\left((1+z)^{1-2a}+(1-z)^{1-2a}\right). \tag{16}$$

Setting $a = -\dfrac{n}{2}$ and $z = \sqrt{5}$ in (16), we get

$$\begin{aligned}2F1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{3}{2}\middle|\,5\right) &= \frac{1}{2(n+1)\sqrt{5}}\left(\left(1+\sqrt{5}\right)^{1+n}+\left(1-\sqrt{5}\right)^{1+n}\right)\\ &= \frac{2^n}{n+1}\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{1+n}+\left(\frac{1-\sqrt{5}}{2}\right)^{1+n}\right) = \frac{2^n}{n+1}F_{n+1},\end{aligned}$$

where F_n is the Fibonacci number and $F_n = \dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n+\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$ is Binet's formula.

We conclude that

$$\sum_{k=0}^{\infty} \binom{n+1}{2k+1} 5^k = 2^n F_{n+1}.$$

Second Solution (*using Egorychev method*)

The principle behind Egorychev method is to express the binomial coefficient as complex contour integral,

$$\binom{n+1}{2k+1} = \oint_{|z|=3} \frac{(1+z)^{n+1}}{z^{2k+2}} \frac{dz}{2\pi i}.$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \binom{n+1}{2k+1} 5^k &= \sum_{k=0}^{\infty} \oint_{|z|=3} \frac{(1+z)^{n+1} 5^k}{z^{2k+2}} \frac{dz}{2\pi i} \\
&= \oint_{|z|=3} \sum_{k=0}^{\infty} \frac{(1+z)^{n+1} 5^k}{z^{2k+2}} \frac{dz}{2\pi i} = \oint_{|z|=\varphi} \frac{(1+z)^{n+1}}{z^2-5} \frac{dz}{2\pi i} \\
&= \frac{1}{2\sqrt{5}} \oint_{|z|=3} \frac{(1+z)^{n+1}}{z-\sqrt{5}} \frac{dz}{2\pi i} - \frac{1}{2\sqrt{5}} \oint_{|z|=3} \frac{(1+z)^{n+1}}{z+\sqrt{5}} \frac{dz}{2\pi i}
\end{aligned}$$

$$\begin{aligned}
\oint_{|z|=3} \frac{(1+z)^{n+1}}{z-\sqrt{5}} \frac{dz}{2\pi i} &= \operatorname{Res}\left\{ \frac{(1+z)^{n+1}}{z-\sqrt{5}}, \sqrt{5} \right\} = \left(1+\sqrt{5}\right)^{n+1} \\
\oint_{|z|=3} \frac{(1+z)^{n+1}}{z+\sqrt{5}} \frac{dz}{2\pi i} &= \operatorname{Res}\left\{ \frac{(1+z)^{n+1}}{z+\sqrt{5}}, -\sqrt{5} \right\} = \left(1-\sqrt{5}\right)^{n+1}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \binom{n+1}{2k+1} 5^k &= \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2\sqrt{5}} = 2^n \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{1+n} + \left(\frac{1-\sqrt{5}}{2} \right)^{1+n} \right) \\
&= 2^n F_{n+1}.
\end{aligned}$$

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York.

The binomial theorem gives us

$$\begin{aligned}(1+x)^{n+1}-(1-x)^{n+1} &= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k-\sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}x^k\\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}(1-(-1)^k)\,x^k\\ &= 2\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2k+1}x^{2k+1}.\end{aligned}$$

Setting $x=\sqrt{5}$, we see that

$$\begin{aligned}\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k &= \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2\sqrt{5}}\\ &= 2^{n+1}\cdot\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{2\sqrt{5}}\\ &= 2^n\cdot F_{n+1},\end{aligned}$$

where F_n is the nth Fibonacci number.

Solution 6 by Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

$$\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k=2^nF_{n+1},$$

where F_n is the nth term of the Fibonacci sequence, defined by $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for integers $n\geqslant 3$.

