Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before March 1, 2023.

• **5703** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve for real x:

$$x^2 + (x-6)\sqrt{x-7} + 12 = 13x.$$

• **5704** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let a and k be positive integers that are relatively prime and of different parity. Further assume that k is not a perfect square. Let u_n and v_n be integers such that

$$\left(a+\sqrt{k}\right)^n = u_n + v_n\sqrt{k}, \quad n = 1,\ 2,\ \ldots$$

Prove that u_n and v_n are relatively prime for all natural numbers n.

• **5705** *Proposed by Rafael Jakimczuk, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Prove the series $\sum_{n=1}^{\infty} a_n$ converges where the the sequence $(a_n)_{n\geqslant 1}$ is recursively defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{3!} + \frac{a_{n-1}}{5!} + \frac{a_{n-2}}{7!} + \cdots + \frac{a_1}{(2n+1)!} \quad (n \geqslant 1).$$

• **5706** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose $a, b, c > 0$. Prove

$$\prod_{cyc} \frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geqslant \left(1+\sqrt[3]{a^2b^2c^2}\right)^3.$$

• **5707** *Proposed by Narendra Bhandari, Bajura District, Nepal.*

Prove that

$$\int_0^{\frac{\pi}{2}} \Big(\sin x \cdot \operatorname{arctanh}^2(\sin x) - 2\sin x \cdot \operatorname{arctanh}(\sin x)\Big)\, dx = 4G - \pi$$

where $G := \sum_{k=1}^{\infty} (-1)^{k+1}/(2k-1)^2$ is Catalan's constant.

• **5708** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Solve the differential equation

$$y\sqrt{y^2+z^2}\, dz + z\sqrt{y^2+z^2}\, dy = \frac{y\,(x\,dy - y\,dx) + z\,(x\,dz - z\,dx)}{x^2+y^2+z^2}.$$

Solutions

To Formerly Published Problems

• **5685** *Proposed by D.M. Bătineţu-Giurgiu, Bucharest, Romania and Neculai Stanciu, Buzău, Romania.*

Prove: If $x \in \left(0, \pi/2\right)$, then for any triangle $\triangle ABC$ with side lengths a, b, c and area F, the following inequality holds:

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}F\sin(2x) + \sum_{cyc}(a\sin x - b\cos x)^2.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

The maximum of $4\sqrt{3}F\sin(2x) + \sum_{cycl}(a\sin x - b\cos x)^2$ considered as a periodic function of x is assumed at a stationary point of x. We find

$$\frac{d}{dx}\left(4\sqrt{3}F\sin(2x) + \sum_{cycl}(a\sin x - b\cos x)^2\right) =$$

$$= 8\sqrt{3}F\cos(2x) + \sum_{cycl}\Big(-2ab\cos(2x) + a^2\sin(2x) - b^2\sin(2x)\Big) =$$

$$= \Big(8\sqrt{3}F - 2ab - 2bc - 2ca\Big)\cos(2x),$$

and see that the maximum is assumed at x=π/4. It remains to prove that

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}F + \frac{1}{2}\sum_{cycl}(a-b)^2$$

which is equivalent to

$$ab + bc + ca \geqslant 4\sqrt{3}F.$$

However the last inequality is well-known. See for instance

https://en.wikipedia.org/wiki/List_of_triangle_inequalities.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Because

$$\sum_{cyc}(a\sin x - b\cos x)^2 = a^2 + b^2 + c^2 - (ab + bc + ca)\sin(2x)$$

and $\sin(2x) \neq 0$ for $x \in (0, \pi/2)$, the inequality

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}F\sin(2x) + \sum_{cyc}(a\sin x - b\cos x)^2$$

is equivalent to

$$ab + bc + ca \geqslant 4\sqrt{3}F.$$

Now,

$$F = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B,$$

so

$$ab = \frac{2F}{\sin C}, \quad bc = \frac{2F}{\sin A}, \quad \text{and} \quad ca = \frac{2F}{\sin B}.$$

Thus, $ab+bc+ca \geqslant 4\sqrt{3}F$ becomes $\csc A + \csc B + \csc C \geqslant 2\sqrt{3}$. Finally, by Jensen's inequality,

$$\csc A + \csc B + \csc C \geqslant 3\csc\left(\frac{A+B+C}{3}\right) = 3\csc\frac{\pi}{3} = 3\cdot\frac{2}{\sqrt{3}} = 2\sqrt{3}.$$

Solution 3 by Daniel Văcaru, Pitești, Romania.

We have

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}F\sin(2x) + \sum_{\text{cyc}}(a\sin x - b\cos x)^2$$

$$\Leftrightarrow$$

$$a^2+b^2+c^2 \geqslant 4\sqrt{3}F\sin(2x)+a^2+b^2+c^2-\sum ab\sin 2x$$

$$\Leftrightarrow$$

$$\sum ab\sin(2x) \geqslant 4\sqrt{3}F\sin(2x)$$

The last relationship is equivalent to

$$\sum ab \geqslant 4\sqrt{3}F,$$

which is well known. See

https://en.wikipedia.org/wiki/List_of_triangle_inequalities#:~

:text=The%20parameters%20in%20a%20triangle,to%20the%20opposite%20side%2C%20the

Solution 4 by Michel Bataille, Rouen, France.

Since $\sin 2x = 2\sin x\cos x > 0$ and

$$\sum_{cyc}(a\sin x - b\cos x)^2 = a^2+b^2+c^2-(ab+bc+ca)\sin 2x$$

the required inequality is equivalent to

$$ab+bc+ca \geqslant 4\sqrt{3}F. \tag{1}$$

Let α, β, γ be the angles opposite to the sides a, b, c, respectively. Then, $2F = bc\sin\alpha = ca\sin\beta = ab\sin\gamma$ so that

$$ab+bc+ca = 2F\left(\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}\right). \tag{2}$$

Now, the function $f : x \mapsto f(x) = \dfrac{1}{\sin x}$ being convex on $(0,\pi)$ (its second derivative $f''(x) = \dfrac{\sin^2 x + 2\cos^2 x}{\sin^3 x}$ is positive), Jensen's inequality yields

$$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma} \geqslant \frac{3}{\sin((\alpha+\beta+\gamma)/3)} = \frac{3}{\sqrt{3}/2} = 2\sqrt{3},$$

from which, using (2), we deduce (1).

Solution 5 by Moti Levy, Rehovot, Israel.

$$\begin{aligned}
&4\sqrt{3}F\sin(2x)+\sum_{cyc}\left(a\sin(x)-b\cos(x)\right)^2\\
&=4\sqrt{3}F\sin(2x)+a^2+b^2+c^2-2\sin(x)\cos(x)(ab+bc+ca)\\
&=\sin(2x)\left(4\sqrt{3}F-(ab+bc+ca)\right)+a^2+b^2+c^2.
\end{aligned}$$

Plugging this in the original inequality, we get

$$a^2 + b^2 + c^2 \geqslant \sin(2x)\left(4\sqrt{3}F - (ab + bc + ca)\right) + a^2 + b^2 + c^2,$$

or

$$\sin(2x)\left(4\sqrt{3}F - (ab + bc + ca)\right) \leqslant 0.$$

Since $\sin(2x) \geqslant 0$ for $x \in \left(0, \frac{\pi}{2}\right)$, it follows that the original inequality is equivalent to the following inequality:

$$ab + bc + ca \geqslant 4\sqrt{3}F.$$

This inequality is well known. A proof can be found in the classic book by Bottemi et al., "*Geometric inequalities*", Wolters-Noordhoff Publishing, Gronigen 1969, entry **4.5**, on page **43**.

