Problems and Solutions

Albert Natian, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at the Department of Mathematics, Los Angeles Valley College, CA. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before June 1, 2023.

• **5721** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Triangle △ABC has angles α, β, γ (expressed all in radians), inradius r and circumradius R. Prove that

$$\left(\frac{\sin\alpha}{\alpha}\right)^{\frac{3}{2}} + \left(\frac{\sin\beta}{\beta}\right)^{\frac{3}{2}} + \left(\frac{\sin\gamma}{\gamma}\right)^{\frac{3}{2}} \geqslant 2 + \frac{r}{2R}.$$

• **5722** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Let p_n be the n-th prime number. Prove the following inequality

$$p_{n+1} < 3p_{\lfloor \frac{n}{2} \rfloor + 1} \quad \text{for} \quad n \geqslant 1$$

where $\lfloor . \rfloor$ denotes the integer part function.

Hint: Use the Rosser-Schoenfeld inequalities $p_n < n\log n + n\log\log n - \dfrac{1}{2}n$ for $n \geqslant 20$ and $p_n > n\log n$ for $n \geqslant 1$ along with a small table of primes.

• **5723** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.*

For real x, solve the equation $\left(\sqrt[5]{x+2} - \sqrt[5]{2x+1} - \sqrt[5]{4x+7}\right)^5 = 3x + 8.$

• **5724** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$\int_0^{\infty} \frac{\sqrt{x}\ln^2 x}{x^4 + x^2 + 1} dx.$$

• **5725** *Proposed by Narendra Bhandari, Bajura District, Nepal.*

Prove

$$\sum_{n=1}^{\infty} \frac{n \cdot 4^n}{(2n-1)^2(4n+5)} \left[\frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} \right] = \frac{1559\sqrt{2} - 1216}{58800}.$$

• **5726** *Proposed by Toyesh Prakash Sharma (Student) Agra College, Agra, India.*

Calculate

$$I := \int_0^1 \int_0^1 \int_0^1 \frac{\sqrt{x^3y^2z}\, dxdydz}{x\left(\sqrt{y} + \sqrt{z}\right) + y\left(\sqrt{z} + \sqrt{x}\right) + z\left(\sqrt{x} + \sqrt{y}\right)}.$$

Solutions

To Formerly Published Problems

• **5697** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve the following equation in real numbers:

$$\left(x^2 - 6x + 5\right)^5 + \left(x^2 - 9x + 14\right)^5 - \left(2x^2 - 15x + 19\right)^5 = 0.$$

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$\because \quad a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

let $a = 2x^2 + 5x + 19,\ b = x^2 - 9x + 14,\ c = x^2 - 6x + 5$

then the equation can also be expressed by $c^5 = c(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

divided by $c \qquad (x \neq 1,\ x \neq 5)$

$c^4 = a^4 + a^3b + a^2b^2 + ab^3 + b^4$

$c^4 - b^4 = (c^2 + b^2)(c - b)(c + b) = a^4 + a^3 b + a^2 b^2 + ab^3$

$= (c^2 + b^2)(3x - 9)(2x^2 - 15x + 19) = a(a^3 + a^2 b + ab^2 + b^3)$

divided by a $\quad (x \neq \dfrac{15 \pm \sqrt{73}}{4})$

$(c^2 + b^2)(3x - 9) = (a + b)(a^2 + b^2) = (a^2 + b^2)(3x^2 - 24x + 33)$

i.e. $(c^2 + b^2)(x - 3) = (a^2 + b^2)(x^2 - 8x + 11)$

$c^2(x - 3) + b^2(x - 3) = a^2(x^2 - 8x + 11) + b^2(x^2 - 8x + 11)$

i.e. $b^2(x^2-9x+14) = c^2(x-3) - a^2(x^2-8x+11) = (x^2-9x+14)\left(-4x^4 + 57x^3 - 271x^2 + 483x - 289\right)$

divided by b $\quad (x \neq 2,\ x \neq 7)$

$(x^2 - 9x + 14)^2 + 4x^4 - 57x^3 + 271x^2 - 483x + 289 = 0$

i.e. $5x^4 - 75x^3 + 380x^2 - 735x + 485 = 0$ **no real roots**

$\therefore \quad x = 1,\ 5,\ 2,\ 7,\ \dfrac{15 \pm \sqrt{73}}{4}$

Solution 2 by Trey Smith, Angelo State University, San Angelo, TX.

