Problems and Solutions

Albert Natian, Section Editor

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This section of the SSMA Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Prof. Albert Natian, Department of Mathematics, Los Angeles Valley College, 5800 Fulton Avenue, Valley Glen, CA, 91401, USA. It's highly preferable that you send your contributions via email.

To propose problems, email them to: **problems4ssma@gmail.com**

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Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before April 1, 2023.

• **5709** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1, \ldots, x_n > 0$ be real numbers such that $x_1 + x_2 + \ldots + x_n = 1$. Prove the following inequality

$$1 + \sum_{i=1}^{n} x_i^2 \geqslant \prod_{i=1}^{n} (1 + x_i)^{x_i}.$$

When does equality occur?

• **5710** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania..*

Define the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ by $a_n = \sum_{k=1}^{n} \arctan\left(\frac{1}{k^2 - k + 1}\right)$ and $\lim_{n\to\infty} \frac{b_{n+1}}{nb_n} = \pi$. Compute $\lim_{n\to\infty}\left(\frac{\pi}{2} - a_n\right)\sqrt[n]{b_n}$.

• **5711** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy .*

Let a, b, c, d be nonnegative real numbers. Prove that

$$(a^5 - a + 4)(b^5 - b + 4)(c^5 - c + 4)(d^5 - d + 4) \geqslant (a + b + c + d)^4.$$

• **5712** *Proposed by Syed Shahabudeen, Ernakulam, Kerala, India.*
Prove that

$$\lim_{n\to\infty}\sum_{q=0}^{n-3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}}=\frac{\pi^2}{12}-\frac{1}{2}$$

• **5713** *Proposed by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

Prove

$$\int_0^1 \frac{\log^2(x)}{1+x^2+x^4}\,dx=\frac{7}{9}\zeta(3)+\frac{\pi^3}{18\sqrt{3}}.$$

Here $\zeta(3)$ is Apéry's constant defined by $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$.

• **5714** *Proposed by Peter Fulop, Gyomro, Hungary.*

Prove

$$\sum_{k=1}^{\infty}\ln\left(\frac{k}{\sqrt{k^2+1}}\right)=\frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right).$$

Solutions

To Formerly Published Problems

• **5691** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.*

Solve for real numbers $x \geqslant 1$:

$$2+2^x+4^x+\log_7\left(\frac{3^x+5^x}{2+2^x+4^x}\right)=3^x+5^x.$$

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

The only solution is $x=1$.

Let $u=2+2^x+4^x$ and $v=3^x+5^x$. Then the equation to solve is

$$u+\log_7\frac{v}{u}=v,$$

or

$$u-\log_7 u=v-\log_7 v.$$

If $f(x) = x - \log_7 x$, then $f'(x) = 1 - \dfrac{1}{x\ln 7} > 0$ for $x \geqslant 1$, so f is increasing, and hence injective, over the interval $[1, \infty)$. Thus, if $f(u) = f(v)$, then $u = v$, so

$$2 + 2^x + 4^x = 3^x + 5^x.$$

Notice that if x and c are real numbers with $x > 1$ and $0 < c < 1$, then $c^x < c$, so that

$$\frac{2^x + 1}{3^x} = \left(\frac{2}{3}\right)^x + \left(\frac{1}{3}\right)^x < \frac{2}{3} + \frac{1}{3} = 1$$

and

$$\frac{4^x + 1}{5^x} = \left(\frac{4}{5}\right)^x + \left(\frac{1}{5}\right)^x < \frac{4}{5} + \frac{1}{5} = 1.$$

Thus, $2^x + 1 < 3^x$ and $4^x + 1 < 5^x$, so that

$$2 + 2^x + 4^x < 3^x + 5^x$$

for all real numbers $x > 1$. Therefore, the only solution to $2 + 2^x + 4^x = 3^x + 5^x$, and to the original equation, for real numbers $x \geqslant 1$ is $x = 1$.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Let us rename $v = 3^x + 5^x$ and $z = 2 + 2^x + 4^x$, so the problem becomes $z + \log_7 \dfrac{v}{z} = v$, that is $7^{v-z} = \dfrac{v}{z}$, or $\dfrac{7^v}{v} = \dfrac{7^z}{z}$. Now, since function $f(x) = \dfrac{7^x}{x}$ is injective, if follows that $v = z$, or $3^x + 5^x = 2 + 2^x + 4^x$. By simple inspection it may be checked that $x = 1$ is a solution to the equation. In addition, since $3^x + 5^x - 2 - 2^x - 4^x$ is increasing, there are no more solutions.

Solution 3 by Moti Levy, Rehovot, Israel.

Let

$$f(x) := \frac{1}{\ln(7)} \ln(x) - x.$$

Then the equation is equivalent to

$$f(3^x + 5^x) = f(2 + 2^x + 4^x).$$

The function $f(x)$ is monotonically decreasing for $x \geqslant \dfrac{1}{\ln(7)} \cong 0.513\,90$ and since $3^x + 5^x \geqslant \dfrac{1}{\ln(7)}$ for $x \geqslant 1$ and $2 + 2^x + 4^x \geqslant \dfrac{1}{\ln(7)}$or $x \geqslant 1$, the equation is satisfied only if

$$3^x + 5^x = 2 + 2^x + 4^x.$$

Let

$$g(x) := 3^x + 5^x - (2 + 2^x + 4^x).$$

Then $g(1) = 0$. It follows from the inequality

$$(a+1)^x \geqslant a^x + 1, \ \ x \geqslant 1, \ a \geqslant 1 \ ,$$

that

$$3^x \geqslant 2^x + 1, \ \ x \geqslant 1,$$
$$5^x \geqslant 4^x + 1, \ \ x \geqslant 1.$$

Hence, $g(x) = 0$ for $x > 1$. We conclude that only the real number $x = 1$ is a solution.

Solution 4 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We show that the unique real solution with $x \geqslant 1$ is $x = 1$.

It is straightforward to verify that the equation holds when $x = 1$. For $x \geqslant 1$, we let $f(x) = 3^x + 5^x - 2 - 2^x - 4^x$ and $g(x) = \log_7(x)$. Then for $x \geqslant 1$,

$$f'(x) = (3^x \ln 3 - 2^x \ln 2) + (5^x \ln 5 - 4^x \ln 4) > 0.$$

Thus on $[1, \infty)$, f is increasing with $f(1) = 0$, which implies

$$3^x + 5^x > 2 + 2^x + 4^x \quad \text{for } x > 1.$$

Next, we let $a = 3^x + 5^x$ and $b = 2 + 2^x + 4^x$. Since g is concave down on $[1, \infty)$ and $a > b > 0$, we have

$$g(a - b) \geqslant g(a) - g(b).$$

Then applying the inequality $a - b > g(a - b)$ for $a > b$ yields

$$3^x + 5^x - 2 - 2^x - 4^x > \log_7(3^x + 5^x) - \log_7(2 + 2^x + 4^x) = \log_7\left(\frac{3^x + 5^x}{2 + 2^x + 4^x}\right)$$

for $x > 1$. Hence $x = 1$ is the unique real solution of the given equation on $[1, \infty)$.

Solution 5 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $f(t) = \log_7 t$, $a = 2 + 2^x + 4^x$, $b = 3^x + 5^x$, and $x \geqslant 1$. By the Mean Value Theorem, for each $x \geqslant 1$, there exists ξ between a and b such that

$$\log_7\left(\frac{3^x + 5^x}{2 + 2^x + 4^x}\right) = f(b) - f(a) = f'(\xi)(b - a) = f'(\xi)(3^x + 5^x - 2 - 2^x - 4^x).$$

Now,

$$f'(\xi) = \frac{1}{\xi \ln 7}.$$

With $x \geqslant 1$, both a and b are greater than or equal to 8, so $\xi \geqslant 8$ and $f'(\xi) < 1$. It then follows that

$$\log_7\left(\frac{3^x + 5^x}{2 + 2^x + 4^x}\right) \leqslant 3^x + 5^x - 2 - 2^x - 4^x,$$

with equality if and only if $3^x + 5^x - 2 - 2^x - 4^x = 0$. Now, let

$$g(x) = 3^x + 5^x - 2 - 2^x - 4^x.$$

Then $g(1) = 0$, and

$$g'(x) = 3^x \ln 3 + 5^x \ln 5 - 2^x \ln 2 - 4^x \ln 4 > 0$$

for $x > 1$. Thus, $g(x) > 0$ for all $x > 1$. Finally, the only solution to the equation

$$2 + 2^x + 4^x + \log_7\left(\frac{3^x + 5^x}{2 + 2^x + 4^x}\right) = 3^x + 5^x$$

with $x \geqslant 1$ is $x = 1$.

Solution 6 by Michel Bataille, Rouen, France..