If $x=a+b\sqrt{5}$, where a and b are rational numbers, then we denote the irrational part of x by $\mathrm{Irr}(x)=b\sqrt{5}$, and the (Galois) conjugate of x as $\bar{x}=a-b\sqrt{5}$, so that $x-\bar{x}=2b\sqrt{5}=2\mathrm{Irr}(x)$. Since $\overline{xy}=\bar{x}\cdot\bar{y}$, then

$$\overline{\left(1+\sqrt{5}\right)^{n+1}}=\left(\overline{1+\sqrt{5}}\right)^{n+1}=\left(1-\sqrt{5}\right)^{n+1},$$

and

$$\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}=\left(1+\sqrt{5}\right)^{n+1}-\overline{\left(1+\sqrt{5}\right)^{n+1}}=2\mathrm{Irr}\left(\left(1+\sqrt{5}\right)^{n+1}\right).$$

Using the Binomial Theorem,

$$\left(1+\sqrt{5}\right)^{n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}(\sqrt{5})^k=\sum_{k=0}^{\lceil n/2\rceil}\binom{n+1}{2k}5^k+\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}5^k,$$

so that

$$\mathrm{Irr}\left(\left(1+\sqrt{5}\right)^{n+1}\right) = \sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}5^k = \sqrt{5}\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k.$$

Finally, we use Binet's formula to see that

$$\begin{aligned}
2^n\cdot F_{n+1} &= 2^n\cdot\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}\\
&= \frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}}{2\sqrt{5}}\\
&= \frac{\mathrm{Irr}\left(\left(1+\sqrt{5}\right)^{n+1}\right)}{\sqrt{5}}\\
&= \sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k.
\end{aligned}$$

Solution 7 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Note

$$\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}5^k.$$

By the binomial theorem

$$(1+x)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k \quad\text{and}\quad (1-x)^{n+1} = \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}x^k,$$

so

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}x^{2k} = \frac{(1+x)^{n+1}-(1-x)^{n+1}}{2x}.$$

Thus,

$$\begin{aligned}
\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k &= \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n+1}{2k+1}(\sqrt{5})^{2k}\\
&= \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2\sqrt{5}}\\
&= 2^n\cdot\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}\\
&= 2^nF_{n+1},
\end{aligned}$$

where F_{n+1} is the $(n+1)$-st Fibonacci number.

Solution 8 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We denote the given sum by S_n and show that for every nonnegative integer n,

$$S_n = 2^n F_{n+1},$$

where F_k denotes the kth Fibonacci number.

First, we rewrite the sum in the form

$$S_n = \frac{1}{\sqrt{5}} \sum_{k \geqslant 0} \binom{n+1}{2k+1} (\sqrt{5})^{2k+1}.$$

Then by the Binomial Theorem, we have

$$S_n = \frac{1}{2\sqrt{5}} \left[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1} \right] = 2^n \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} \right),$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. Hence we conclude that $S_n = 2^n F_{n+1}$ as claimed.

Addendum. The inspiration for this proof comes from the solution to Problem B-1286 in *The Fibonacci Quarterly*, Volume 60, Number 2 (May 2022), pages 178-179. The problem was posed by Michel Bataille and solved by Jason L. Smith.

Solution 9 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Note that if $f(x) = \sum_{k \geqslant 0} \binom{n}{k} x^k = (1+x)^n$, then

$$\begin{aligned}
\sum_{k \geqslant 0} \binom{n}{2k+1} 5^k &= \frac{1}{\sqrt{5}} \sum_{k \geqslant 0} \binom{n}{2k+1} \left(\sqrt{5}\right)^{2k+1} \\
&= \frac{f(\sqrt{5}) - f(-\sqrt{5})}{2\sqrt{5}} \\
&= \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2\sqrt{5}} \\
&= 2^n \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{2\sqrt{5}} \\
&= 2^{n-1} F_n,
\end{aligned}$$

and, therefore $\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k = 2^n F_{n+1}$.

Solution 10 by Albert Stadler, Herrliberg, Switzerland.