Solution 6 by Péter Fülöp, Gyömrő, Hungary.

It is known that

$$\sum_{cyc}(a\sin(x) - b\cos(x))^2 = (a\sin(x) - b\cos(x))^2 + (b\sin(x) - c\cos(x))^2 + (c\sin(x) - a\cos(x))^2$$

After performing cancellations, we get:

$$\sum_{cyc}(a\sin(x) - b\cos(x))^2 = a^2 + b^2 + c^2 - \sin(2x)[ab + bc + ac]$$

Put this result into the statement we have:

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}F\sin(2x) + a^2 + b^2 + c^2 - \sin(2x)[ab + bc + ac]$$

$$0 \geqslant \sin(2x)\Big[4\sqrt{3}F - [ab + bc + ac]\Big]$$

As $x \in (0, \pi/2)$ follows that $2x \in (0, \pi)$ where $\sin(2x) > 0$, we can do further simplification in the inequality:

$$0 \geqslant 4\sqrt{3}F - [ab + bc + ac]$$

On the other hand the area (F) can be expressed by the sides and with the angles subtended by their sides (α, β and γ):

$$F = \frac{1}{2}ab\sin(\gamma)$$

$$F = \frac{1}{2}bc\sin(\alpha)$$

$$F = \frac{1}{2}ac\sin(\beta)$$

Let's express ab, bc, ac and put them back to the inequality:

$$0 \geqslant 4\sqrt{3}F - [\frac{2F}{\sin(\alpha)} + \frac{2F}{\sin(\beta)} + \frac{2F}{\sin(\gamma)}]$$

$$\frac{1}{\sin(\alpha)} + \frac{1}{\sin(\beta)} + \frac{1}{\sin(\gamma)} \geqslant 2\sqrt{3}$$

The left hand side of the inequality can be considered as a bivariete function: $\mathrm{G}(\alpha, \beta, \pi - \alpha - \beta)$

Performing examination of the extreme value of G, we get that it has a minimum value at:

$$\alpha = \beta = \gamma = \frac{\pi}{3}$$

Substitute back to the last inequality we get that LHS equals to $3\frac{2}{\sqrt{3}} = 2\sqrt{3}$. It means that the statement is proved. The equality occurs when the triangle is equilateral.

Also solved by the problem proposer.

• **5686** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let n be a natural number. Prove that the following three statements are equivalent:

1. The n-th central trinomial coefficient is divisible by 3.
2. The n-th central binomial coefficient is divisible by 3.
3. The base 3 representation of n has at least one digit "2".

Note: the n-th central trinomial coefficient is the coefficient of x^n in the expansion of $\left(1 + x + x^2\right)^n$, while the n-th central binomial coefficient is the coefficient of x^n in the expansion of $(1 + x)^{2n}$ and equals $\binom{2n}{n}$.

Solution 1 by Moti Levy, Rehovot, Israel.

Let us denote the central trinomial coefficient by T_n. We begin by showing that (**1**) is equivalent to (**2**).
The ordinary generating function (OGF) of the sequence $(T_n)_{n\geqslant 0}$ is

$$\frac{1}{\sqrt{(1+x)}\sqrt{1-3x}} = \sum_{k=0}^{\infty} T_k x^k. \qquad (1)$$

There are several proofs for (1). The first one is due to Euler.

By binomial expansion and the identity $\binom{-\frac{1}{2}}{k} = \left(\frac{-1}{4}\right)^k \binom{2k}{k}$, we get

$$\frac{1}{\sqrt{1-fx}} = (1-fx)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k} f^k x^k = \sum_{k=0}^{\infty} \frac{1}{4^k} \binom{2k}{k} f^k x^k \tag{2}$$

Setting $f = -1$ in (2), we get the OGF of the sequence $\left(\frac{1}{4^n}\binom{2n}{n}(-1)^n\right)_{n\geqslant 0}$,

$$\frac{1}{\sqrt{1+x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \binom{2k}{k} x^k. \tag{3}$$

Setting $f = 3$ in (2) we get the OGF of the sequence $\left(\frac{3^n}{4^n}\binom{2n}{n}\right)_{n\geqslant 0}$,

$$\frac{1}{\sqrt{1-3x}} = \sum_{k=0}^{\infty} \frac{1}{4^k} \binom{2k}{k} 3^k x^k. \tag{4}$$

By (1), (3) and (4), the sequence $(T_n)_{n\geqslant 0}$ is the convolution of the sequence $\left(\frac{(-1)^n}{4^n}\binom{2n}{n}\right)_{n\geqslant 0}$ with the sequence $\left(\left(\frac{3}{4}\right)^n \binom{2n}{n}\right)_{n\geqslant 0}$,

$$T_n = \sum_{m=0}^{n} \left(\frac{3}{4}\right)^m \binom{2m}{m} \frac{(-1)^{n-m}}{4^{n-m}} \binom{2n-2m}{n-m} = \frac{(-1)^n}{4^n} \sum_{m=0}^{n} (-3)^m \binom{2m}{m}\binom{2n-2m}{n-m}$$

or,

$$(-1)^n\, 4^n T_n = \binom{2n}{n} - 3\sum_{m=1}^{n} (-3)^{m-1} \binom{2m}{m}\binom{2n-2m}{n-m} \tag{5}$$

Equation (5) implies that $4^n T_n$ is divisible by 3 if and only if $\binom{2n}{n}$ is divisible by 3, hence T_n is divisible by 3 if and only if $\binom{2n}{n}$ is divisible by 3.

Now we show that (**1**) is equivalent to (**3**).

Lucas's Theorem: Let p be a prime number, and let r, c integers which can be written in p-ary notation as:

$$r = r_0 + r_1 p + r_2 p^2 + \cdots + r_k p^k, \quad 0 \leqslant r_i < p,$$
$$c = c_0 + c_1 p + c_2 p^2 + \cdots + c_k p^k, \quad 0 \leqslant c_i < p.$$

Then

$$\binom{r}{c} = \binom{r_0}{c_0}\binom{r_1}{c_1}\binom{r_2}{c_2}\cdots\binom{r_k}{c_k} \pmod{p}.$$

The convention here is $\binom{r}{c} = 0$ if $c > r$. ■

Proof of Lucas's theorem can be found in Wikipedia at the entry "*Lucas's theorem*".

Let $p = 3$, and let the representation of n in base 3 be

$$n = n_0 + 3n_1 + 3^2 n_2 + \cdots + 3^k n_k.$$

Suppose that

$$n_i \in \{0, 1\}$$

Then the representation of $2n$ in base 3 is

$$2n = n_0' + 3n_1' + 3^2 n_2' + \cdots + 3^k n_k',$$

where

$$n_i' = 2n_i \in \{0, 2\}.$$

By Lucas's theorem,

$$\binom{2n}{n} = \binom{n_0'}{n_0}\binom{n_1'}{n_1}\binom{n_2'}{n_2}\cdots\binom{n_k'}{n_k} \pmod{3}.$$

$$\binom{n_i'}{n_i} = \begin{cases} \binom{0}{0} = 1 & if \quad n_i = 0 \\ \binom{2}{1} = 2 & if \quad n_i = 1 \end{cases}.$$

It follows that $\binom{2n}{n} = 1 \pmod{3}$, or $\binom{2n}{n} = 2 \pmod{3}$; hence $\binom{2n}{n}$ is not divisible by 3.