We show that 1 is the unique solution for x. Clearly, 1 is a solution. We now prove that if $x > 1$, then x is not a solution.
Let $x > 1$ and, for the purpose of a contradiction, assume that x is a solution. Then $f(3^x + 5^x) = f(2 + 2^x + 4^x)$ where f is the function defined by $f(t) = t - \log_7(t)$. Since $3^x + 5^x$ and $2 + 2^x + 4^x$ are in $(8, \infty)$ and because f is strictly increasing on this interval (its derivative $f'(t) = 1 - \dfrac{1}{t\ln(7)}$ is positive), we must have

$$3^x + 5^x = 2 + 2^x + 4^x. \tag{1}$$

Now, a quick study of the function $t \mapsto (1 + t)^p - t^p - 1$ shows that if $t > 0$ and $p > 1$, then $(1 + t)^p - t^p > 1$. Therefore we have $5^x - 4^x = (1 + 4)^x - 4^x > 1$ and $3^x - 2^x = (1 + 2)^x - 2^x > 1$ so that $3^x + 5^x - 2^x - 4^x > 2$, contradicting (1). This completes the proof.

Also solved by Albert Stadler, Herrliberg, Switzerland; Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Daniel Vacaru, Pitesti, Romania and the problem proposer.

• **5692** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

Prove that

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}}{e^{\left(\frac{n+1}{2}\right)}(n+1)^{-\frac{3}{2}}} - \frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\left(\frac{n}{2}\right)}(n)^{-\frac{3}{2}}}\right) = \frac{e}{\sqrt{2\pi}}.$$

Solution 1 by Kaushik Mahanta, National Institute of Technology, Assam, India.

Recall $\prod_{k=1}^{n}\binom{n}{k} = \frac{(H(n))^2}{(n!)^{n+1}}$ and $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ where $H(n)$ denotes the hyperfactorial function.

By Stirling's approximation

$$H(n) \sim e^{-\frac{n^2}{4}} n^{\frac{6n^2+6n+1}{12}}\left(A+O\left(\frac{1}{n^2}\right)\right) \text{ and } n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Therefore

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{H^2(n+1)}{((n+1)!))^{n+2}}}(n+1)^{\frac{3}{2}}e^{-\frac{n+1}{2}} - \sqrt[n]{\frac{H^2(n)}{((n)!))^{n+1}}}(n)^{\frac{3}{2}}e^{-\frac{n}{2}}\right)$$

$$= \lim_{n\to\infty}\left(e^{-\frac{n+1}{2}}(n+1)^{\frac{3}{2}}\sqrt[n+1]{A^2\frac{e^{-\frac{(n+1)^2}{2}}(n+1)^{\frac{6(n+1)^2+6(n+1)+1}{6}}}{(\sqrt{2\pi(n+1)})^{n+2}\left(\frac{n+1}{e}\right)^{(n+1)(n+2)}}} - e^{-\frac{n}{2}}n^{\frac{3}{2}}\sqrt[n]{A^2\frac{e^{-\frac{n^2}{2}}n^{\frac{6n^2+6n+1}{6}}}{(\sqrt{2\pi n}))^{n+1}\left(\frac{n}{e}\right)^{n(n+1)}}}\right)$$

$$= \lim_{n\to\infty}\left(e^{-n-1+n+2}(n+1)^{\frac{3}{2}+n+1+1+\frac{1}{6(n+1)}-n-2-\frac{1}{2}-\frac{1}{n+1}}(\sqrt{2\pi})^{-1} - e^{-n+n+1}n^{\frac{3}{2}+n+1+\frac{1}{6n}-n-1-\frac{1}{2}-\frac{1}{n}}(\sqrt{2\pi})^{-1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{e}{\sqrt{2\pi}}(n+1-n) = \frac{e}{\sqrt{2\pi}}\right).$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let

$$a_n = \frac{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}{e^{\left(\frac{1}{2}n^2\right)}n^{-\frac{3}{2}n}}.$$

Because

$$\frac{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}} = (n+1)\prod_{k=1}^{n}\frac{\binom{n+1}{k+1}}{\binom{n}{k}} = (n+1)\prod_{k=1}^{n}\frac{n+1}{k+1} = \frac{(n+1)^{n+1}}{(n+1)!},$$

it follows that

$$\begin{aligned}
\lim_{n\to\infty}\frac{a_{n+1}}{na_n} &= \lim_{n\to\infty}\frac{(n+1)^{n+1}n^{-\frac{3}{2}n}}{n(n+1)!e^{n+\frac{1}{2}}(n+1)^{-\frac{3}{2}(n+1)}} \\
&= \lim_{n\to\infty}\frac{(n+1)^{n+1}n^{-\frac{3}{2}n}}{n(n+1)^{n+\frac{3}{2}}\sqrt{2\pi}e^{-\frac{1}{2}}(n+1)^{-\frac{3}{2}(n+1)}} \quad \text{by Stirling's approximation} \\
&= \frac{\sqrt{e}}{\sqrt{2\pi}}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{-\frac{3}{2}n-1} \\
&= \frac{e^2}{\sqrt{2\pi}},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} &= \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_n} \\
&= \lim_{n\to\infty}\frac{a_{n+1}}{na_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{e^2}{\sqrt{2\pi}}\cdot\frac{1}{e} = \frac{e}{\sqrt{2\pi}}.
\end{aligned}$$

Now, let

$$u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}.$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} u_n &= \lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}}{n+1}\cdot\frac{n}{\sqrt[n]{a_n}}\cdot\frac{n+1}{n} = \frac{e}{\sqrt{2\pi}}\cdot\frac{\sqrt{2\pi}}{e}\cdot 1 = 1, \\
\lim_{n\to\infty}\frac{u_n-1}{\ln u_n} &= 1, \text{ and} \\
\lim_{n\to\infty} u_n^n &= \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}} = \lim_{n\to\infty}\frac{a_{n+1}}{na_n}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1} \\
&= \frac{e^2}{\sqrt{2\pi}}\cdot\frac{\sqrt{2\pi}}{e}\cdot 1 = e.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}}{e^{\binom{n+1}{2}}(n+1)^{-\frac{3}{2}}} - \frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\binom{n}{2}}n^{-\frac{3}{2}}}\right) \\
&\qquad = \lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) \\
&\qquad = \lim_{n\to\infty}\sqrt[n]{a_n}(u_n-1) = \lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n \\
&\qquad = \frac{e}{\sqrt{2\pi}}\cdot 1\cdot\ln e = \frac{e}{\sqrt{2\pi}}.
\end{aligned}$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}=\frac{(n!)^{n-1}}{\left(\prod_{j=1}^{n-1}j!\right)\left(\prod_{j=1}^{n-1}(n-j)!\right)}=\frac{(n!)^{n-1}}{\left(\prod_{j=1}^{n-1}j!\right)^2}=$$

$$=\frac{(n!)^{n+1}}{\left(\prod_{j=1}^{n}j!\right)^2}=\frac{(n!)^{n+1}}{1^{2n}2^{2n-2}\cdots n^2}=\frac{\left(1^1 2^2\cdots n^n\right)^2}{(n!)^{n+1}}.$$

Stirling's asymptotic formula for the factorial states that

$$\log n!\ =\log\sqrt{2\pi}\ +\left(n+\frac{1}{2}\right)\log n\ -n+\frac{1}{12n}+O\left(\frac{1}{n^2}\right),$$

as n tends to infinity. The hyperfactorial H(n) of a positive integer n is defined as

$$H(n)=\prod_{k=1}^{n}k^k.$$

Glaisher provided an asymptotic formula for the hyperfactorials, analogous to Stirling's formula for the factorials, namely

$$H(n)=An^{\frac{6n^2+6n+1}{12}}e^{-\frac{n^2}{4}}\left(1+\frac{1}{720n^2}-\frac{1433}{7257600n^4}+\dots\right),$$

where A≈1.28243 is the Glaisher-Kinkelin constant (see the references https://en.wikipedia.org/wiki/Hyperfactorial and https://en.wikipedia.org/wiki/Glaisher%E2%80%93Kinkelin_constant).
Hence

$$\frac{1}{n}\log\left(\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}\right)-\frac{n}{2}+\frac{3}{2}\log n\ =\frac{2}{n}\log H(n)\ -\frac{n+1}{n}\log n!\ -\frac{n}{2}+\frac{3}{2}\log n\ =$$

$$=\frac{2}{n}\left(\log A\ +\frac{6n^2+6n+1}{12}\log n\ -\frac{n^2}{4}+O\left(\frac{1}{n^2}\right)\right)$$

$$-\frac{n+1}{n}\left(\log\sqrt{2\pi}\ +\left(n+\frac{1}{2}\right)\log n\ -n+\frac{1}{12n}+O\left(\frac{1}{n^2}\right)\right)-\frac{n}{2}+\frac{3}{2}\log n\ =$$

$$=(\log n\ )+1-\log\sqrt{2\pi}\ -\frac{\log n}{3n}+\frac{1}{n}\left(2\log A\ -\log\sqrt{2\pi}\ -\frac{1}{12}\right)+O\left(\frac{1}{n^2}\right)$$

and

$$\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\left(\frac{n}{2}\right)}(n)^{-\frac{3}{2}}}=n\frac{e}{\sqrt{2\pi}}e^{-\frac{\log n}{3n}+\frac{1}{n}\left(2\log A\ -\log\sqrt{2\pi}\ -\frac{1}{12}\right)+O\left(\frac{1}{n^2}\right)}$$

$$= n\frac{e}{\sqrt{2\pi}} + \frac{e}{\sqrt{2\pi}}\left(-\frac{\log n}{3} + 2\log A \; - \log\sqrt{2\pi} \; - \frac{1}{12}\right) + O\left(\frac{\log^2 n}{n}\right).$$