Let $A = A(x) = x^2 - 6x + 5$, $B = B(x) = x^2 - 9x + 14$, and $C = C(x) = 2x^2 - 15x + 19$. Then $A + B = C$. So

$$(x^2 - 6x + 5)^5 + (x^2 - 9x + 14)^5 - (2x^2 - 15x + 19)^5 = 0$$

$$\implies A^5 + B^5 = (A + B)^5$$

$$\implies A^5 + B^5 = A^5 + 5A^4 B + 10A^3 B^2 + 10A^2 B^3 + 5AB^4 + B^5$$

$$\implies 0 = 5A^4 B + 10A^3 B^2 + 10A^2 B^3 + 5AB^4$$

$$\implies 0 = AB(A^3 + 2A^2 B + 2AB^2 + B^3)$$

$$\implies 0 = AB(A + B)(A^2 + AB + B^2)$$

$$\implies 0 = ABC(A^2 + AB + B^2).$$

The above equation is satisfied when $A = 0,\ B = 0,\ C = 0$, or $A^2 + AB + B^2 = 0$. Hence, six of the ten possible roots are simply 1, 5, 2, 7, and $(15 \pm \sqrt{73})/4$.
If $A^2 + AB + B^2 = 0$, we have that

$$A = \frac{-B \pm \sqrt{B^2 - 4B^2}}{2}$$

$$\implies A = B\left(\frac{-1 \pm i\sqrt{3}}{2}\right).$$

If $x = r$ is a real solution not given above, then $A(r) \neq 0$ is real. Also $B(r) \neq 0$ is real which means that $B(r)((-1 + i\sqrt{3})/2)$ is complex. Thus $A(r)$ is complex. This is, of course, a contradiction. So there are no additional real solutions.

Solution 3 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

Let $P = x^2 - 6x + 5$ and $Q = x^2 - 9x + 14$. Observe that we wish to solve the equation $P^5 + Q^5 - (P + Q)^5 = 0$. Right away, we can factor

$$\begin{aligned}
P^5 + Q^5 - (P + Q)^5 &= (P + Q)(P^4 - P^3Q + P^2Q^2 - PQ^3 + Q^4) - (P + Q)^5 \\
&= (P + Q)(P^4 - P^3Q + P^2Q^2 - PQ^3 + Q^4 - (P + Q)^4) \\
&= (P + Q)(P^4 - P^3Q + P^2Q^2 - PQ^3 + Q^4 - (P^4 + 4P^3Q + 6P^2Q^2 + 4PQ^3 + Q^4)) \\
&= (P + Q)(-5P^3Q - 5P^2Q^2 - 5PQ^3) \\
&= -5PQ(P + Q)(P^2 + PQ + Q^2).
\end{aligned}$$

We now find the roots of each factor:

$$\begin{aligned}
P = x^2 - 6x + 5 = (x - 1)(x - 5) = 0 &\Rightarrow x = 1 \text{ or } x = 5, \\
Q = x^2 - 9x + 14 = (x - 2)(x - 7) = 0 &\Rightarrow x = 2 \text{ or } x = 7, \\
P + Q = 2x^2 - 15 + 19 = 0 &\Rightarrow x = \frac{15 \pm \sqrt{(-15)^2 - 4(2)(19)}}{2(2)} = \frac{15 \pm \sqrt{73}}{4}.
\end{aligned}$$

As for $P^2 + PQ + Q^2$, completing the square reveals that

$$P^2 + PQ + Q^2 = \left(P + \frac{1}{2}Q\right)^2 + \frac{3}{4}Q^2 \geqslant 0$$

for real P and Q. Thus, this quadratic form has real roots only when $P + \frac{1}{2}Q = 0$ and $Q = 0$ simultaneously. Of course, Q equalling 0 would then force $P = 0$. Since P and Q do not share any roots, this last factor does not contribute any new real roots. It follows that the real solutions to the original equation are

$$x = 1, x = \frac{15}{4} - \frac{1}{4}\sqrt{73}, x = 2, x = 5, x = \frac{15}{4} + \frac{1}{4}\sqrt{73}, x = 7.$$

Solution 4 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

The real solutions are $1, 2, 5, 7$, and $\dfrac{15 \pm \sqrt{73}}{4}$. If we let $A = x^2 - 6x + 5 = (x-1)(x-5)$ and $B = x^2 - 9x + 14 = (x-2)(x-7)$, then the equation has the form $A^5 + B^5 - (A+B)^5 = 0$; the left side of which can be factored as $-5AB(A+B)(A^2 + AB + B^2) = 0$. Thus, any solution will have $A = 0$, $B = 0$, $A + B = 0$, or $A^2 + AB + B^2 = 0$. From the factorizations above, $A = 0$ if and only if $x = 1$ or $x = 5$; while $B = 0$ if and only if $x = 2$ or $x = 7$. From the quadratic formula, $A + B = 0$ if and only if $x = \dfrac{15 \pm \sqrt{73}}{4}$.