Now suppose that at least one the digits $n_i = 2$. Then the corresponding n_i' in the representation of $2n$ in base 3 is equal to 1. Hence $\binom{n_i'}{n_i} = \binom{1}{2} = 0$, which implies (by Lucas's theorem) that $\binom{2n}{n}$ is divisible by 3.

Also solved by the problem proposer .

• **5687** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Find complex numbers u, v such that:

$$\left\{ \begin{array}{c} \dfrac{|u|^2}{3} + \dfrac{|v|^2}{4} = \dfrac{|u+v|^2}{7} \\ \\ 8u + v = 7 + 7i \end{array} \right\}.$$

Solution 1 by Hyunbin Yoo, South Korea.

Let $u = a + bi$ and $v = c + di$ where c and d are real numbers. Substitution gives

$$\frac{a^2+b^2}{3} + \frac{c^2+d^2}{4} = \frac{(a+c)^2+(b+d)^2}{7}$$
$$8a + c = 7,\ 8b + d = 7$$

We substitute $c = 7 - 8a$ and $d = 7 - 8b$ to get:

$$\begin{aligned}
&\frac{a^2+b^2}{3} + \frac{(7-8a)^2+(7-8b)^2}{4} = \frac{(7-7a)^2+(7-7b)^2}{7} \\
&\Leftrightarrow \frac{a^2+b^2}{3} + \frac{(64a^2-112a+49)+(64b^2-112b+49)}{4} = 7((a-1)^2+(b-1)^2) \\
&\Leftrightarrow \frac{49}{3}a^2 - 28a + \frac{49}{3}b^2 - 28b + \frac{49}{2} = 7(a^2 - 2 + b^2 - 2b + 2) \\
&\Leftrightarrow \frac{7}{3}a^2 - 4a + \frac{7}{3}b^2 - 4b + \frac{7}{2} = a^2 - 2a + b^2 - 2b + 2 \\
&\Leftrightarrow \frac{4}{3}a^2 - 2a + \frac{4}{3}b^2 - 2b + \frac{3}{2} = 0 \\
&\Leftrightarrow 8a^2 - 12a + 8b^2 - 12b + 9 = 0 \\
&\Leftrightarrow 2\left(2a - \frac{3}{2}\right)^2 + 2\left(2b - \frac{3}{2}\right)^2 = 0
\end{aligned}$$

In conclusion, $a = \frac{3}{4},\ b = \frac{3}{4},\ c = 1,\ d = 1.$

Solution 2 by Andrew Siefker, Angelo State University, San Angelo, TX.

Let $u = a + ib$ and $v = c + id$ where $a, b, c, d \in \mathbb{R}$. Then the given equations become

$$\frac{a^2+b^2}{3} + \frac{c^2+d^2}{4} = \frac{(a+c)^2+(b+d)^2}{7} \tag{1}$$

and

$$\begin{pmatrix} 8a + c = 7 \\ 8b + d = 7 \end{pmatrix} \tag{2}$$

respectively. Expanding the right hand side of equation (1), clearing the denominators, and moving everything to one side yields

$$(16a^2 - 24ac + 9c^2) + (16b^2 - 24bd + 9d^2) = 0.$$

Factoring the grouped terms yields

$$(4a - 3c)^2 + (4b - 3d)^2 = 0.$$

This equation is true iff $4a - 3c = 0$ and $4b - 3d = 0$. Solving for c in terms of a and for d in terms of b in equations (2) and substituting into $4a - 3c = 0$ and $4b - 3d = 0$ respectively results in

$$\begin{aligned} 4a &= 3(7 - 8a) \Longrightarrow a = \frac{3}{4} \Longrightarrow c = 1 \quad \text{and} \\ 4b &= 3(7 - 8b) \Longrightarrow b = \frac{3}{4} \Longrightarrow d = 1 \end{aligned}$$

$$\therefore \quad u = \frac{3}{4} + i\frac{3}{4} \quad \text{and} \quad v = 1 + i \qquad \blacksquare$$

Solution 3 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

The unique solution is $u = \frac{3}{4}(1 + i)$ and $v = 1 + i$.

Let $u = a + bi$ and $v = c + di$, where $a, b, c,$ and d are real numbers. Multiplying each side of the first equation by 84 gives

$$\begin{aligned} 28(a^2 + b^2) + 21(c^2 + d^2) &= 12\left((a + c)^2 + (b + d)^2\right) \\ 16(a^2 + b^2) + 9(c^2 + d^2) &= 24(ac + bd) \\ (16a^2 - 24ac + 9c^2) + (16b^2 - 24bd + 9d^2) &= 0 \\ (4a - 3c)^2 + (4b - 3d)^2 &= 0, \end{aligned}$$

from which we see that $4a = 3c$ and $4b = 3d$.

From the second equation, $8a + c = 7 = 8b + d$, so that $8a + c = 7c = 7$ and $8b + d = 7d = 7$. Consequently, $c = d = 1$ and $a = b = \frac{3}{4}$; thus $u = \frac{3}{4}(1 + i)$ and $v = 1 + i$.

Solution 4 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX.

Let $u = a + bi$ and $v = c + di$ with $a, b, c, d \in \mathbb{R}$. Then, the equation

$$8u + v = 7 + 7i$$

becomes

$$(8a + c) + (8b + d)\, i = 7 + 7i$$

and hence,

$$8a + c = 7 \tag{1}$$

and

$$8b + d = 7. \tag{2}$$

Further,

$$|u|^2 = a^2 + b^2, \;\; |v|^2 = c^2 + d^2, \; and \;\; |u + v|^2 = (a + c)^2 + (b + d)^2 .$$

Therefore, the equation

$$\frac{|u|^2}{3} + \frac{|v|^2}{4} = \frac{|u + v|^2}{7}$$

becomes

$$\frac{a^2 + b^2}{3} + \frac{c^2 + d^2}{4} = \frac{(a + c)^2 + (b + d)^2}{7}.$$

If we clear the denominators, we obtain

$$28\left(a^2 + b^2\right) + 21\left(c^2 + d^2\right) = 12\,(a + c)^2 + 12\,(b + d)^2$$

which reduces to

$$\left(16a^2 - 24ac + 9c^2\right) + \left(16b^2 - 24bd + 9d^2\right) = 0,$$

or

$$(4a - 3c)^2 + (4b - 3d)^2 = 0. \tag{3}$$

Equation (3) yields

$$4a - 3c = 0 \tag{4}$$

and

$$4b - 3d = 0. \tag{5}$$

By combining (1) and (4), we get

$$4a - 3\,(7 - 8a) = 0$$

which reduces to

$$a = \frac{21}{28} = \frac{3}{4}.$$

Then,

$$c = 7 - 8a = 7 - 6 = 1.$$

Similar steps using (2) and (5) lead to

$$b = \frac{3}{4} \text{ and } d = 1.$$

Therefore, our result is $u = a + bi = \frac{3}{4} + \frac{3}{4}i = \frac{3}{4}(1+i)$ and $v = c + di = 1 + i$. It is easily seen that these numbers satisfy