We conclude that

$$\frac{\sqrt[n+1]{\begin{pmatrix} n+1 \\ 1 \end{pmatrix}\begin{pmatrix} n+1 \\ 2 \end{pmatrix}\cdots\begin{pmatrix} n+1 \\ n+1 \end{pmatrix}}}{e^{\binom{n+1}{2}}(n+1)^{-\frac{3}{2}}} - \frac{\sqrt[n]{\begin{pmatrix} n \\ 1 \end{pmatrix}\begin{pmatrix} n \\ 2 \end{pmatrix}\cdots\begin{pmatrix} n \\ n \end{pmatrix}}}{e^{\binom{n}{2}}(n)^{-\frac{3}{2}}} =$$

$$= \frac{e}{\sqrt{2\pi}} + \frac{e}{\sqrt{2\pi}}\left(\frac{\log n}{3} - \frac{\log(n+1)}{3}\right) + O\left(\frac{\log^2 n}{n}\right) =$$

$$= \frac{e}{\sqrt{2\pi}} + O\left(\frac{\log^2 n}{n}\right)$$

which is a more precise statement of what needs to be established.

Solution 4 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

The partial results I), II), III) below are needed.

I)

$$\frac{1}{n}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{\frac{-3}{2}}} = \left(\frac{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}n^{\frac{3n}{2}}}{n^n e^{\frac{n^2}{2}}}\right)^{\frac{1}{n}}$$

I use the Cesąro–Stolz theorem (https://it.wikipedia.org/wiki/Teorema_di_Stolz-Cesaro)

$$\frac{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}\frac{(n+1)^{\frac{3(n+1)}{2}}}{n^{\frac{3n}{2}}}\frac{e^{\frac{n^2}{2}}}{e^{\frac{(n+1)^2}{2}}}\frac{n^n}{(n+1)^{n+1}} =$$

$$= \frac{(n+1)^n}{n!}\frac{(n+1)^{\frac{3n+1}{2}}}{n^{\frac{3n}{2}}}\frac{e^{\frac{n^2}{2}}}{e^{\frac{(n+1)^2}{2}}}\frac{1}{(1+\frac{1}{n})^n} = \frac{(n+1)^n}{n!e^n}\frac{(n+1)^{\frac{3n+1}{2}}}{n^{\frac{3n}{2}}}\frac{1}{\sqrt{e}}\frac{1}{(1+\frac{1}{n})^n} =$$

$$= \frac{(n+1)^n}{n^n}\frac{n^n}{n!e^n}\sqrt{n+1}\left(1+\frac{1}{n}\right)^{\frac{3n}{2}}\frac{1}{\sqrt{e}}\frac{1}{(1+\frac{1}{n})^n} =$$

$$= \frac{n^n}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}(1+o(1))e^n}\sqrt{n+1}\left(1+\frac{1}{n}\right)^{\frac{3n}{2}}\frac{1}{\sqrt{e}} \to \frac{e}{\sqrt{2\pi}}.$$

II) Let $P_n \doteq \dfrac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{\frac{-3}{2}}}$.

$$\lim_{n\to\infty}\frac{P_{n+1}}{P_n} = \frac{P_{n+1}}{n+1}\frac{n}{P_n}\frac{n+1}{n} = \frac{e}{\sqrt{2\pi}}\frac{\sqrt{2\pi}}{e}\cdot 1 = 1.$$

III)

$$\left(\frac{P_{n+1}}{P_n}\right)^n = \frac{\left(\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}\right)}{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}\left(\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}\right)^{-\frac{1}{n+1}}\cdot$$

$$\cdot\frac{e^{\frac{n^2}{2}}}{e^{\frac{n(n+1)}{2}}}\frac{(n+1)^{\frac{3n}{2}}}{n^{\frac{3n}{2}}} =$$

$$= \frac{(n+1)^n}{n!}\left[\frac{ne^{\frac{n+1}{2}}}{\left(\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}\right)^{\frac{1}{n+1}}(n+1)^{\frac{3}{2}}}\right]\frac{(n+1)^{\frac{3}{2}}}{ne^n\sqrt{e}}\left(1+\frac{1}{n}\right)^{\frac{3n}{2}} =$$

$$= \frac{(n+1)^n}{n^n}\frac{n^n}{n!e^n}\frac{(n+1)^{\frac{3}{2}}}{n\sqrt{e}}\left[\frac{ne^{\frac{n+1}{2}}}{\left(\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}\right)^{\frac{1}{n+1}}(n+1)^{\frac{3}{2}}}\right]\left(1+\frac{1}{n}\right)^{\frac{3n}{2}} =$$

$$= \frac{(n+1)^n}{n^n}\frac{n^n}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}(1+o(1))e^n}\frac{(n+1)^{\frac{3}{2}}}{n\sqrt{e}}\left(1+\frac{1}{n}\right)^{\frac{3n}{2}}\cdot$$

$$\cdot\left[\frac{ne^{\frac{n+1}{2}}}{\left(\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}\right)^{\frac{1}{n+1}}(n+1)^{\frac{3}{2}}}\right] =$$

$$= \frac{(n+1)^n}{n^n}\frac{1}{\sqrt{2\pi}(1+o(1))}\frac{(n+1)^{\frac{3}{2}}}{\sqrt{n}n\sqrt{e}}\left(1+\frac{1}{n}\right)^{\frac{3n}{2}}\cdot$$

$$\cdot\left[\frac{ne^{\frac{n+1}{2}}}{\left(\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}\right)^{\frac{1}{n+1}}(n+1)^{\frac{3}{2}}}\right] \to e\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{e}}\cdot e\sqrt{e}\cdot\frac{\sqrt{2\pi}}{e} = e.$$

Finally our limit is

$$\lim_{n\to\infty} P_{n+1}-P_n = \lim_{n\to\infty}\frac{P_n}{n}\frac{\frac{P_{n+1}}{P_n}-1}{\ln\frac{P_{n+1}}{P_n}}\ln\left(\frac{P_{n+1}}{P_n}\right)^n = \frac{e}{\sqrt{2\pi}}\cdot 1\cdot\ln(e) = \frac{e}{\sqrt{2\pi}}.$$

Also solved by Moti Levy, Rehovot, Israel; Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia; Michel Bataille, Rouen, France and the problem proposer .

• **5693** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$\int_0^\infty \frac{\ln x}{x^4 + x^2 + 1} dx.$$

Solution 1 by Péter Fülöp, Gyömrő, Hungary.

Let's denote the value of the integral by I.

Steps of the calculation

1. Substitution: $x^2 = t$
2. Using the fact: $t^2 + t + 1 = \dfrac{t^3 - 1}{t - 1}$
3. Substitution: $z = t^3$
4. Substitution: $z = \dfrac{r}{r - 1}$
5. Using the fact: $\ln(r) = \dfrac{d(r^a)}{da}|_{a=0}$
6. Applying the definiton of β function: $\beta(x, y) = \int\limits_0^1 t^{x-1}(1 - t)^{y-1} dt$
7. Using the relation between β and Γ functions: $\beta(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$
8. Euler's reflection formula for Γ function was used
9. Derivation according to a and calculation at the point $a = 0$.