Suppose $x \in \mathbb{R} - \{1, 2, 5, 7\}$. Then $A = (x-1)(x-5)$ and $B = (x-2)(x-7)$ are both nonzero. If A and B have the same sign, then $AB > 0$ and $A^2 + AB + B^2 > 0$. If A and B have opposite signs, then $2AB < AB < 0$, so that

$$A^2 + AB + B^2 > A^2 + 2AB + B^2 = (A+B)^2 \geqslant 0.$$

Thus, $A^2 + AB + B^2 > 0$ for all $x \in \mathbb{R} - \{1, 2, 5, 7\}$, and the only real solutions to the original equation are $1, 2, 5, 7$, and $\dfrac{15 \pm \sqrt{73}}{4}$.

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

We first check if the given equation has integer roots. Fermat's last theorem (for the exponent 5) states that if $a^5+b^5-c^5 = 0$ in integers a, b, c then either a=0 or b=0 or c=0. Hence either $x^2 - 6x + 5 = 0$ or $x^2 - 9x + 14 = 0$ or $2x^2 - 15x + 19 = 0$. The first equation has the roots 1 and 5, the second the roots 2 and 7 and the third the roots $\frac{1}{4}\left(15 \pm \sqrt{73}\right)$, and it is easily verified that all 6 roots are roots of the given equation taking into account that

$$\left(x^2 - 6x + 5\right) + \left(x^2 - 9x + 14\right) = 2x^2 - 15x + 19$$

and a^5+b^5 is divisible by a+b. Hence we have the factorization

$$\left(x^2 - 6x + 5\right)^5 + \left(x^2 - 9x + 14\right)^5 - \left(2x^2 - 15x + 19\right)^5 =$$

$$= -15\,(x-1)\,(x-2)\,(x-5)\,(x-7)\left(2x^2 - 15x + 19\right)\left(x^4 - 15x^3 + 76x^2 - 147x + 97\right).$$

We note that

$$x^4 - 15x^3 + 76x^2 - 147x + 97 = \left(x - \frac{5}{3}\right)^2\left(x - \frac{35}{6}\right)^2 + \frac{99}{324}\left(x - \frac{189}{99}\right)^2 + \frac{1216}{891} > 0.$$

So the only real roots of the given equation are $\left\{1, 2, 5, 7, \frac{1}{4}\left(15 \pm \sqrt{73}\right)\right\}$.

Solution 6 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By doing $a = x^2 - 6x + 5$, and $b = x^2 - 9x + 14$, the proposed equation reads as $a^5 + b^5 - (a+b)^5 = 0$, Since $a^5 + b^5 - (a+b)^5 = -5ab(a+b)(a^2 + ab + b^2)$, it follows that $a = 0$, $b = 0$ and $a + b = 0$ are solutions, that is $x^2 - 6x + 5 = 0$, $x^2 - 9x + 14 = 0$, and $2x^2 - 16x + 9 = 0$, which imply $x = 1, 5, 2, 7, 4 - \sqrt{23/2}, 4 + \sqrt{23/2}$. Finally, notice that $a^2 + ab + b^2 = \frac{1}{2}\left((a+b)^2 + a^2 + b^2\right) \geqslant 0$, with value 0 if and only if $a = b = 0$. This implies that the given equation does not have more real solutions.

Solution 7 by Brian D. Beasley, Presbyterian College, Clinton, SC.

For each real number x, we let $f = x^2 - 6x + 5$ and $g = x^2 - 9x + 14$. Then $f + g = 2x^2 - 15x + 19$, so the given equation becomes

$$f^5 + g^5 - (f + g)^5 = 0,$$

or equivalently

$$-5f^4g - 10f^3g^2 - 10f^2g^3 - 5fg^4 = 0.$$

Thus we obtain

$$-5fg(f + g)(f^2 + fg + g^2) = 0,$$

where $f^2 + fg + g^2 > 0$ unless $f = g = 0$. Hence the solutions of the equation occur when $f = 0$ or $g = 0$ or $f + g = 0$. This yields the solution set