$$8u + v = 8\left[\frac{3}{4}(1+i)\right] + (1+i) = 7(1+i).$$

Further, since $|u|^2 = \frac{9}{16}|1+i|^2 = \frac{9}{16}(2) = \frac{9}{8}$, $|v|^2 = |1+i|^2 = 2$, and $|u+v|^2 = \left|\frac{7}{4}(1+i)\right|^2 = \frac{49}{16}(2) = \frac{49}{8}$, we have

$$\frac{|u|^2}{3} + \frac{|v|^2}{4} = \frac{3}{8} + \frac{1}{2} = \frac{7}{8} = \frac{1}{7}\left(\frac{49}{8}\right) = \frac{|u+v|^2}{7}$$

as well. ■

Solution 5 by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

We shall show that

$$u = \frac{3}{4}(1+i) \quad \text{and} \quad v = 1 + i.$$

Since

$$|u+v|^2 = (u+v)\overline{(u+v)} = (u+v)(\overline{u}+\overline{v}) = |u|^2 + |v|^2 + u\overline{v} + v\overline{u},$$

the first of the given equations

$$\frac{|u|^2}{3} + \frac{|v|^2}{4} = \frac{|u+v|^2}{7},$$

becomes

$$\frac{4}{21}|u|^2 + \frac{3}{28}|v|^2 = \frac{1}{7}u\overline{v} + \frac{1}{7}v\overline{u}. \tag{6}$$

From the second of the given equations, we have

$$v = 7 + 7i - 8u \quad \text{and} \quad \overline{v} = 7 - 7i - 8\overline{u}.$$

So we see that

$$u\overline{v} + v\overline{u} = 7(u + \overline{u}) - 7i(u - \overline{u}) - 16|u|^2. \tag{7}$$

Recalling that if z is a complex number, then

$$\text{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \overline{z}}{2i}.$$

So one can rewrite the expression in (7) as

$$u\overline{v} + v\overline{u} = 14\,\text{Re}(u) + 14\,\text{Im}(u) - 16|u|^2. \tag{8}$$

Combining (6) with (8), after simplifying we arrive at

$$\frac{2}{3}|u|^2 + \frac{3}{4} = \text{Re}(u) + \text{Im}(u).$$

If we now let $u = x + iy$ where $x, y \in \mathbb{R}$, then

$$\frac{2}{3}(x^2 + y^2) + \frac{3}{4} = x + y,$$

or after rearranging

$$\left(x - \frac{3}{4}\right)^2 + \left(y - \frac{3}{4}\right)^2 = 0.$$

This equation can only be true provided $x = y = \frac{3}{4}$. So

$$u = x + iy = \frac{3}{4}(1 + i),$$

and

$$v = 7 + 7i - 8u = 7 + 7i - 8 \cdot \frac{3}{4}(1 + i) = 1 + i,$$

as announced.

Solution 6 by Albert Stadler, Herrliberg, Switzerland.

We write u=a+ib, v=c+id with a, b, c, d real. The system of equations is then equivalent to

$$\left\{ \begin{array}{c} \frac{1}{3}\left(a^2 + b^2\right) + \frac{1}{4}\left(c^2 + d^2\right) = \frac{1}{7}\left((a + c)^2 + (b + d)^2\right) \\ 8a + c = 7 \\ 8b + d = 7 \end{array} \right\}.$$

The first equation is a quadratic equation with respect to the variable a with discriminant $-\frac{4}{441}(4b - 3d)^2$ and at the same time a quadratic equation with respect to the variable b with discriminant $-\frac{4}{441}(4a - 3c)^2$. However the two discriminants must be ⩾0, since a and b are both real. So 4b=3d and 4a=3c, giving the solution (a,b,c,d)=(¾,¾,1,1) or equivalently

$$u = \frac{3}{4}\left(1 + i\right),\ v = 1 + i.$$

Solution 7 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We write $u = a + bi$ and $v = c + di$ for real numbers a, b, c, and d. Then the given system of equations becomes

$$\left\{ \begin{array}{c} \dfrac{a^2+b^2}{3} + \dfrac{c^2+d^2}{4} = \dfrac{(a+c)^2+(b+d)^2}{7} \\ \\ (8a+c)+(8b+d)i = 7+7i \end{array} \right\}.$$

This implies $c = 7 - 8a$ and $d = 7 - 8b$, so we have

$$\frac{a^2+b^2}{3} + \frac{(7-8a)^2+(7-8b)^2}{4} = \frac{(7-7a)^2+(7-7b)^2}{7}$$

and thus

$$\frac{28}{3}\left(a-\frac{3}{4}\right)^2 + \frac{28}{3}\left(b-\frac{3}{4}\right)^2 = 0.$$

Since a and b are real, we conclude that $a = b = 3/4$. Hence $c = d = 1$, so

$$u = \frac{3}{4} + \frac{3}{4}i = \frac{3}{4}(1+i) \quad \text{and} \quad v = 1+i.$$

Solution 8 by David A. Huckaby, Angelo State University, San Angelo, TX.

Let $u = a + bi$ and $v = c + di$, with a, b, c, and d real numbers. Then the second equation above becomes $8(a + bi) + c + di = 7 + 7i$, which gives $8a + c = 7$ and $8b + d = 7$, so that $c = 7 - 8a$ and $d = 7 - 8b$. The first equation above becomes

$$\begin{aligned} \frac{|a+bi|^2}{3} + \frac{|c+di|^2}{4} &= \frac{|(a+c)+(b+d)i|^2}{7} \\ \frac{a^2+b^2}{3} + \frac{c^2+d^2}{4} &= \frac{(a+c)^2+(b+d)^2}{7} \\ \frac{a^2+b^2}{3} + \frac{(7-8a)^2+(7-8b)^2}{4} &= \frac{(a+7-8a)^2+(b+7-8b)^2}{7}. \end{aligned}$$

Simplifying gives the equation $112a^2 + 112b^2 - 168a - 168b + 126 = 0$. Solving for a yields

$$a = \frac{3 \pm \sqrt{-(4b-3)^2}}{4}.$$

Since a is real, $4b - 3 = 0$, so that $b = \dfrac{3}{4}$ and $a = \dfrac{3}{4}$. Then $c = 7 - 8a = 7 - 8\left(\dfrac{3}{4}\right) = 1$; likewise $d = 7 - 8b = 7 - 8\left(\dfrac{3}{4}\right) = 1$. So $u = \dfrac{3}{4} + \dfrac{3}{4}i$ and $v = 1 + i$.

Solution 9 by Michel Bataille, Rouen, France.

Let u, v satisfying the equations. Note that $u \neq 0$ and $v \neq 0$ (if, say, $u = 0$, then the first equation implies that $v = 0$ as well, contradicting the second equation).