Execution

Perform the substitution (1): $x^2 = t$ and
using the fact (2) that $t^2 + t + 1 = \dfrac{t^3 - 1}{t - 1}$, we get:

$$\frac{1}{4}\int\limits_0^\infty \frac{\ln(t)}{\sqrt{t}(t^2 + t + 1)} dt = \frac{1}{4}\int\limits_0^\infty \frac{(t - 1)\ln(t)}{\sqrt{t}(t^3 - 1)} dt = \frac{1}{4}\int\limits_0^\infty \frac{\sqrt{t}\ln(t)}{(t^3 - 1)} dt - \frac{1}{4}\int\limits_0^\infty \frac{\ln(t)}{\sqrt{t}(t^3 - 1)} dt.$$

Substitution (3): $z = t^3$

$$I = \frac{1}{36}\int\limits_0^\infty \frac{z^{-\frac{1}{2}}\ln(z)}{(z - 1)} dz - \frac{1}{36}\int\limits_0^\infty \frac{\ln(z)}{z^{\frac{5}{6}}(z - 1)} dz.$$

After performing the substitution (4): $z = \dfrac{r}{r - 1}$, let's apply the fact (5) $\ln(r) = \dfrac{d(r^a)}{da}|_{a=0}$ and then exchanging the order of the derivation and integration, we get the following:

$$I = -\frac{1}{36}\frac{d}{da}\int\limits_0^1 r^{a-\frac{1}{2}}(r - 1)^{-a-\frac{1}{2}} dr|_{a=0} + \frac{1}{36}\frac{d}{da}\int\limits_0^1 r^{a-\frac{5}{6}}(r - 1)^{-a-\frac{1}{6}} dr|_{a=0}.$$

Based on the definition of the β function (6) we have:

$$I=-\frac{1}{36}\frac{d}{da}\Big[(-1)^{-a-\frac{1}{2}}\beta(a+\frac{1}{2},\frac{1}{2}-a)\Big]+\frac{1}{36}\frac{d}{da}\Big[(-1)^{-a-\frac{1}{6}}\beta(a+\frac{1}{6},\frac{5}{6}-a)\Big].$$

Moving to Γ function (7):

$$I=-\frac{1}{36}\frac{d}{da}\Big[(-1)^{-a-\frac{1}{2}}\Gamma(a+\frac{1}{2})\Gamma(\frac{1}{2}-a)\Big]+\frac{1}{36}\frac{d}{da}\Big[(-1)^{-a-\frac{1}{6}}\Gamma(a+\frac{1}{6})\Gamma(\frac{5}{6}-a)\Big].$$

We can realize that both expressions have the reflection relation (8) $\Gamma(a+\frac{1}{2})\Gamma(\frac{1}{2}-a)$ and $\Gamma(a+\frac{1}{6})\Gamma(\frac{5}{6}-a)$. So

$$I=-\frac{1}{36}\frac{d}{da}\Big[(-1)^{-a-\frac{1}{2}}\frac{\pi}{\sin[\pi(a+\frac{1}{2})]}\Big]+\frac{1}{36}\frac{d}{da}\Big[(-1)^{-a-\frac{1}{6}}\frac{\pi}{\sin[\pi(a+\frac{1}{6})]}\Big].$$

Performing the derivation according to a and quite a lot calculations with complex numbers at the point $a=0$ (9), I got the result:

$$I=\int_0^\infty\frac{\ln(x)}{x^4+x^2+1}dx=-\frac{\pi^2}{12}.$$

Solution 2 by Moti Levy, Rehovot, Israel.

$$\begin{aligned}I:&=\int_0^\infty\frac{\ln(x)}{x^4+x^2+1}dx=\int_1^\infty\frac{\ln(x)}{x^4+x^2+1}dx+\int_0^1\frac{\ln(x)}{x^4+x^2+1}dx\\&=-\int_0^1\frac{x^2\ln(x)}{x^4+x^2+1}dx+\int_0^1\frac{\ln(x)}{x^4+x^2+1}dx=\int_0^1\frac{\left(1-x^2\right)\ln(x)}{x^4+x^2+1}dx\\&=\int_0^1\frac{\left(1-2x^2+x^4\right)\ln(x)}{1-x^6}dx\end{aligned}$$

$$\int_0^1 x^n\ln(x)\,dx=-\frac{1}{(n+1)^2}.$$

$$\begin{aligned}\int_0^1\frac{\ln(x)}{1-x^6}dx&=\int_0^1\sum_{k=0}^\infty x^{6k}\ln(x)\,dx=\sum_{k=0}^\infty\int_0^\infty x^{6k}\ln(x)\,dx\\&=-\frac{1}{36}\sum_{k=0}^\infty\frac{1}{\left(k+\frac{1}{6}\right)^2}=-\frac{1}{36}\psi^{(1)}\left(\frac{1}{6}\right),\end{aligned}$$

where $\psi^{(1)}(x)$ is the polygamma function of order 1.

$$\int_0^1 \frac{x^2 \ln(x)}{1-x^6} dx = \sum_{k=0}^{\infty} \int_0^{\infty} x^{6k+2} \ln(x)\, dx = -\frac{1}{36} \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{2}\right)^2} = -\frac{1}{36} \psi^{(1)}\left(\frac{1}{2}\right).$$

$$\int_0^1 \frac{x^4 \ln(x)}{1-x^6} dx = \sum_{k=0}^{\infty} \int_0^{\infty} x^{6k+4} \ln(x)\, dx = -\frac{1}{36} \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{5}{6}\right)^2} = -\frac{1}{36} \psi^{(1)}\left(\frac{5}{6}\right).$$

$$I = -\frac{1}{36}\left(\psi^{(1)}\left(\frac{1}{6}\right) + \psi^{(1)}\left(\frac{5}{6}\right) - 2\psi^{(1)}\left(\frac{1}{2}\right)\right).$$

From the reflection relation of the polygamma function,

$$\psi^{(1)}(1-z) + \psi^{(1)}(z) = -\pi \frac{d\left(\cot(\pi z)\right)}{dz} = \pi^2\left(\cot^2 \pi z + 1\right).$$

$$\psi^{(1)}\left(\frac{5}{6}\right) + \psi^{(1)}\left(\frac{1}{6}\right) = \pi^2\left(\cot^2\left(\frac{\pi}{6}\right) + 1\right) = 4\pi^2,$$

$$\psi^{(1)}\left(\frac{1}{2}\right) = \frac{\pi^2}{2}.$$

$$I = -\frac{\pi^2}{12} \cong -0.822\,47.$$

Solution 3 by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

Denoting the integral by I, we begin by writing it as

$$I = \int_0^1 \frac{\log(x)}{x^4 + x^2 + 1}\, dx + \int_1^{\infty} \frac{\log(x)}{x^4 + x^2 + 1}\, dx.$$

In the second of the integrals, enforcing a substitution of $x \mapsto \dfrac{1}{x}$ leads to

$$I = \int_0^1 \frac{(1-x^2)\log(x)}{x^4 + x^2 + 1}\, dx.$$

Now let $u = x^2$. This gives

$$I = \frac{1}{4} \int_0^1 \frac{(1-u)\log(u)}{\sqrt{u}(u^2 + u + 1)}\, du.$$

From the algebraic identity

$$u^2 + u + 1 = \frac{1-u^3}{1-u},$$

using this result we may rewrite the integral as

$$I = \frac{1}{4}\int_0^1 \frac{(1-u)^2\log(u)}{\sqrt{u}(1-u^3)}\,du.$$

Letting $t = u^3$, we see that

$$I = \frac{1}{36}\int_0^1 \frac{(t^{-\frac{5}{6}} - 2t^{-\frac{1}{2}} + t^{-\frac{1}{6}})\log(t)}{1-t}\,dt. \tag{1}$$

Now consider the parametric integral

$$I(a) = \int_0^1 \frac{(t^{-\frac{5}{6}} - 2t^{-\frac{1}{2}} + t^{-\frac{1}{6}})t^a}{1-t}\,dt,$$

where $a > 0$. As

$$I'(a) = \int_0^1 \frac{(t^{-\frac{5}{6}} - 2t^{-\frac{1}{2}} + t^{-\frac{1}{6}})t^a\log(t)}{1-t}\,dt,$$

we observe that

$$I = \frac{1}{36}\lim_{a\to 0} I'(a).$$

Evaluating $I(a)$ we have

$$\begin{aligned} I(a) &= \int_0^1 \frac{(t^{a-\frac{5}{6}} - 2t^{a-\frac{1}{2}} + t^{a-\frac{1}{6}})}{1-t}\,dt \\ &= -\int_0^1 \frac{1-t^{a-\frac{5}{6}}}{1-t}\,dt + 2\int_0^1 \frac{1-t^{a-\frac{1}{2}}}{1-t}\,dt \\ &\quad - \int_0^1 \frac{1-t^{a-\frac{1}{6}}}{1-t}\,dt. \end{aligned}$$

Recalling the integral definition for the digamma function $\psi(z)$, namely

$$\psi(z+1) = -\gamma + \int_0^1 \frac{1-t^z}{1-t}\,dt,$$

where γ denotes the Euler–Mascheroni constant, in terms of this function the above integral can be expressed as

$$I(a) = -\psi\left(a+\frac{5}{6}\right) + 2\psi\left(a+\frac{1}{2}\right) - \psi\left(a+\frac{1}{6}\right).$$