$$\left\{1, 5, 2, 7, \frac{15 \pm \sqrt{73}}{4}\right\}.$$

Solution 8 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let

$$u = x^2 - 6x + 5 \quad \text{and} \quad v = x^2 - 9x + 14.$$

Then

$$u + v = 2x^2 - 15x + 19,$$

and

$$\begin{aligned}
(x^2 - 6x + 5)^5 + (x^2 - 9x + 14)^5 - (2x^2 - 15x + 19)^5 \\
&= u^5 + v^5 - (u + v)^5 \\
&= -5u^4v - 10u^3v^2 - 10u^2v^3 - 5uv^4 \\
&= -5uv(u^3 + 2u^2v + 2uv^2 + v^3) \\
&= -5uv(u + v)(u^2 + uv + v^2) \\
&= -5uv(u + v)\left[\left(u + \frac{1}{2}v\right)^2 + \frac{3}{4}v^2\right].
\end{aligned}$$

This last expression is equal to zero when

- $u = 0$, which occurs when $x = 1$ or $x = 5$;
- $v = 0$, which occurs when $x = 2$ or $x = 7$;
- $u + v = 0$, which occurs when $x = \dfrac{15 \pm \sqrt{73}}{4}$.

The equation

$$(x^2 - 6x + 5)^5 + (x^2 - 9x + 14)^5 - (2x^2 - 15x + 19)^5 = 0$$

therefore has six solutions:

$$x = 1, \quad \frac{15 - \sqrt{73}}{4}, \quad 2, \quad 5, \quad \frac{15 + \sqrt{73}}{4}, \quad 7.$$

Solution 9 by David A. Huckaby, Angelo State University, San Angelo, TX.

Letting $a = x^2 - 6x + 5$ and $b = x^2 - 9x + 14$, the given equation is

$$a^5 + b^5 - (a + b)^5 = 0.$$

That $a = 0$, $b = 0$, and $a = -b$ are solutions of this equation is clear by inspection. These three solutions can also be obtained by expanding $(a + b)^5$ and simplifying, which yields the equation $a^4b + 2a^3b^2 + 2a^2b^3 + ab^4 = 0$. Factoring out ab gives $ab(a^3 + 2a^2b + 2ab^2 + b^3) = 0$. The third factor has a familiar factorization, $(a^3 + 2a^2b + 2ab^2 + b^3) = (a + b)(a^2 + ab + b^2)$. So

$$ab(a + b)(a^2 + ab + b^2) = 0, \qquad (1)$$

from which it is again easy to see that $a = 0$, $b = 0$, and $a = -b$ are solutions.

Now $a = x^2 - 6x + 5 = (x - 1)(x - 5)$, and $b = x^2 - 9x + 14 = (x - 2)(x - 7)$. Since we seek only solutions for which x is real, a and b must both be real, since each is a product of real numbers. So from equation (1), the only possibilities for real solutions x are when $a = 0$, $b = 0$, or $a + b = 0$.

The condition $a = 0$ implies $x = 1$ or $x = 5$, whereas $b = 0$ implies $x = 2$ or $x = 7$. Now $a + b = 2x^2 - 15x + 19$. An application of the quadratic formula yields the factorization $2x^2 - 15x + 19 = 2\left(x - \dfrac{15 + \sqrt{73}}{4}\right)\left(x - \dfrac{15 - \sqrt{73}}{4}\right)$. So $a + b = 0$ implies $x = \dfrac{15 + \sqrt{73}}{4}$ or $x = \dfrac{15 - \sqrt{73}}{4}$.

So the six real solutions to the original equation are $x = 1$, $x = 2$, $x = 5$, $x = 7$, $x = \dfrac{15 + \sqrt{73}}{4}$, and $x = \dfrac{15 - \sqrt{73}}{4}$.

Solution 10 by G. C. Greubel, Newport News, VA .

The equation can be seen in the form

$$(x-1)^5(x-5)^5 + (x-2)^5(x-7)^5 = (2x^2 - 15x + 19)^5.$$

Note that the factors on the left-hand side are $\{1, 2, 5, 7\}$. Let $f(x) = 2x^2 - 15x + 19$ then $f(1) = 6$, $f(2) = -3$, $f(5) = -6$, $f(7) = 12$. It is quickly noticed that $x = \{1, 2, 5, 7\}$ are solutions. Now factoring $f(x)$ yields

$$(x-1)^5(x-5)^5 + (x-2)^5(x-7)^5 = \left(x - \frac{15+\sqrt{73}}{4}\right)^5 \left(x - \frac{15-\sqrt{73}}{4}\right)^5.$$

It is noticed that when $4x = 15 \pm \sqrt{73}$ the left-hand side results in a value of zero. It turns out that the real valued roots of the equation are the factors of the equation. This gives

$$x \in \left\{1, 2, 5, 7, \frac{15+\sqrt{73}}{4}, \frac{15-\sqrt{73}}{4}\right\}$$

are the real valued solutions.