Since $|u+v|^2 = (u+v)(\overline{u}+\overline{v}) = |u|^2 + |v|^2 + (u\overline{v} + \overline{u}v)$, the first equation shows that the real number $u\overline{v} + \overline{u}v$ is equal to $\frac{4}{3}|u|^2 + \frac{3}{4}|v|^2$. It follows that

$$u\overline{v} + \overline{u}v \geqslant 2\left(\frac{4}{3}|u|^2 \cdot \frac{3}{4}|v|^2\right)^{1/2} = 2|u||v|$$

so that

$$2|u||v| \leqslant u\overline{v} + \overline{u}v \leqslant |u\overline{v} + \overline{u}v| \leqslant |u\overline{v}| + |\overline{u}v| = 2|u||v|.$$

Thus, $|u\overline{v} + \overline{u}v| = |u\overline{v}| + |\overline{u}v|$, which implies that $u\overline{v} = \lambda\overline{u}v$ for some positive real umber λ. Taking conjugates, we also have $\overline{u}v = \lambda u\overline{v} = \lambda^2\overline{u}v$, hence $\lambda = 1$, that is, $\overline{u}v = u\overline{v}$ is a nonzero real number. Since $u\overline{v} + \overline{u}v = 2|u||v|$, we have $\left(\frac{2}{\sqrt{3}}|u| - \frac{\sqrt{3}}{2}|v|\right)^2 = \frac{4}{3}|u|^2 + \frac{3}{4}|v|^2 - (u\overline{v} + \overline{u}v) = 0$, hence $|u| = \frac{3}{4}|v|$.

Now, the second equation gives $(8u+v)(8\overline{u}+\overline{v}) = 98$ and it follows that $64|u|^2+|v|^2+16|u||v| = 98$. With $|u| = \frac{3}{4}|v|$, this leads to $|v|^2 = 2$, so that $|v| = \sqrt{2}$, $|u| = \frac{3\sqrt{2}}{4}$. Let us set $u = \frac{3\sqrt{2}}{4}e^{i\alpha}$, $v = \sqrt{2}e^{i\beta}$ where $\alpha, \beta \in \mathbb{R}$. Since $u\overline{v} = |u||v|$ is a positive real number, we must have $\alpha \equiv \beta \pmod{2\pi}$ and therefore $7 + 7i = 8u + v = 7\sqrt{2}e^{i\alpha}$. Thus $\alpha = \frac{\pi}{4}$ and

$$u = \frac{3\sqrt{2}}{4} \cdot \frac{1+i}{\sqrt{2}} = \frac{3}{4}(1+i), \quad v = \sqrt{2} \cdot \frac{1+i}{\sqrt{2}} = 1+i.$$

Conversely, it is easily checked that these complex numbers satisfy the two equations.
We conclude that the system has a unique solution

$$(u,v) = \left(\frac{3}{4}(1+i), 1+i\right).$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Daniel Văcaru, Pitești, Romania; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA and the problem proposer.

• **5688** *Proposed by Kenneth Korbin, New York, NY.*

Three convex hexagons with integer side lengths are all inscribed in the same circle. The hexagons have perimeters p, $p+1$ and $p+2$. Find the lengths of the sides of each hexagon.

Solution by the problem proposer.

Answers:

$$(2,2,2,2,7,7)$$

$$(2,2,4,4,4,7)$$

$$(4,4,4,4,4,4)$$

with diameter equal to 8.

• **5689** *Proposed by Rafael Jakimczuk, Universidad National de Lujá, Buenos Aires, Argentina.*

Let $(F_n)_{n\geqslant 1}$ denote the Fibonacci sequence defined by the recursion $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. Find $\lim\limits_{n\to\infty} P_n$ where the sequence $(P_n)_{n\geqslant 1}$ is defined by

$$P_n := \prod_{k=1}^{n}\left(1+\frac{1}{nF_k}\right)^{F_{k+1}}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

In the following solution, we shall apply three well-known results: (a) $x/(1+x) \leqslant \ln(1+x) \leqslant x$ for $x > -1$; (b) If $a_n \to L$ as $n \to \infty$, then $(a_1 + a_2 + \cdots + a_n)/n \to L$ as $n \to \infty$; (c) $\lim\limits_{n\to\infty} F_{k+1}/F_k \to (1+\sqrt{5})/2$ as $n \to \infty$.

Taking the natural logarithm of P_n, we have

$$\sum_{k=1}^{n}\frac{F_{k+1}}{nF_k+1} \overset{(a)}{\leqslant} \sum_{k=1}^{n} F_{k+1}\ln\left(1+\frac{1}{nF_k}\right) \overset{(a)}{\leqslant} \frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}. \tag{9}$$

Since $(1/n)F_{k+1}/(F_k+1) = F_{k+1}/(nF_k+n) \leqslant F_{k+1}/(nF_k+1)$, we can rewrite (9) as

$$\frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k+1} \leqslant \ln P_n \leqslant \frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}.$$

But $F_{k+1}/(F_k+1) = (F_{k+1}/F_k)\cdot(1+1/F_k) \to (1+\sqrt{5})/2$ as $n\to\infty$ by (c). Finally, result (b) and the squeeze principle imply that $\ln P_n \to (1+\sqrt{5})/2 = \varphi$, so that $\lim\limits_{n\to\infty} P_n = e^{\varphi} \approx 5.04317$.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let $(F_n)_{n\geqslant 1}$ be the Fibonacci sequence, and let

$$P_n = \prod_{k=1}^{n}\left(1+\frac{1}{nF_k}\right)^{F_{k+1}}.$$

Then,

$$\begin{aligned}
\ln P_n &= \sum_{k=1}^{n} F_{k+1}\ln\left(1+\frac{1}{nF_k}\right)\\
&= \sum_{k=1}^{n} F_{k+1}\left(\frac{1}{nF_k}-\frac{1}{2n^2F_k^2}+\frac{1}{3n^3F_k^3}-+\cdots\right)\\
&= \frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}-\frac{1}{2n^2}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k^2}+\frac{1}{3n^3}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k^3}-+\cdots .
\end{aligned}$$

Now, for large k,

$$F_k \sim \frac{1}{\sqrt{5}}\varphi^k, \quad \text{where} \quad \varphi = \frac{1+\sqrt{5}}{2}.$$

It follows that

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \varphi, \quad \text{so} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k} = \varphi.$$

Additionally, for $j \geqslant 2$,

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k^j} = 0, \quad \text{so} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k^j} = 0.$$

Finally,

$$\lim_{n\to\infty} \ln P_n = \varphi, \quad \text{and} \quad \lim_{n\to\infty} P_n = e^{\varphi}.$$

Solution 3 by Moti Levy, Rehovot, Israel.

$$\ln(P_n) = \sum_{k=1}^{n} F_{k+1} \ln\left(1 + \frac{1}{nF_k}\right). \tag{10}$$

Taylor series of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m}. \tag{11}$$

Plugging (11) in (10) gives

$$\ln(P_n) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} (-1)^{m-1} \frac{F_{k+1}}{mn^m F_k^m} = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{mn^m} \left(\sum_{k=1}^{n} \frac{F_{k+1}}{F_k^m}\right). \tag{12}$$

We express the limit of (12) as sum of two limits,

$$\lim_{n\to\infty} \ln(P_n) = \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k}\right) + \lim_{n\to\infty} \left(\sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{mn^m} \sum_{k=1}^{n} \frac{F_{k+1}}{F_k^m}\right). \tag{13}$$

Now we show that the second limit is zero:

$$\left|\sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{k=1}^{n} \frac{F_{k+1}}{n^m F_k^m}\right| \leqslant \left|\sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{1}{n^2}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k^2}\right)\right| \leqslant \big(1 - \ln(2)\big) \left(\frac{1}{n^2}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k^2}\right) \tag{14}$$

$$\lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} \frac{F_{k+1}}{F_k^m}\right) = \lim_{k\to\infty} \frac{F_{k+1}}{F_k^m} = \begin{cases} \varphi, & m = 1 \\ 0, & m \geqslant 2 \end{cases} \tag{15}$$

It follows from (14) and (15) that $\lim_{n\to\infty}\left(\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{mn^m}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k^m}\right) = 0$. We are left with the first limit which by (15) is

$$\lim_{n\to\infty}\ln(P_n) = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}\right) = \varphi.$$

We conclude that $\lim_{n\to\infty} P_n = e^{\varphi} \cong 5.0432$.