On differentiating this result with respect to a we find

$$I'(a) = -\psi^{(1)}\left(a+\frac{5}{6}\right) + 2\psi^{(1)}\left(a+\frac{1}{2}\right) - \psi^{(1)}\left(a+\frac{1}{6}\right),$$

where $\psi^{(1)}(z)$ denotes the trigamma function. Thus

$$\begin{aligned}
I &= \frac{1}{36}\lim_{a\to 0} I'(a) \\
&= \frac{1}{36}\lim_{a\to 0}\left[-\psi^{(1)}\left(a+\frac{5}{6}\right)+2\psi^{(1)}\left(a+\frac{1}{2}\right)-\psi^{(1)}\left(a+\frac{1}{6}\right)\right] \\
&= \frac{1}{36}\left[-\psi^{(1)}\left(\frac{5}{6}\right)+2\psi^{(1)}\left(\frac{1}{2}\right)-\psi^{(1)}\left(\frac{1}{6}\right)\right] \\
&= \frac{1}{36}\left[\psi^{(1)}\left(1-\frac{1}{2}\right)+\psi^{(1)}\left(\frac{1}{2}\right)-\psi^{(1)}\left(1-\frac{1}{6}\right)-\psi^{(1)}\left(\frac{1}{6}\right)\right] \\
&= \frac{1}{36}\left[\pi^2\csc^2\left(\frac{\pi}{2}\right)-\pi^2\csc^2\left(\frac{\pi}{6}\right)\right].
\end{aligned}$$

Note in the last line the reflexion formula for the trigamma function of

$$\psi^{(1)}(1-z)+\psi^{(1)}(z)=\pi^2\csc^2(\pi z),$$

has been used. Since

$$\csc\left(\frac{\pi}{2}\right)=1 \quad\text{and}\quad \csc\left(\frac{\pi}{6}\right)=2,$$

we see that

$$I=\frac{\pi^2}{36}\left(1^2-2^2\right)=-\frac{\pi^2}{12},$$

the required value for the integral.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Break the integration interval at $x = 1$ and make the substitution $x \to 1/x$ over the interval $[1, \infty)$. This yields

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{x^4+x^2+1}\,dx &= \int_0^1 \frac{\ln x}{x^4+x^2+1}\,dx+\int_1^\infty \frac{\ln x}{x^4+x^2+1}\,dx \\
&= \int_0^1 \frac{\ln x}{x^4+x^2+1}\,dx-\int_0^1 \frac{x^2\ln x}{x^4+x^2+1}\,dx \\
&= \int_0^1 \frac{(1-x^2)\ln x}{x^4+x^2+1}\,dx.
\end{aligned}$$

Now, multiply numerator and denominator of the integrand by $1 - x^2$ to obtain

$$\int_0^\infty \frac{\ln x}{x^4+x^2+1}\,dx=\int_0^1 \frac{1-2x^2+x^4}{1-x^6}\ln x\,dx.$$

Next, use the power series representation

$$\frac{1}{1-x^6} = \sum_{k=0}^{\infty} x^{6k}$$

and then integrate by parts term-by-term:

$$\begin{aligned}\int_0^\infty \frac{\ln x}{x^4+x^2+1}\,dx &= \sum_{k=0}^{\infty}\int_0^1 \left(x^{6k}-2x^{6k+2}+x^{6k+4}\right)\ln x\,dx \\ &= -\sum_{k=0}^{\infty}\left[\frac{1}{(6k+1)^2}-\frac{2}{(6k+3)^2}+\frac{1}{(6k+5)^2}\right] \\ &= -\frac{1}{36}\left(\psi_1\left(\frac{1}{6}\right)-2\psi_1\left(\frac{1}{2}\right)+\psi_1\left(\frac{5}{6}\right)\right),\end{aligned}$$

where $\psi_1(x)$ is the trigamma function. Using the reflection formula

$$\psi_1(x)+\psi_1(1-x) = -\pi\frac{d}{dx}\cot\pi x = \pi^2\csc^2\pi x = \frac{\pi^2}{\sin^2\pi x},$$

this becomes

$$\int_0^\infty \frac{\ln x}{x^4+x^2+1}\,dx = -\frac{\pi^2}{36}\left(\frac{1}{\sin^2\frac{\pi}{6}}-\frac{1}{\sin^2\frac{\pi}{2}}\right) = -\frac{\pi^2}{12}.$$

Solution 5 by Daniel Văcaru, Pitești, Romania.

We have

$$\frac{\ln x}{x^4+x^2+1} = \frac{\frac{1}{2}x+\frac{1}{2}}{x^2+x+1}\ln x + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2-x+1}\ln x.$$

On the other hand, we have

$$\int_0^\infty \frac{x\ln x}{x^2+x+1}\mathrm{dx} = \int_0^1 \frac{x\ln x}{x^2+x+1}\mathrm{dx} + \int_1^\infty \frac{\mathrm{x}\ln \mathrm{x}}{\mathrm{x}^2+\mathrm{x}+1}\mathrm{dx} = 2\int_0^1 \frac{\mathrm{x}\ln \mathrm{x}}{\mathrm{x}^2+\mathrm{x}+1}\mathrm{dx}$$

and three more relationships.

It follows (Gradshteyn and Ryzhik 4.233) that

$$\int_0^\infty \frac{\ln x}{x^4+x^2+1}\mathrm{dx} = \int_0^1 \frac{x\ln x}{x^2+x+1}\mathrm{dx}+\int_0^1 \frac{\ln x}{x^2+x+1}\mathrm{dx}-\int_0^1 \frac{x\ln x}{x^2-x+1}\mathrm{dx}+\int_0^1 \frac{\ln \mathrm{x}}{\mathrm{x}^2-\mathrm{x}+1}\mathrm{dx} =$$

$$-\frac{1}{9}\left[\frac{7\pi^2}{6}-\psi'\left(\frac{1}{3}\right)\right]+\frac{2}{9}\left[\frac{2\pi^2}{3}-\psi'\left(\frac{1}{3}\right)\right]-\frac{1}{6}\left[\frac{5\pi^2}{6}-\psi'\left(\frac{1}{3}\right)\right]+\frac{1}{3}\left[\frac{2\pi^2}{3}-\psi'\left(\frac{1}{3}\right)\right] = \frac{11\pi^2}{108}-\frac{5}{18}\psi'\left(\frac{1}{3}\right).$$

Solution 6 by Michel Bataille, Rouen, France.

Let I be the given integral. We apply a formula deduced from a suitable contour integration, as detailed in **[1]**: setting $f(z) = \dfrac{1}{z^4 + z^2 + 1}$, we have

$$\int_0^\infty \frac{\ln x}{x^4 + x^2 + 1}\,dx = -\frac{1}{2}Re(\sigma)$$

where σ is the sum of the residues of $f(z)(\log z)^2$ at the poles of $f(z)$ and $\log z = \ln(|z|) + i\theta$ when $0 \leqslant \theta = \arg(z) < 2\pi$.
The poles of $f(z)$ are $-\omega^2, \omega, \omega^2, -\omega$ where $\omega = \exp(2\pi i/3)$. The residues of $f(z)(\log z)^2$ at these poles respectively are

$$\frac{(i\pi/3)^2}{4(-\omega^2)^3 - 2\omega^2} = \frac{\pi^2/9}{4 + 2\omega^2}$$

$$\frac{(2i\pi/3)^2}{4(\omega)^3 + 2\omega} = \frac{-4\pi^2/9}{4 + 2\omega}$$

$$\frac{(4i\pi/3)^2}{4(\omega^2)^3 + 2\omega^2} = \frac{-16\pi^2/9}{4 + 2\omega^2}$$

$$\frac{(5i\pi/3)^2}{-4 - 2\omega} = \frac{25\pi^2/9}{4 + 2\omega}$$

and their sum is

$$\frac{7\pi^2/3}{4 + 2\omega} - \frac{5\pi^2/3}{4 + 2\omega^2} = \frac{\pi^2}{6}\left(\frac{7}{2 + \omega} - \frac{5}{2 + \omega^2}\right) = \frac{\pi^2}{18} \cdot (3 - 6i\sqrt{3})$$

so that $Re(\sigma) = \dfrac{\pi^2}{6}$. It follows that

$$I = -\frac{\pi^2}{12}.$$

[1] H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Dover, 1995, ch. III, p. 109.

Also solved by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy; Albert Stadler, Herrliberg, Switzerland; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Kaushik Mahanta, National Institute of Technology, Assam, India; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain and the problem proposer.