Solution 11 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We show that there are only six real solutions:

$$x \;=\; 1,\; 2,\; 5,\; 7,\; (15+\sqrt{73})/4,\; (15-\sqrt{73})/4.$$

Let $a = x^2 - 6x + 5$ and $b = x^2 - 9x + 14$, so that the given equation becomes $a^5 + b^5 - (a+b)^5 = 0$. Then the binomial expansion yields

$$\begin{aligned}
a^5 + b^5 - (a+b)^5 &= a^5 + b^5 - \left(a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\right) \\
&= -\left(5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4\right) \\
&= -5ab\left(a^3 + 2a^2b + 2ab^2 + b^3\right) \\
&= -5ab(a+b)(a^2 + ab + b^2) \\
&= -15(x^2 - 6x + 5)(x^2 - 9x + 14)(2x^2 - 15x + 19) \cdot P(x),
\end{aligned}$$

where $P(x) = x^4 - 15x^3 + 76x^2 - 147x + 97 = \left(x^2 - \frac{15}{2}x + \frac{19}{2}\right)^2 + \frac{3}{4}(x-3)^2$ does not vanish for any real value of x. Solving each of the first three quadratic equations yields the six real roots listed above.

Solution 12 by Michel Bataille, Rouen, France.

Let $A = x^2 - 6x + 5$ and $B = x^2 - 9x + 14$. The equation is successively equivalent to

$$A^5 + B^5 - (A + B)^5 = 0$$

$$5AB(A^3 + B^3) + 10A^2B^2(A + B) = 0$$

$$5AB(A + B)(A^2 + AB + B^2) = 0.$$

The latter is equivalent to $A = 0$ or $B = 0$ or $A + B = 0$ or $A = B = 0$. The solutions of $A = 0$ are 1 and 5, of $B = 0$ are 2 and 7, of $A + B = 0$ are $\dfrac{15 + \sqrt{73}}{4}$ and $\dfrac{15 - \sqrt{73}}{4}$; $A = B = 0$ has no solution.

In conclusion, the solutions of the proposed equation are $1, 5, 2, 7, \dfrac{15 + \sqrt{73}}{4}, \dfrac{15 - \sqrt{73}}{4}$.

Solution 13 by Moti Levy, Rehovot, Israel.

Let $a := x^2 - 6x + 5,\;\; b := x^2 - 9x + 14$ then the equation can be rewritten as

$$a^5 + b^5 - (a + b)^5 = 0,$$

or

$$ab\,(a + b)\left(ab + a^2 + b^2\right) = 0.$$

It follows that the roots of the original polynomial are the same as the roots of the following four polynomials:

$$\begin{aligned}
a &= x^2 - 6x + 5 = (x - 1)(x - 5),\\
b &= x^2 - 9x + 14 = (x - 2)(x - 7),\\
a + b &= 2x^2 - 15x + 19 = 2\left(x - \left(\frac{1}{4}\sqrt{73} + \frac{15}{4}\right)\right)\left(x - \left(-\frac{1}{4}\sqrt{73} + \frac{15}{4}\right)\right),\\
ab + a^2 + b^2 &= 3\left(x^4 - 15x^3 + 76x^2 - 147x + 97\right).
\end{aligned}$$

Now we show, by Sturm's theorem, that the polynomial $x^4 - 15x^3 + 76x^2 - 147x + 97$ has no real roots:

$$\begin{aligned}
P_0 &= x^4 - 15x^3 + 76x^2 - 147x + 97\\
P_1 &= 4x^3 - 45x^2 + 152x - 147\\
P_2 &= -rem\,(P_0, P_1) = -\frac{67}{16}x^2 + \frac{129}{4}x - \frac{653}{16}\\
P_3 &= -rem\,(P_1, P_2) = -\frac{16\,608}{4489}x + \frac{38\,880}{4489}\\
P_4 &= -rem\,(P_2, P_3) = \frac{1405\,057}{119\,716}
\end{aligned}$$

The difference in sign variations is

$$V(-\infty) - V(+\infty) = 2 - 2 = 0,$$

hence the polynomial $x^4 - 15x^3 + 76x^2 - 147x + 97$ has no real roots. We conclude that the real roots of $\left(x^2 - 6x + 5\right)^5 + \left(x^2 - 9x + 14\right)^5 - \left(2x^2 - 15x + 19\right)^5$ are $\left\{1, 2, 5, 7, \frac{15 - \sqrt{73}}{4}, \frac{15 + \sqrt{73}}{4}\right\}$.