We express the binomial coefficient as a complex integral and then evaluate the resulting sum by means of the residue theorem. This approach is known as Egorychev's method (https://en.wikipedia.org/wiki/Egory
By Cauchy's theorem of residues,

$$\binom{n+1}{2k+1}=\frac{1}{2\pi i}\int_{|z|=3}\frac{(1+z)^{n+1}}{z^{2k+2}}dz,$$

where |z|=3 denotes the circle with center 0 and radius 3 that is run through once in the positive direction. Hence

$$\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k=\sum_{k\geqslant 0}\frac{1}{2\pi i}\int_{|z|=3}\frac{(1+z)^{n+1}}{z^{2k+2}}5^k dz=\frac{1}{2\pi i}\int_{|z|=3}\frac{(1+z)^{n+1}}{z^2}\frac{1}{1-\frac{5}{z^2}}dz =$$

$$= \frac{1}{2\pi i}\int_{|z|=3}\frac{(1+z)^{n+1}}{z^2-5}dz = \frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}}{2\sqrt{5}}.$$

The last term equals the sum of the residues of the integrand at z=$\sqrt{5}$ and z=-$\sqrt{5}$.
Of course there is also a direct elementary approach, namely

$$\sum_{k\geqslant 0}\binom{n+1}{2k+1}5^k = \frac{1}{\sqrt{5}}\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1-(-1)^k}{2}\left(\sqrt{5}\right)^k =$$

$$=\frac{1}{2\sqrt{5}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\sqrt{5}\right)^k-\frac{1}{2\sqrt{5}}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(-\sqrt{5}\right)^k =$$

$$= \frac{\left(1+\sqrt{5}\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}}{2\sqrt{5}}.$$

Also solved by the proposer.

• **5684** *Proposed by Goran Conar, Varaždin, Croatia.*

Let α,β,γ be angles of an acute triangle. Prove that the arithmetic mean of sines of half-angles is bounded between sine of harmonic mean of that half-angles and sine of arithmetic mean of that half-angles. In other words, prove that the following inequalities hold

$$\sin\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)}\right)\leqslant\frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}}{3}\leqslant\frac{1}{2}\,.$$

When does equality occur?

Solution 1 by Michel Bataille, Rouen, France.

Note that $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ are all in $(0, \frac{\pi}{4})$ and so are their classical means.
The sine function being strictly concave on $(0, \frac{\pi}{4})$, we have

$$\frac{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}}{3} \leqslant \sin\left(\frac{\alpha/2 + \beta/2 + \gamma/2}{3}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

with equality if and only if $\frac{\alpha}{2} = \frac{\beta}{2} = \frac{\gamma}{2}$, that is, if and only $\alpha = \beta = \gamma$.
From HM-GM inequality, we have

$$\frac{3}{2\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)} \leqslant \sqrt[3]{\frac{\alpha}{2}\cdot\frac{\beta}{2}\cdot\frac{\gamma}{2}} = \exp\left(\frac{\ln(\alpha/2) + \ln(\beta/2) + \ln(\gamma/2)}{3}\right).$$

Since the sine function is increasing on $(0, \frac{\pi}{4})$, we deduce that the left inequality will follow if we prove that

$$\sin\left(\sqrt[3]{\frac{\alpha}{2}\cdot\frac{\beta}{2}\cdot\frac{\gamma}{2}}\right) \leqslant \frac{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2}}{3}. \qquad (1)$$

Now, for $x \in (-\infty, \ln(\pi/4))$, let $f(x) = \sin(e^x)$.
An easy calculation gives $f''(x) = e^{2x}\cos(e^x)(e^{-x} - \tan(e^x))$, hence $f''(x) > 0$ (since $e^{-x} \in (4/\pi, \infty)$ and $\tan(e^x) \in (0, 1)$) and f is strictly convex. As a result, we obtain

$$f\left(\frac{\ln(\alpha/2) + \ln(\beta/2) + \ln(\gamma/2)}{3}\right) \leqslant \frac{1}{3}\left(f(\ln(\alpha/2)) + f(\ln(\beta/2)) + f(\ln(\gamma/2))\right),$$

which is (1).
Clearly, equality holds in the required left inequality if $\alpha = \beta = \gamma$. Conversely, this equality holds only if equality holds in (1), hence only if $\alpha = \beta = \gamma$ (since f is strictly convex and ln is one-to-one). Thus, the case of equality on the left is $\alpha = \beta = \gamma$ as well.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York.

Since the triangle is acute, we must have $0 < \alpha, \beta, \gamma < \pi/2$, implying that $2/\alpha, 2/\beta, 2/\gamma > 4/\pi > 1$. Now we note that $f(x) = \sin(1/x)$ is convex on $[1, \infty)$: $f''(x) = (2x - \tan(1/x))\cos(1/x)/x^4 > 0$ for $x \geqslant 1$.