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

Binet's explicit formula for the Fibonacci numbers states that

$$F_k = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right).$$

Thus $\lim_{k\to\infty}\frac{F_{k+1}}{F_k} = \frac{1+\sqrt{5}}{2}$. We conclude that

$$\log P_n = \sum_{k=1}^{n} F_{k+1}\log\left(1+\frac{1}{nF_k}\right) = \sum_{k=1}^{n}\frac{F_{k+1}}{nF_k} + O\left(\sum_{k=1}^{n}\frac{F_{k+1}}{(nF_k)^2}\right) =$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1+\sqrt{5}}{2}+o(1)\right) + O\left(\frac{1}{n^2}\sum_{k=1}^{n}1\right) = \frac{1+\sqrt{5}}{2} + o(1)$$

and

$$\lim_{n\to\infty} P_n = e^{\frac{1+\sqrt{5}}{2}}.$$

Solution 5 by Michel Bataille, Rouen, France.

We have

$$\ln(P_n) = \sum_{k=1}^{n} F_{k+1}\ln\left(1+\frac{1}{nF_k}\right)$$

and for positive x,

$$x - \frac{x^2}{2} \leqslant \ln(1+x) \leqslant x.$$

We deduce that for $k = 1, 2, \ldots, n$,

$$\frac{1}{n}\cdot\frac{F_{k+1}}{F_k} - \frac{1}{2n^2}\cdot\frac{F_{k+1}}{F_k^2} \leqslant F_{k+1}\ln\left(1+\frac{1}{nF_k}\right) \leqslant \frac{1}{n}\cdot\frac{F_{k+1}}{F_k}$$

so that

$$\frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k} - \frac{1}{2n}\cdot\frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k^2} \leqslant \ln(P_n) \leqslant \frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}. \quad (1)$$

We know that $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$ where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. It follows that $\lim\limits_{n\to\infty} F_n = \infty$ and that $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n} = \alpha$ (since

$$\frac{F_{n+1}}{F_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} = \alpha \cdot \frac{1 - (\beta/\alpha)^{n+1}}{1 - (\beta/\alpha)^n}.)$$

Therefore $\lim\limits_{n\to\infty} \dfrac{F_{n+1}}{F_n^2} = \lim\limits_{n\to\infty} \dfrac{F_{n+1}/F_n}{F_n} = 0$ and, using Cesaro's Theorem, we obtain

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}\right) = \alpha, \qquad \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k^2}\right) = 0.$$

Now, (1) and the Squeeze Theorem give $\lim\limits_{n\to\infty} \ln(P_n) = \alpha$.
We conclude

$$\lim_{n\to\infty} P_n = e^{\alpha}.$$

Solution 6 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

We use the following facts I and II

I. $\lim\limits_{n\to\infty} \dfrac{F_{k+1}}{F_k} = \dfrac{1+\sqrt{5}}{2} \doteq \alpha,$

II. $\ln(1+x) = x + O(x^2)$ for $x \to 0$.

Now

$$\ln P_n = \sum_{k=1}^{n} F_{k+1}\ln\left(1 + \frac{1}{nF_k}\right) = \sum_{k=1}^{n}\frac{F_{k+1}}{nF_k} + \sum_{k=1}^{n} F_{k+1} O\left(\frac{1}{n^2F_k^2}\right)$$

$$\sum_{k=1}^{n}\frac{F_{k+1}}{nF_k} = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{F_{k+1}}{F_k} - \alpha\right) + \alpha = \frac{1}{n}\sum_{k=1}^{n_0}\left(\frac{F_{k+1}}{F_k} - \alpha\right) + \frac{1}{n}\sum_{k=n_0+1}^{n}\left(\frac{F_{k+1}}{F_k} - \alpha\right) + \alpha$$

We know that $\forall \varepsilon > 0 \ \exists\, n_0 : \ k > n_0 \implies \left|\dfrac{F_{k+1}}{F_k} - \alpha\right|$:

$$\frac{1}{n}\left|\sum_{k=n_0+1}^{n}\left(\frac{F_{k+1}}{F_k} - \alpha\right)\right| \leqslant \frac{1}{n}\sum_{k=n_0+1}^{n}\varepsilon = \varepsilon\frac{n-n_0}{n} < \varepsilon$$

$$\frac{1}{n}\left|\sum_{k=1}^{n_0}\left(\frac{F_{k+1}}{F_k} - \alpha\right)\right| \leqslant \frac{1}{n}\max_{1\leqslant n\leqslant n_0}\left|\frac{F_{k+1}}{F_k} - \alpha\right| \to 0.$$

It follows that $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\frac{F_{k+1}}{nF_k}=\alpha$

$$\left|\sum_{k=1}^{n}F_{k+1}O\left(\frac{1}{n^2F_k^2}\right)\right|\leqslant C\left|\sum_{k=1}^{n}\frac{1}{n^2}\right|\to 0.$$

It follows $\lim\limits_{n\to\infty}\ln P_n=\alpha$ and then $P_n\to e^{\alpha}$

Also solved by the problem proposer.

• **5690** *Proposed by Toyesh Prakash Sharma (Student) St. C.F. Andrews School, Agra, India.*

Find the value of

$$\int_0^{1/\sqrt{2}}\sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)dx-\int_{\pi/4}^{\pi/2}\sin\left(\cos^{-1}(\sin x)\right)dx.$$

Solution 1 by Mohammad Bakkar, Tishreen University, Latakia, Syria.

First let's establish that

$$x\in\left[0,\pi/2\right]\implies\cos\left(\sin^{-1}\ (x)\right)=\sqrt{1-x^2},\ \sin\left(\cos^{-1}\ (x)\right)=\sqrt{1-x^2}.$$

Using the latter in the given expression, we get

$$\int_0^{1/\sqrt{2}}\sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)dx-\int_{\pi/4}^{\pi/2}\sin\left(\cos^{-1}(\sin x)\right)dx$$

$$=\int_0^{1/\sqrt{2}}\sin^{-1}\left(\sqrt{1-x^2}\right)dx-\int_{\pi/4}^{\pi/2}\sin\left(\sqrt{1-x^2}\right)dx.$$

Implementing the substitution $x=\cos(u)$, with $dx=-\sin(u)\,du$, in the first integral of the preceding expression, we get

$$\int_{\pi/4}^{\pi/2}\left[-u\sin(u)\right]du-\int_{\pi/4}^{\pi/2}\sin\left(\sqrt{1-x^2}\right)dx$$

$$\int_{\pi/4}^{\pi/2}\left[-u\cos(u)\right]'du=-u\cos(u)\Big|_{\pi/4}^{\pi/2}=\frac{\pi}{4\sqrt{2}}.$$

Solution 2 by Charles Diminnie and Andrew Siefker, Angelo State University, San Angelo, TX.