• **5694** *Proposed by Michel Bataille, Rouen, France.*

Given a positive integers m and n, let $S_m(n) = \dfrac{1}{n^{m+1}}\displaystyle\sum_{k=1}^{n} k^m H_k$ where $H_k = \displaystyle\sum_{i=1}^{k}\frac{1}{i}$. Find real numbers λ_m and μ_m such that

$$\lim_{n\to\infty}(S_m(n) - \lambda_m \ln n - \mu_m) = 0.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We use the well-known asymptotic expansion of the harmonic numbers (see for instance https://en.wikipedia.org/w

$$H_k = \ln k \; + +O\left(\frac{1}{k}\right),$$

as k tends to infinity. Here γ denotes the Euler-Mascheroni constant. Then

$$S_m(n) = \frac{1}{n^{m+1}}\sum_{k=1}^{n} k^m H_k = \frac{1}{n^{m+1}}\sum_{k=1}^{n} k^m \left(\ln k \; + +O\left(\frac{1}{k}\right)\right) =$$

$$= \frac{1}{n^{m+1}}\sum_{k=1}^{n} k^m \ln k \; + \frac{\gamma}{n^{m+1}}\sum_{k=1}^{n} k^m + \frac{1}{n^{m+1}} O\left(\sum_{k=1}^{n} k^{m-1}\right).$$

If f(x) is a real-valued monotonically increasing function defined for positive real values then

$$f(k-1) \leqslant \int_{k-1}^{k} f(x)\,dx \leqslant f(k).$$

Hence

$$\sum_{k=1}^{n} k^m \ln k \; = \int_0^n x^m \ln x\,dx + O\left(n^m \ln n\right) = \frac{n^{m+1}\ln n}{m+1} - \frac{n^{m+1}}{(m+1)^2} + O\left(n^m \ln n\right),$$

$$\sum_{k=1}^{n} k^m = \int_0^n x^m dx + O\left(n^m\right) = \frac{n^{m+1}}{m+1} + O\left(n^m\right),$$

$$\sum_{k=1}^{n} k^{m-1} = O\left(n^m\right)$$

and

$$S_m(n) = \frac{\ln n}{m+1} - \frac{1}{(m+1)^2} + O\left(\frac{\ln n}{n}\right) + \frac{\gamma}{m+1} + O\left(\frac{1}{n}\right) + O\left(\frac{1}{n}\right) =$$

$$= \frac{\ln n}{m+1} + \frac{\gamma}{m+1} - \frac{1}{(m+1)^2} + O\left(\frac{\ln n}{n}\right),$$

as n tends to infinity. A comparison then gives

$$_m = \frac{1}{m+1}, \;\; \mu_m = \frac{\gamma}{m+1} - \frac{1}{(m+1)^2}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Interchanging the order of summation,

$$\begin{aligned}
\sum_{k=1}^{n} k^m H_k &= \sum_{k=1}^{n} k^m \sum_{i=1}^{k} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} \sum_{k=i}^{n} k^m \\
&= \sum_{i=1}^{n} \frac{1}{i} \left(\frac{1}{m+1} n^{m+1} + O(n^m) - \frac{1}{m+1}(i-1)^{m+1} + O(i^m) \right) \\
&= \left(\frac{1}{m+1} n^{m+1} + O(n^m) \right) H_n - \frac{1}{m+1} \sum_{i=1}^{n} i^m + \sum_{i=1}^{n} O(i^{m-1}) \\
&= \left(\frac{1}{m+1} n^{m+1} + O(n^m) \right) H_n - \frac{1}{(m+1)^2} n^{m+1} + O(n^m),
\end{aligned}$$

so

$$\begin{aligned}
S_m(n) &= \frac{1}{m+1} H_n - \frac{1}{(m+1)^2} + O\left(\frac{H_n}{n}\right) \\
&= \frac{1}{m+1} \ln n + \frac{(m+1)\gamma - 1}{(m+1)^2} + O\left(\frac{H_n}{n}\right).
\end{aligned}$$

Now, let

$$\lambda_m = \frac{1}{m+1} \quad \text{and} \quad \mu_m = \frac{(m+1)\gamma - 1}{(m+1)^2};$$

then

$$\lim_{n\to\infty} (S_m(n) - \lambda_m \ln n - \mu_m) = 0.$$

Solution 3 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

It is known that (https://en.wikipedia.org/wiki/Harmonic_number)

$$H_k = \ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + O(1/k^4)$$

$$\frac{1}{n^{m+1}} \sum_{k=1}^{n} k^m H_k = \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^m \left(\ln k + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + O(1/k^4) \right).$$

I)

$$\begin{aligned}
&\frac{C_1 n^m}{n^{m+1}} \leqslant \frac{C_2}{n^{m+1}} \int_1^n x^{m-1}\, dx \leqslant \\
&\qquad \leqslant \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^{m-1} \leqslant \frac{C_3}{n^{m+1}} \int_1^n x^{m-1}\, dx \leqslant \frac{C_4 n^m}{n^{m+1}}.
\end{aligned}$$

C_i, $i = 1, 2, 3, 4$ are positive constants independent on n. The limits of the r.h.s and l.h.s. are equal to zero thus

$$\lim_{n\to\infty} \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^{m-1} = 0 \implies \lim_{n\to\infty} \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^{m-2} = \lim_{n\to\infty} \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^{m} O(\frac{1}{k^4}) = 0.$$

II)

Cesąro–Stolz yields

$$\lim_{n\to\infty} \frac{1}{n^{m+1}} \sum_{k=1}^{n} k^{m} = \lim_{n\to\infty} \frac{(n+1)^m}{(m+1)^{m+1} - n^{m+1}} = \lim_{n\to\infty} \frac{1}{n} \left(1 + \frac{1}{n}\right)^m \frac{1}{\left(1+\frac{1}{n}\right)^{m+1} - 1} =$$

$$= \frac{1}{m+1}.$$

By I) and II) we are left with proving

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} (k^m \ln k) - \lambda_m n^{m+1} \ln n}{n^{m+1}} - \mu_m + \frac{\gamma}{m+1} = 0$$

provided that λ_m and μ_m are suitably chosen. By applying the Cesąro–Stolz theorem again we come to

$$\lim_{n\to\infty} \frac{(n+1)^m \ln(n+1) - \lambda_m((n+1)^{m+1} \ln(n+1) - n^{m+1} \ln n)}{(n+1)^{m+1} - n^{m+1}} = \mu_m - \frac{\gamma}{m+1}$$

$$(n+1)^{m+1} - n^{m+1} = (m+1)n^m + O(n^{m-1})$$

III)

$$(n+1)^m \ln(n+1) = (n+1)^m (\ln n + \ln(1 + \frac{1}{n})) = (n+1)^m (\ln n + \frac{1}{n} + O(\frac{1}{n^2})).$$

$$\frac{(m+1)^m \ln(n+1)}{(n+1)^{m+1} - n^{m+1}} = \frac{(n+1)^m \ln n}{(m+1)n^m(1 + O(1/n))} + \frac{(n+1)^m \frac{1}{n} + (n+1)^m O(1/n^2)}{(m+1)n^m(1+O(1/n))} =$$

$$= \frac{\ln n}{m+1} + o(1).$$

IV)

$$\begin{aligned}
&(n+1)^{m+1} \ln(n+1) - n^{m+1} \ln n = \\
&= (n+1)^{m+1}\big(\ln n + \ln(1+\frac{1}{n})\big) - n^{m+1} \ln n = \\
&= \big((n+1)^{m+1} - n^{m+1}\big) \ln n + (n+1)^{m+1} \ln(1 + \frac{1}{n}) = \\
&= ((n+1)^{m+1} - n^{m+1}) \ln n + \frac{(n+1)^{m+1}}{n} + O(n^{m-1}).
\end{aligned}$$

III) and IV) yield

$$\frac{(n+1)^m \ln(n+1) - \lambda_m((n+1)^{m+1}\ln(n+1) - n^{m+1}\ln n)}{(n+1)^{m+1} - n^{m+1}} =$$

$$= \frac{\ln n}{m+1} - \lambda_m \ln n - \lambda_m \frac{\frac{(n+1)^{m+1}}{n}}{(m+1)n^m(1+O(\frac{1}{n}))} + o(1)$$

and we must define λ_m, μ_m such that

$$\lim_{n\to\infty} \frac{\ln n}{m+1} - \lambda_m \ln n - \lambda_m \frac{\frac{(n+1)^{m+1}}{n}}{(m+1)n^m(1+O(\frac{1}{n}))} + o(1) = \mu_m - \frac{\gamma}{m+1}.$$

It follows

$$\lambda_m = \frac{1}{m+1}, \qquad \mu_m = \frac{-1}{(m+1)^2} + \frac{\gamma}{m+1}.$$

Also solved by Moti Levy, Rehovot, Israel and the problem proposer.