Jensen's inequality gives us

$$\sin\left(\frac{1}{\frac{x+y+z}{3}}\right) \leqslant \frac{\sin\left(\frac{1}{x}\right) + \sin\left(\frac{1}{y}\right) + \sin\left(\frac{1}{z}\right)}{3}.$$

With $x = 2/\alpha$, $y = 2/\beta$, $z = 2/\gamma$, this inequality becomes

$$\sin\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)}\right) \leqslant \frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}}{3},$$

with equality if and only if $\alpha = \beta = \gamma = \pi/3$.

The right-hand inequality follows from the concavity of $\sin x$ on $[0, \pi/2]$ and Jensen's inequality:

$$\frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}}{3} \leqslant \sin\left(\frac{\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}}{3}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Again, equality holds if and only if $\alpha = \beta = \gamma = \pi/3$.

Solution 3 by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata," Rome, Italy.

The symmetry of the two inequalities allows us to take $\alpha \geqslant \beta \geqslant \gamma$ from which $\pi/3 \leqslant \alpha \leqslant \pi/2$, $\beta \geqslant \pi/4$.

R.h.s. By concavity of $\sin x$, $0 \leqslant x \leqslant \pi/2$ we have

$$\frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}}{3} \leqslant \sin\left(\frac{\alpha}{6}+\frac{\beta}{6}+\frac{\gamma}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$$

L.h.s.

$$f(\alpha) = \frac{\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\pi-\beta-\alpha}{2}}{3} - \sin\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\beta-\alpha}\right)}\right)$$

$$\begin{aligned}
f'(\alpha) &= \frac{1}{6}\cos\frac{\alpha}{2} - \frac{1}{6}\cos\frac{\pi-\alpha-\beta}{2} + \\
&-\cos\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\alpha-\beta}\right)}\right)\frac{-3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\alpha-\beta}\right)^2}\left(\frac{1}{(\pi-\alpha-\beta)^2}-\frac{1}{\alpha^2}\right) = \\
&= \frac{-1}{3}\sin\frac{\pi-\beta}{4}\sin\frac{2\alpha+\beta-\pi}{4} + \\
&+\frac{3}{2}\cos\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\alpha-\beta}\right)}\right)\frac{1}{\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\alpha-\beta}\right)^2}\frac{(2\alpha+\beta-\pi)(\pi-\beta))}{\alpha^2(\pi-\alpha-\beta)^2}
\end{aligned}$$

$$\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)} \leqslant \frac{1}{2}\frac{1}{3}(\alpha+\beta+\gamma) = \frac{\pi}{6} \implies \cos\left(\frac{3}{2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\alpha-\beta}\right)}\right) \geqslant \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

The arithmetic–harmonic–mean has been used in the first $\leqslant$.

γ is to be intended as $\pi - \alpha - \beta$ in the following

$$\alpha^2(\pi-\alpha-\beta)^2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\pi-\alpha-\beta}\right)^2 = \alpha^2\gamma^2\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^2}{(\alpha\beta\gamma)^2} \leqslant$$
$$\leqslant \frac{(\alpha^2+\beta^2+\gamma^2)^2}{\beta^2} \leqslant \frac{(\alpha+\beta+\gamma)^2}{\left(\frac{\pi}{4}\right)^2} = 16$$

This means that $(2\alpha+\beta-\pi \geqslant \alpha+\beta+\gamma-\pi \geqslant 0)$

$$f'(\alpha) \geqslant \frac{-1}{3}\sin\frac{\pi-\beta}{4}\sin\frac{2\alpha+\beta-\pi}{4}+\frac{3}{2}\frac{\sqrt{3}}{2}16(2\alpha+\beta-\pi)(\pi-\beta) \geqslant$$
$$\geqslant (2\alpha+\beta-\pi)(\pi-\beta)\left(12\sqrt{3}-\frac{1}{3}\right) \geqslant 0$$

It follows
$$f'(\alpha) \geqslant f(\pi/3) = 0$$
because $\alpha = \pi/3$ implies $\beta = \gamma = \pi/3$. The equality case is $\alpha = \beta = \gamma$ in both the l.h.s. and the r.h.s. The proof is complete.

Also solved by Toyesh Prakash Sharma (Student), Agra College, India; Albert Stadler, Herrliberg, Switzerland and the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated! . . . And don't worry about making a mistake. All is well!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