For $0 \leqslant x \leqslant \frac{\pi}{2}$, we have $0 \leqslant \cos^{-1}(\sin x) \leqslant \frac{\pi}{2}$ and hence,

$$\begin{aligned} \sin\left(\cos^{-1}(\sin x)\right) &= \sqrt{1-\cos^2\left(\cos^{-1}(\sin x)\right)} \\ &= \sqrt{1-\sin^2 x} \\ &= \cos x. \end{aligned} \tag{1}$$

It follows that

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(\cos^{-1}(\sin x)\right) dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x dx \\ &= \sin x \mid_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= 1 - \frac{\sqrt{2}}{2}. \end{aligned} \tag{2}$$

Note that condition (1) implies that

$$\sin^{-1}(\cos x) = \cos^{-1}(\sin x) \tag{3}$$

when $0 \leqslant x \leqslant \frac{\pi}{2}$. When $0 \leqslant x \leqslant 1$, it follows that $0 \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$ and condition (3) yields

$$\begin{aligned} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right) &= \cos^{-1}\left(\sin\left(\sin^{-1} x\right)\right) \\ &= \cos^{-1} x. \end{aligned} \tag{4}$$

This implies that

$$\int_0^{\frac{\sqrt{2}}{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right) dx = \int_0^{\frac{\sqrt{2}}{2}} \cos^{-1} x dx. \tag{5}$$

As recommended in calculus textbooks, $\int_0^{\frac{\sqrt{2}}{2}} \cos^{-1} x dx$ can be evaluated by using Integration by Parts with $u = \cos^{-1} x$ and $dv = dx$. Then,
$du = -\frac{1}{\sqrt{1-x^2}} dx$ and $v = x$ and thus,

$$\begin{aligned} \int_0^{\frac{\sqrt{2}}{2}} \cos^{-1} x dx &= x \cos^{-1} x \mid_0^{\frac{\sqrt{2}}{2}} - \int_0^{\frac{\sqrt{2}}{2}} x \left(-\frac{1}{\sqrt{1-x^2}}\right) dx \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} - \sqrt{1-x^2} \mid_0^{\frac{\sqrt{2}}{2}} \\ &= \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + 1. \end{aligned} \tag{6}$$

Conditions (5) and (6) imply that

$$\int_0^{\frac{\sqrt{2}}{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right) dx = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + 1. \tag{7}$$

Finally, by conditions (2) and (7), we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{\sqrt{2}}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right) dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(\cos^{-1}\left(\sin x\right)\right) dx \\
&= \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + 1 - \left(1 - \frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2}\pi}{8}. \quad \blacksquare
\end{aligned}$$

Solution 3 by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

Let

$$f(x) = \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right).$$

The function f is a strictly monotonically decreasing function on the interval $[0,1]$ with continuous derivative on $(0,1)$. So on the interval $[0, \frac{1}{\sqrt{2}}]$, f has an inverse function f^{-1} given by

$$f^{-1}(x) = \sin\left(\cos^{-1}\left(\sin x\right)\right).$$

In general, if f is a strictly monotonic function on the interval $[a,b]$ $(a < b)$ with continuous derivative on the interval (a,b), then (see, for example, [**?**, Thm 17.1, p. 233])

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a). \tag{16}$$

Here f^{-1} denotes the inverse of f. In our case $a = 0$ and $b = \frac{1}{\sqrt{2}}$. Thus

$$f(a) = f(0) = \sin^{-1}(\cos(\sin^{-1}(0))) = \frac{\pi}{2},$$

and

$$f(b) = f\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1}\left(\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)\right) = \frac{\pi}{4}.$$

On applying (16) to our function f we find

$$\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right)\, dx + \int_{\pi/2}^{\pi/4} \sin\left(\cos^{-1}\left(\sin x\right)\right)\, dx = \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} - 0 \cdot \frac{\pi}{2}$$

$$\Rightarrow \int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right)\, dx - \int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}\left(\sin x\right)\right)\, dx = \frac{\pi}{4\sqrt{2}}.$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

Clearly,

$$\cos\left(\arcsin x\right) = \sin\left(\arccos x\right) = \sqrt{1-x^2},\ \ 0 \leqslant x \leqslant \frac{\pi}{2}.$$

So

$$\int_0^{\frac{1}{\sqrt{2}}} \arcsin\left(\cos\left(\arcsin x\right)\right) dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(\arccos\left(\sin x\right)\right) dx =$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \arcsin\left(\sqrt{1-x^2}\right) dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 x}\, dx =$$

$$\overbrace{=}^{x=\cos y} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \arcsin\left(\sin y\right) \sin y\, dy - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \sin x\, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\, dx =$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{d}{dx}\left(x\cos x\right) dx = -x\cos x\, \Big|_{x=\frac{\pi}{4}}^{x=\frac{\pi}{2}} = \frac{\pi}{4\sqrt{2}}.$$

Solution 5 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We let $I = \int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos(\sin^{-1} x)\right) dx$ and $J = \int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}(\sin x)\right) dx$, and we show that

$$I - J = \frac{\pi\sqrt{2}}{8}.$$

For I, since $0 \leqslant x \leqslant 1/\sqrt{2}$, we have $\cos(\sin^{-1} x)) = \sqrt{1-x^2}$ and thus

$$\sin^{-1}\left(\cos(\sin^{-1} x)\right) = \sin^{-1}(\sqrt{1-x^2}) = \cos^{-1} x.$$

This yields

$$I = \int_0^{1/\sqrt{2}} \cos^{-1} x\, dx = \left[x\cos^{-1} x - \sqrt{1-x^2}\right]_0^{1/\sqrt{2}} = \frac{\pi\sqrt{2}}{8} - \frac{\sqrt{2}}{2} + 1.$$

For J, since $\pi/4 \leqslant x \leqslant \pi/2$, we have

$$\sin\left(\cos^{-1}(\sin x)\right) = \sqrt{1-(\sin x)^2)} = \cos x.$$

This yields

$$J = \int_{\pi/4}^{\pi/2} \cos x\, dx = \sin x\Big|_{\pi/4}^{\pi/2} = 1 - \frac{\sqrt{2}}{2}.$$

Hence we conclude that

$$I - J = \frac{\pi\sqrt{2}}{8}.$$

Solution 6 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

For $x \in (0, 1/\sqrt{2})$,

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2},$$

so

$$\sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right) = \sin^{-1}\left(\sqrt{1-x^2}\right) = \cos^{-1} x.$$

Thus,

$$\begin{aligned}
\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right)\, dx &= \int_0^{1/\sqrt{2}} \cos^{-1} x\, dx \\
&= x\cos^{-1} x\Big|_0^{1/\sqrt{2}} + \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^2}}\, dx \\
&= \frac{\pi}{4\sqrt{2}} - \sqrt{1-x^2}\Big|_0^{1/\sqrt{2}} = \frac{\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}} + 1.
\end{aligned}$$

Next, for $x \in (\pi/4, \pi/2)$,

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - x,$$

so

$$\sin\left(\cos^{-1}(\sin x)\right) = \sin\left(\frac{\pi}{2} - x\right) = \cos x.$$

Thus,

$$\int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}(\sin x)\right)\, dx = \int_{\pi/4}^{\pi/2} \cos x\, dx = \sin x\Big|_{\pi/4}^{\pi/2} = 1 - \frac{1}{\sqrt{2}}.$$

Hence,

$$\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right)\, dx - \int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}(\sin x)\right)\, dx = \frac{\pi}{4\sqrt{2}}.$$

Solution 7 by David A. Huckaby, Angelo State University, San Angelo, TX.