Solution 14 by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

We observe that

$$\begin{aligned} a^5 + b^5 &= (a+b)^5 - 5ab(a^3 + b^3) - 10a^2b^2(a+b) = \\ &= (a+b)^5 - 5ab((a+b)^3 - 3ab(a+b)) - 10a^2b^2(a+b) \end{aligned}$$

thus letting

$$x^2 - 6x + 5 \doteq a, \quad x^2 - 9x + 14 \doteq b, \quad 2x^2 - 15x + 19 = a + b$$

the equation $a^5 + b^5 = (a+b)^5$ hold true if and only if

$$-5ab((a+b)^3 - 3ab(a+b)) - 10a^2b^2(a+b) = 0 \iff (a+b)ab(-5(a+b)^2 + 15ab - 10a^2b^2) = 0$$

$a = 0$ implies $x_1 = 1$ $x_2 = 5$

$b = 0$ implies $x_3 = 2$, $x_4 = 7$

$a + b = 0$ implies $x_5 = \frac{15 + \sqrt{73}}{4}$, $x_6 = \frac{15 - \sqrt{73}}{4}$

The equation $-5(a+b)^2 + 15ab - 10a^2b^2 = 0$ does not have solutions.

$$-5(a+b)^2 + 15ab - 10a^2b^2 = 0 \iff 5a^2 + 5b^2 - 5ab + 10a^2b^2 = 0$$

$$5a^2 + 5b^2 - 5ab + 10a^2b^2 \geqslant 2\sqrt{50}|ab|\sqrt{a^2 + b^2} - 5ab$$

but $\sqrt{a^2 + b^2} > 1$ thus

$$5a^2 + 5b^2 - 5ab + 10a^2b^2 \geqslant 2\sqrt{50}|ab| - 5ab > 0$$

unless $ab = 0$ but the solutions of the equation $ab = 0$ have already been found. The last step is to show $a^2 + b^2 > 1$.

$$a^2 + b^2 = 2x^4 - 30x^3 + 155x^2 - 312x + 221 \doteq f(x) \quad f'(x) = 2(x-4)(4x^2 - 29x + 39) = 0$$

hence $f(x)$ ha two minima at $x_1 = \frac{29 - \sqrt{217}}{9}$ and $x_2 = \frac{29 + \sqrt{217}}{9}$.

The absolute minimum is attained at x_1 and its value is circa 7.6 so concluding the proof.

Solution 15 by Péter Fülöp, Gyömrő, Hungary.

If $a = (x^2 - 6x + 5)$ and $b = (x^2 - 9x + 14)$ then the equation can be rewritten as follows:

$$a^5 + b^5 - (a + b)^5 = 0$$

from which the trivial solutions can be read: $a = -b$ and $a = b = 0$

That is
$x^2 - 6x + 5 = 0, \rightarrow \qquad x_1 = 5, \qquad x_2 = 1,$

$x^2 - 9x + 14 = 0, \rightarrow \qquad x_3 = 7, \qquad x_4 = 2,$

$2x^2 - 15x + 19 = 0, \rightarrow \qquad x_5 = \dfrac{15 + \sqrt{73}}{4}, \qquad x_6\dfrac{15 - \sqrt{73}}{4},$

We have to check whether the remaining four roots are reals or complexes.
Let's calculate $(a + b)^5$ and put it back to the equation expressed by a, b.

$$a^5 + b^5 - (a^5 + \binom{5}{1}ab^4 + \binom{5}{2}a^2b^3 + + \binom{5}{3}a^3b^2 + \binom{5}{4}a^4b^1 + b^5) = 0$$

After further simplification of the last equality we get:

$$-5ab(b^3 + 2ab^2 + 2a^2b + a^3) = 0$$

$$-5ab(a + b)\left(a^2 + b^2 + ab\right) = 0$$

We can see the so called trivial roots again and the two (four regarding x) missing roots they are complexes.