• **5695** *Proposed by Narendra Bhandari and Yogesh Joshi, Nepal.*

Prove that

$$\int_1^{\sqrt{2}} \frac{xdx}{(1+x)\sqrt{(2-x^2)(x^2-1)}} = \frac{\pi}{2}\left(\frac{\Gamma^4\left(\frac{1}{4}\right) - 8\pi^2}{(2\pi)^{3/2}\Gamma^2\left(\frac{1}{4}\right)}\right)$$

where $\Gamma(z) = \displaystyle\int_0^\infty x^{z-1}e^{-x}dx$ is Gamma function where the real part $\Re(z)$ of the complex number z is positive.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

With the substitution $x = \sqrt{1+u^2}$,

$$\begin{aligned}
\int_1^{\sqrt{2}} & \frac{x\,dx}{(1+x)\sqrt{(2-x^2)(x^2-1)}} \\
&= \int_0^1 \frac{du}{(1+\sqrt{1+u^2})\sqrt{1-u^2}} \\
&= \int_0^1 \frac{\sqrt{1+u^2}-1}{u^2\sqrt{1-u^2}}\,du \\
&= \int_0^1 \left(\frac{1+u^2}{u^2\sqrt{1-u^4}} - \frac{1+u^4}{u^2\sqrt{1-u^4}} + \frac{1+u^4}{u^2\sqrt{1-u^4}} - \frac{1}{u^2\sqrt{1-u^2}} \right) du \\
&= \int_0^1 \left(\frac{1-u^2}{\sqrt{1-u^4}} + \frac{1+u^4}{u^2\sqrt{1-u^4}} - \frac{1}{u^2\sqrt{1-u^2}} \right) du.
\end{aligned}$$

Now, with the substitution $u = \sin\theta$,

$$\int \frac{1}{u^2\sqrt{1-u^2}}\,du = \int \csc^2\theta\,d\theta = -\cot\theta + C = -\frac{\sqrt{1-u^2}}{u} + C,$$

and with the substitution $u^2 = \sin\theta$,

$$\int \frac{1+u^4}{u^2\sqrt{1-u^4}}\,du = \int \frac{1+\sin^2\theta}{2\sin^{3/2}\theta}\,d\theta = -\frac{\cos\theta}{\sqrt{\sin\theta}} + C = -\frac{\sqrt{1-u^4}}{u} + C,$$

so

$$\begin{aligned}
\int_0^1 \left(\frac{1+u^4}{u^2\sqrt{1-u^4}} - \frac{1}{u^2\sqrt{1-u^2}} \right) du &= \left. \frac{\sqrt{1-u^2} - \sqrt{1-u^4}}{u} \right|_0^1 \\
&= 0 - \lim_{u\to 0^+} \frac{\sqrt{1-u^2} - \sqrt{1-u^4}}{u} = 0.
\end{aligned}$$

Additionally,

$$\begin{aligned}
\int_0^1 \frac{1-u^2}{\sqrt{1-u^4}}\,du &= \int_0^1 \frac{1}{\sqrt{1-u^4}}\,du - \int_0^1 \frac{u^2}{\sqrt{1-u^4}}\,du \\
&= \frac{1}{4}\int_0^1 \frac{x^{-3/4}}{\sqrt{1-x}}\,dx - \frac{1}{4}\frac{x^{-1/4}}{\sqrt{1-x}}\,dx \\
&= \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) - \frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right) \\
&= \frac{1}{4}\left(\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} - \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)} \right),
\end{aligned}$$

where $B(x,y)$ denotes the beta function. Finally, with

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{5}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right), \quad \text{and} \quad \Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)},$$

it follows that

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{x\,dx}{(1+x)\sqrt{(2-x^2)(x^2-1)}} &= \frac{\sqrt{\pi}}{4}\left(\frac{\Gamma^2\left(\frac{1}{4}\right)}{\pi\sqrt{2}} - \frac{4\pi\sqrt{2}}{\Gamma^2\left(\frac{1}{4}\right)}\right)\\ &= \frac{\pi}{2}\left(\frac{\Gamma^4\left(\frac{1}{4}\right)-8\pi^2}{(2\pi)^{3/2}\Gamma^2\left(\frac{1}{4}\right)}\right).\end{aligned}$$

Solution 2 by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.

Denoting the integral by I. Let $x^2 = \cos^2 t + 2\sin^2 t$. Then

$$\begin{aligned}x\,dx &= \sin t\cos t\,dt\\ 1+x &= 1+\sqrt{1+\sin^2 t}\\ (2-x^2)(x^2-1) &= (\sin t\cos t)^2,\end{aligned}$$

while the limits of integration are mapped from $(1, \sqrt{2})$ to $(0, \frac{\pi}{2})$. Under such a substitution the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{dt}{1+\sqrt{1+\sin^2 t}}.$$

Enforcing a substitution of $t = \dfrac{\pi}{2} - x$ yields

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{1+\cos^2 x}} = -\int_0^{\frac{\pi}{2}} \frac{1-\sqrt{1+\cos^2 x}}{\cos^2 x}\,dx. \tag{2}$$

The integrand will now be rewritten. Here

$$\begin{aligned}\frac{1-\sqrt{1+\cos^2 x}}{\cos^2 x} &= \frac{1-\sqrt{1+\cos^2 x}}{\cos^2 x} - \sqrt{1+\cos^2 x} + \sqrt{1+\cos^2 x}\\ &= \frac{1-\sqrt{1+\cos^2 x}}{\cos^2 x} - \frac{1+\cos^2 x}{\sqrt{1+\cos^2 x}} + \sqrt{1+\cos^2 x}\\ &= \frac{1-\sqrt{1+\cos^2 x}}{\cos^2 x} + \left(\frac{1-\cos^2 x}{\sqrt{1+\cos^2 x}} - \frac{2}{\sqrt{1+\cos^2 x}}\right) + \sqrt{1+\cos^2 x}\\ &= \frac{1}{\cos^2 x} - \frac{\sqrt{1+\cos^2 x}}{\cos^2 x} - \frac{2}{\sqrt{1+\cos^2 x}} + \frac{\sin^2 x}{\sqrt{1+\cos^2 x}} + \sqrt{1+\cos^2 x}\\ &= \sec^2 x - \frac{2}{\sqrt{1+\cos^2 x}} + \sqrt{1+\cos^2 x} - \frac{\sqrt{1+\cos^2 x}}{\cos^2 x} + \frac{\sin^2 x}{\sqrt{1+\cos^2 x}}\\ &= \sec^2 x - \frac{2}{\sqrt{1+\cos^2 x}} + \sqrt{1+\cos^2 x} + \frac{\sin^2 x\cos^2 x - 1 - \cos^2 x}{\cos^2 x\sqrt{1+\cos^2 x}}.\end{aligned}$$

Also, since

$$\frac{d}{dx}\left(\frac{-1}{\sqrt{2}}\tan x\sqrt{\cos^2 x+3}\right) = \frac{\sin^2 x\cos^2 x - 1 - \cos^2 x}{\cos^2 x\sqrt{1+\cos^2 x}},$$

the integrand can be expressed as

$$\frac{1-\sqrt{1+\cos^2 x}}{\cos^2 x} = \frac{d}{dx}\left(\tan x - \frac{1}{\sqrt{2}}\tan x\sqrt{\cos^2 x+3}\right) - \frac{2}{\sqrt{1+\cos^2 x}} + \sqrt{1+\cos^2 x}.$$

The integral in (2) can therefore be written as

$$\begin{aligned} I = &-\int_0^{\frac{\pi}{2}} \frac{d}{dx}\left(\tan x - \frac{1}{\sqrt{2}}\tan x\sqrt{\cos^2 x+3}\right) dx + \int_0^{\frac{\pi}{2}} \frac{2}{\sqrt{1+\cos^2 x}}\,dx \\ &- \int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 x}\,dx = -I_1 + I_2 - I_3. \end{aligned} \tag{3}$$

For the first of the integrals

$$\begin{aligned} I_1 &= \lim_{a\to\frac{\pi}{2}} \left[\tan x - \frac{1}{\sqrt{2}}\tan x\sqrt{\cos^2 x+3}\right]_0^a \\ &= \lim_{a\to\frac{\pi}{2}} \left(\tan a - \frac{1}{\sqrt{2}}\tan a\sqrt{\cos^2 a+3}\right) \\ &= \frac{1}{\sqrt{2}}\lim_{a\to\frac{\pi}{2}} \frac{\sqrt{2}-\sqrt{\cos^2 a+3}}{\cot a} = 0. \end{aligned}$$

Here the limit has been evaluated using de l'Hôpital's rule.