In the following we will use the facts that for acute angle θ, $\sin^{-1}(\cos\theta) = \cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$, and that $\int \sin^{-1} x\, dx = \sqrt{1-x^2} + x\sin^{-1} x + C$, where C is a constant.

The first integral is

$$\begin{aligned}
&\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1} x\right)\right)\, dx \\
&\quad = \int_0^{1/\sqrt{2}} \left(\frac{\pi}{2} - \sin^{-1} x\right)\, dx \\
&\quad = \left[\frac{\pi}{2}x - \left(\sqrt{1-x^2} + x\sin^{-1} x\right)\right]_0^{1/\sqrt{2}} \\
&\quad = \frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} - \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} - \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\
&\qquad\qquad - \frac{\pi}{2}(0) + \sqrt{1-0^2} - 0\cdot\sin^{-1} 0 \\
&\quad = \frac{\pi}{2\sqrt{2}} - \sqrt{1-\frac{1}{2}} - \frac{1}{\sqrt{2}}\cdot\frac{\pi}{4} + 1 \\
&\quad = \frac{\pi}{2\sqrt{2}} - \frac{\sqrt{2}}{2} - \frac{\pi}{4\sqrt{2}} + 1.
\end{aligned}$$

The second integral is

$$\begin{aligned}
&\int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}(\sin x)\right)\, dx \\
&\quad = \int_{\pi/4}^{\pi/2} \sin\left(\frac{\pi}{2} - x\right)\, dx \\
&\quad = \left[\cos\left(\frac{\pi}{2} - x\right)\right]_{\pi/4}^{\pi/2} \\
&\quad = 1 - \frac{\sqrt{2}}{2}.
\end{aligned}$$

So

$$\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\,dx - \int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}(\sin x)\right)\,dx$$
$$= \left(\frac{\pi}{2\sqrt{2}} - \frac{\sqrt{2}}{2} - \frac{\pi}{4\sqrt{2}} + 1\right) - \left(1 - \frac{\sqrt{2}}{2}\right)$$
$$= \frac{\pi}{2\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$$
$$= \frac{\pi}{4\sqrt{2}}.$$

Solution 8 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Setting $f(x) = \sin(\cos^{-1}(\sin x))$, we see that f is monotonically decreasing since

$$f'(x) = -\sin x \cos x / \sqrt{1 - \sin^2 x} < 0$$

for $x \in [\pi/4, \pi/2]$, which implies f has an inverse. Furthermore, since

$$(F \circ G \circ H)^{-1} = H^{-1} \circ G^{-1} \circ F^{-1},$$

we have

$$f^{-1}(x) = \sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right).$$

Now we use a known formula for integrating inverse functions:

$$\int_c^d f^{-1}(x)\,dx \;+\; \int_a^b f(x)\,dx \;=\; bd \;-\; ac, \quad \text{where } f(a) = c,\ f(b) = d.$$

Noting that $f(\pi/4) = 1/\sqrt{2}$ and $f(\pi/2) = 0$, we apply this formula to f, with $a = \pi/2$, $b = \pi/4$, $c = 0$, $d = 1/\sqrt{2}$, to get

$$\int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\,dx \;+\; \int_{\pi/2}^{\pi/4} \sin\left(\cos^{-1}(\sin x)\right)\,dx$$
$$= \int_0^{1/\sqrt{2}} \sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\,dx \;-\; \int_{\pi/4}^{\pi/2} \sin\left(\cos^{-1}(\sin x)\right)\,dx$$
$$= \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} \;-\; \frac{\pi}{2}\cdot 0 \;=\; \frac{\sqrt{2}\,\pi}{8}.$$

Comment: Although the formula cited above was published in 1905 by Charles-Ange Laisant, it doesn't seem to be well-known. See the Wikipedia article titled "Integral of inverse functions" for some discussion and references.

Solution 9 by Michel Bataille, Rouen, France.

Let Δ be the difference to be evaluated. We show that $\Delta = \dfrac{\pi\sqrt{2}}{8}$.

For $x \in [0, 1/\sqrt{2}]$, we have $\sin^{-1} x \in [0, \pi/4]$ and

$$\sin^{-1}\left(\cos(\sin^{-1} x)\right) = \sin^{-1}(\sin(\pi/2 - \sin^{-1} x)) = \pi/2 - \sin^{-1} x = \cos^{-1} x.$$

For $x \in [\pi/4, \pi/2]$, we have

$$\sin\left(\cos^{-1}(\sin x)\right) = \sin(\pi/2 - \sin^{-1}(\sin x)) = \sin(\pi/2 - x) = \cos x.$$

It follows that

$$\Delta = \int_0^{1/\sqrt{2}} \cos^{-1} x\, dx - \int_{\pi/4}^{\pi/2} \cos x\, dx = -\left(\int_{\pi/4}^{\pi/2} \cos x\, dx + \int_{\cos(\pi/4)}^{\cos(\pi/2)} \cos^{-1} x\, dx\right),$$

hence

$$\Delta = -\left(\frac{\pi}{2}\cos\frac{\pi}{2} - \frac{\pi}{4}\cos\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}$$

using the following general result: if f is a differentiable, strictly monotone function on $[a,b]$ $(a < b)$ and f^{-1} is its inverse, then

$$\int_a^b f(t)\, dt + \int_{f(a)}^{f(b)} f^{-1}(t)\, dt = bf(b) - af(a).$$

Proof. Let $F(x) = \displaystyle\int_a^x f(t)\, dt + \int_{f(a)}^{f(x)} f^{-1}(t)\, dt - xf(x)$ for $x \in [a,b]$. Then, $F'(x) = f(x) + f^{-1}(f(x))f'(x) - f(x) - xf'(x) = 0$, hence $F(x) = F(a) = -af(a)$ for all $x \in [a,b]$. In particular, $F(b) = -af(a)$.

Note. The result can also be reached *via*

$$\int_0^{1/\sqrt{2}} \cos^{-1} x\, dx = \left[x\cos^{-1} x\right]_0^{1/\sqrt{2}} + \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^2}}\, dx = \frac{\pi}{4\sqrt{2}} - \left[\sqrt{1-x^2}\right]_0^{1/\sqrt{2}} = \frac{\pi\sqrt{2}}{8} - \frac{1}{\sqrt{2}} + 1$$

and $\displaystyle\int_{\pi/4}^{\pi/2} \cos x\, dx = 1 - \frac{1}{\sqrt{2}}$.

Also solved by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA; Moti Levy, Rehovot, Israel and the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

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2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