$$a_{1,2} = \frac{b}{2}(-1 \pm i\sqrt{3})$$

Using a and b expressions we have:

$$2x^2 + x(-15 \pm \sqrt{3}i) + 19 \mp 3\sqrt{3}i = 0$$

As a result, we get four complex roots.
Summarize we have the following real roots:

$$1;2;5;7; \frac{15 \pm \sqrt{73}}{4}$$

Solution 16 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Direct calculation confirms that the left-hand side of the equation can be represented in the form

$$-15\,(x-1)\,(x-2)\,(x-5)\,(x-7)\,p\,(x)\,q\,(x)\,,$$

where

$$p\,(x) = 2x^2 - 15x + 19 = 2\left(x - \frac{15-\sqrt{73}}{4}\right)\left(x - \frac{15+\sqrt{73}}{4}\right)$$

and

$$q\,(x) = x^4 - 15x^3 + 76x^2 - 147x + 97.$$

We are going to show, that q has no real zeros. We have

$$q\left(x+\frac{15}{4}\right) = x^4 - \frac{67}{8}x^2 + \frac{9}{8}x + \frac{5437}{256} = \left(x^2 - \frac{67}{16}\right)^2 + \frac{9}{8}x + \frac{237}{64} > 0,$$

if $(9/8)\,x > -237/64$, i.e., if $x > -79/24 =: x_0 \approx -3.29167$. For $x \leqslant x_0$, we obviously have $x < -3$ and, therefore, $x^2 - 67/16 > 9 - 5 = 4$. Hence,

$$\frac{d}{dx}q\left(x+\frac{15}{4}\right) = 4x\left(x^2 - \frac{67}{16}\right) + \frac{9}{8} < 4\cdot(-3)\cdot 4 + 2 = -46 < 0,$$

such that $q\left(x+\frac{15}{4}\right)$ is strictly decreasing on $(-\infty, x_0]$. Consequently, $q\left(x+\frac{15}{4}\right) \geqslant q\left(x_0+\frac{15}{4}\right) = \left(x_0^2 - \frac{67}{16}\right)^2 > 0$, for $x \leqslant x_0$. Combining both estimates we see that $q\,(x) > 0$ on the whole real axis. Therefore, the polynomial q has no real zeros. Summarizing, the equation has exactly 6 real solutions:

$$x \in \left\{1, 2, 5, 7, \left(15-\sqrt{73}\right)/4, \left(15+\sqrt{73}\right)/4\right\}$$

Also solved by Zaur Rajabov, ADA University, Baku, Azerbaijan; Adil Allahveranov, ADA University, Baku, Azerbaijan; Bruno Salgueiro Fanego, Viveiro, Spain; Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia; and the proposer.

• **5698** *Proposed by Florică Anastase,"Alexandru Odobescu" high school, Lehliu-Gară, Călăraşi, Romania.*

Prove

$$\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}dx \geqslant \log(1+\sqrt{2})\log(e\sqrt{2}).$$

Solution 1 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

Make the trigonometric substitution $x = e\tan\theta$ so that $dx = e\sec^2\theta\,d\theta$, $\sqrt{x^2+e^2} = e|\sec\theta|$, and $0 \leqslant x \leqslant e$ if $0 \leqslant \theta \leqslant \pi/4$. Since secant is positive over this interval,

$$\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}\,dx = \int_0^{\pi/4} \frac{\log(e+e\tan\theta)}{e\sec\theta}\cdot e\sec^2\theta\,d\theta = \int_0^{\pi/4} \log\left(\frac{e(\cos\theta+\sin\theta)}{\cos\theta}\right)\sec\theta\,d\theta.$$

Due to the harmonic addition formula

$$a\cos\theta + b\sin\theta = \operatorname{sgn}(a)\sqrt{a^2+b^2}\cos\left(\tan^{-1}\left(\frac{b}{a}\right)-\theta\right),$$

the integral can be further transformed into

$$\int_0^{\pi/4} \sec\theta\log\left(\frac{e\sqrt{2}\cos(\pi/4-\theta)}{\cos\theta}\right)d\theta = \int_{-\pi/8}^{\pi/8} \sec(\pi/8+\theta)\log\left(\frac{e\sqrt{2}\cos(\pi/8-\theta)}{\cos(\pi/8+\theta)}\right)d\theta.$$

Note that

$$\int_{-\pi/8}^{0} \sec(\pi/8+\theta)\log\left(\frac{e\sqrt{2}\cos(\pi/8-\theta)}{\cos(\pi/8+\theta)}\right)d\theta = -\int_0^{\pi/8} \sec(\pi/8-\theta)\log\left(\frac{e\sqrt{2}\cos(\pi/8-\theta)}{\cos(\pi/8+\theta)}\right)d\theta.$$

Thus, the original integral can be written as

$$\int_0^{\pi/8} \big(\sec(\pi/8+\theta)-\sec(\pi/8-\theta)\big)\log\left(\frac{e\sqrt{2}\cos(\pi/8-\theta)}{\cos(\pi/8+\theta)}\right)d\theta.$$