For the second of the integrals, enforcing a substitution of $u = \cos x$ produces

$$I_2 = 2\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1+u^2}} = 2\int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

Letting $t = u^4$ gives

$$I_2 = \frac{1}{2}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}\,dt = \frac{1}{2}\,\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}.$$

Here $\mathrm{B}(\cdot,\cdot)$ denotes the beta function. Now $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$\Gamma\left(\frac{3}{4}\right) = \Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\Gamma(\frac{1}{4})\sin(\frac{\pi}{4})} = \frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})},$$

where Euler's reflexion formula $\Gamma(1-z)\Gamma(z) = \dfrac{\pi}{\sin(\pi z)}$ has been used. Thus

$$I_2 = \frac{1}{2}\cdot\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)\cdot\frac{\Gamma(\frac{1}{4})}{\pi\sqrt{2}} = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{2\pi}}.$$

And for the third of the integrals, enforcing a substitution of $u = \cos x$ produces

$$I_3 = \int_0^1 \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{1+x^2}{\sqrt{1-x^4}}\,dx$$
$$= \int_0^1 \frac{dx}{\sqrt{1-x^4}} + \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx = J_1 + J_2.$$

For the first of these integrals we see that $J_1 = \dfrac{1}{2}I_2$. For the second of the integrals, letting $t = u^4$ produces

$$J_2 = \frac{1}{4}\int_0^1 t^{-\frac{1}{4}}(1-t)^{-\frac{1}{2}}\,dt = \frac{1}{4}\,\mathrm{B}\left(\frac{3}{4},\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{4})}.$$

And since

$$\Gamma\left(\frac{5}{4}\right) = \Gamma\left(1+\frac{1}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) \quad \text{and} \quad \Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})},$$

we have

$$J_2 = \frac{\sqrt{\pi}}{4}\cdot\frac{\pi\sqrt{2}}{\Gamma(\frac{1}{4})}\cdot\frac{4}{\Gamma(\frac{1}{4})} = \frac{\pi\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})}.$$

Thus

$$I_3 = J_1 + J_2 = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{2\pi}} + \frac{\pi\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})}.$$

Returning to the integral I in (3) we see that

$$I = -I_1 + I_2 - I_3 = \frac{\Gamma^2(\frac{1}{4})}{2\sqrt{2\pi}} - \left(\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{2\pi}} + \frac{\pi\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})}\right) = \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{2\pi}} - \frac{\pi\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})},$$

or

$$I = \int_1^{\sqrt{2}} \frac{x\,dx}{(1+x)\sqrt{(2-x^2)(x^2-1)}} = \frac{\pi}{2}\left(\frac{\Gamma^4\left(\frac{1}{4}\right) - 8\pi^2}{(2\pi)^{3/2}\Gamma^2\left(\frac{1}{4}\right)}\right),$$

as required to prove.

Solution 3 by Moti Levy, Rehovot, Israel.

Let

$$I := \int_1^{\sqrt{2}} \frac{x}{(1+x)\sqrt{(2-x^2)(x^2-1)}}dx,$$

then

$$I = I_1 - I_2,$$

where

$$I_1 := \int_1^{\sqrt{2}} \frac{1}{\sqrt{(2-x^2)(x^2-1)}} dx,$$
$$I_2 := \int_1^{\sqrt{2}} \frac{1}{(1+x)\sqrt{(2-x^2)(x^2-1)}} dx.$$

By changing the integration variable $t = \left(x^2-1\right)^2$, we obtain

$$I_1 = \frac{1}{4}\int_0^1 t^{-\frac{3}{4}}(1-t)^{-\frac{1}{2}}\ dt = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)},$$

where $B(z_1,z_2)$ is the Beta function defined by $B(z_1,z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\ dt$. By changing the integration variable $t = x^2-1$, we obtain

$$I_2 := \int_1^{\sqrt{2}} \frac{1}{(1+x)\sqrt{(2-x^2)(x^2-1)}} dx = \int_0^1 \frac{1}{2\sqrt{t}\left(\sqrt{t+1}+1\right)\sqrt{1-t^2}}\, dt$$

$$I_2 = \int_0^1 \frac{1}{2\sqrt{t}\left(\sqrt{t+1}+1\right)\sqrt{1-t^2}}\, dt = \frac{1}{2}\int_0^1 \left(t^{-\frac{3}{2}}(1-t)^{-\frac{1}{2}} - t^{-\frac{3}{2}}\left(1-t^2\right)^{-\frac{1}{2}}\right) dt.$$

Now we calculate the antiderivatives of $t^{-\frac{3}{2}}(1-t)^{-\frac{1}{2}}$ and of $t^{-\frac{3}{2}}\left(1-t^2\right)^{-\frac{1}{2}}$. The first antiderivative is

$$\int t^{-\frac{3}{2}}(1-t)^{-\frac{1}{2}}\ dt = -\frac{2\sqrt{1-t}}{\sqrt{t}}.$$

For the second antiderivative we use the power series of $\left(1-t^2\right)^{-\frac{1}{2}}$,

$$\begin{aligned}
\left(1-t^2\right)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}t^{2n} = \sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}t^{2n}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-n\right)n!}t^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{\pi}\,(2n)!}{(-4)^n\,n!\,\sqrt{\pi}n!}t^{2n}\\
&= \sum_{n=0}^{\infty}\frac{(2n)!}{4^n\,(n!)^2}t^{2n}.
\end{aligned}$$

$$\begin{aligned}
\int t^{-\frac{3}{2}}\left(1-t^2\right)^{-\frac{1}{2}}\ dt &= \int t^{-\frac{3}{2}}\sum_{n=0}^{\infty}\frac{(2n)!}{4^n\,(n!)^2}t^{2n}\,dt\\
&= \sum_{n=0}^{\infty}\frac{(2n)!}{4^n\,(n!)^2}\int t^{2n-\frac{3}{2}}dt = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n\,(n!)^2\left(2n-\frac{1}{2}\right)}t^{2n-\frac{1}{2}}.
\end{aligned}$$

$$\frac{1}{2}\int\left(t^{-\frac{3}{2}}(1-t)^{-\frac{1}{2}}-t^{-\frac{3}{2}}\left(1-t^2\right)^{-\frac{1}{2}}\right)dt=-\frac{\sqrt{1-t}}{2\sqrt{t}}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2\left(2n-\frac{1}{2}\right)}t^{2n-\frac{1}{2}}$$
$$=\frac{1-\sqrt{1-t}}{\sqrt{t}}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n)!}{4^n(n!)^2\left(2n-\frac{1}{2}\right)}t^{2n-\frac{1}{2}}$$

$$I_2=\frac{1}{2}\int_0^1\left(t^{-\frac{3}{2}}(1-t)^{-\frac{1}{2}}-t^{-\frac{3}{2}}\left(1-t^2\right)^{-\frac{1}{2}}\right)dt=1-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n)!}{4^n(n!)^2\left(2n-\frac{1}{2}\right)}=-\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(4n-1)}.$$

It is quite straightforward to show that

$$-\sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(4n-1)}=2\text{F}1\left(\frac{1}{2},-\frac{1}{4};\frac{3}{4};1\right),$$

where $2\text{F}1(a,b;c;z)$ is the Hypergeometric function.

The Gauss summation formula is

$$2\text{F}1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

hence for our case,

$$I_2=2\text{F}1\left(\frac{1}{2},-\frac{1}{4};\frac{3}{4};1\right)=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)}=\frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right).$$

It follows that

$$I=\frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right)-\frac{1}{4}B\left(\frac{3}{4},\frac{1}{2}\right).$$

Now we can reformulate the expression for I by using the following identities:

$$\Gamma\left(\frac{5}{4}\right)=\frac{1}{4}\Gamma\left(\frac{1}{4}\right),$$
$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\sqrt{2}\pi,$$
$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

We conclude that

$$I=\frac{1}{4}\frac{\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{2}\sqrt{\pi}}-\frac{\sqrt{2}\pi^{\frac{3}{2}}}{\Gamma^2\left(\frac{1}{4}\right)}.$$

Also solved by the problem proposer.

• **5696** *Proposed by Mohsen Soltanifar, University of Toronto, Toronto, Canada.*

The sequence $(A_n)_{n=1}^{\infty}$ of subsets of the set $\mathbb{R}$ of real numbers is said to be convergent if and only if the two sets

$$B_1 := \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m \quad \text{and} \quad B_2 := \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

are identical. Otherwise, we say the sequence is divergent. For each of the following cases, construct as complicated and fanciful a divergent sequence $(A_n)_{n=1}^{\infty}$ as you can muster while using no more than an aggregate of 42 individual symbols (characters):

1. B_1 is empty and B_2 is non-empty.
2. B_1 is bounded and B_2 is unbounded.
3. B_1 is a singleton set and B_2 is a non-singleton set.
4. B_1 is finite and B_2 is infinite.
5. B_1 is countable and B_2 is uncountable.

Editor's Note: This problem was unusual in that it required a non-mathematical — and, at the same time, creative — component to its solution. No solutions (other than that provided by the problem proposer) were received.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