Since cosine is a decreasing function over $[0,\pi/4]$, we see that $\cos(\pi/8-\theta) \geqslant \cos(\pi/8+\theta)$ for $0 \leqslant \theta \leqslant \pi/8$, and so the above integral can be bounded below by

$$\int_0^{\pi/8} \big(\sec(\pi/8+\theta)-\sec(\pi/8-\theta)\big)\log(e\sqrt{2})\,d\theta$$

$$= \log\left(\frac{\tan(\pi/8+\theta)+\sec(\pi/8+\theta)}{\tan(\pi/8-\theta)+\sec(\pi/8-\theta)}\right)\Bigg|_{\theta=0}^{\pi/8}\cdot\log(e\sqrt{2})$$

$$= \log(1+\sqrt{2})\log(e\sqrt{2}),$$

as desired.

Solution 2 by G. C. Greubel, Newport News, VA.

Consider the integral in the form
$$I = \int_0^a \frac{\ln(x+a)}{\sqrt{x^2+a^2}}\,dx$$
which leads to
$$\begin{aligned}
I &= \int_0^a \frac{\ln(x+a)}{\sqrt{x^2+a^2}}\,dx \\
&= \ln(a)\int_0^a \frac{dx}{\sqrt{x^2+a^2}} + \int_0^a \frac{\ln\left(1+\frac{x}{a}\right)}{a\sqrt{1+\left(\frac{x}{a}\right)^2}}\,dx \\
&= \ln(a)\,\coth^{-1}(\sqrt{2}) + \int_0^1 \frac{\ln(1+u)\,du}{\sqrt{1+u^2}} \qquad x = a\,u \\
&= \ln(a)\,\coth^{-1}(\sqrt{2}) + J.
\end{aligned}$$

The integral J is evaluated as follows:
$$\begin{aligned}
J &= \int_0^1 \frac{\ln(1+u)\,du}{\sqrt{1+u^2}} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}\int_0^1 u^{2n}\,\ln(1+u)\,du \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}\left(\frac{\ln(2)}{2n+1} - \phi(-1,1,2n+2)\right) \\
&= \ln(2)\,\sinh^{-1}(1) - \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}\phi(-1,1,2n+2) \\
&\geqslant \ln(2)\,\sinh^{-1}(1) - \frac{1}{2}\ln(2)\,\sinh^{-1}(1) \\
&\geqslant \ln(\sqrt{2})\,\sinh^{-1}(1),
\end{aligned}$$

where $\phi(s,a,x)$ is the Hurwitz Zeta function. Now,
$$\begin{aligned}
I &\geqslant \ln(a)\,\coth^{-1}(\sqrt{2}) + \ln(\sqrt{2})\,\sinh^{-1}(1) \\
&\geqslant (\ln(a) + \ln(\sqrt{2}))\,\sinh^{-1}(1) \\
&\geqslant \ln(a\sqrt{2})\,\ln(1+\sqrt{2}).
\end{aligned}$$

This gives the result
$$\int_0^a \frac{\ln(x+a)}{\sqrt{x^2+a^2}}\,dx \geqslant \ln(a\sqrt{2})\,\ln(1+\sqrt{2}).$$
where $a = e$ the desired result is obtained.

Solution 3 by Moti Levy, Rehovot, Israel.

$$\int_0^e \frac{\ln(e+x)}{\sqrt{x^2+e^2}}dx = \int_0^1 \frac{\ln(x+1)+1}{\sqrt{x^2+1}}dx = \int_0^1 \frac{\ln(x+1)}{\sqrt{x^2+1}}\,dx + \log\left(1+\sqrt{2}\right).$$

It follows that the original inequality is equivalent to

$$\int_0^1 \frac{\ln(x+1)}{\sqrt{x^2+1}}dx \geqslant \frac{1}{2}\log\left(1+\sqrt{2}\right)\ln(2).$$

We have the following inequality

$$\ln(x+1) \geqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}, \quad \text{for } x \in (0,1),$$

hence

$$\int_0^1 \frac{\ln(x+1)}{\sqrt{x^2+1}}\,dx \geqslant \int_0^1 \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}{\sqrt{x^2+1}}\,dx.$$

The following definite integrals are quite straightforward to evaluate:

$$\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{2}-1,$$

$$\int_0^1 \frac{x^2}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln\left(\sqrt{2}+1\right),$$

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx = \frac{2}{3} - \frac{1}{3}\sqrt{2},$$

$$\int_0^1 \frac{x^4}{\sqrt{x^2+1}}\,dx = \frac{3}{8}\ln\left(\sqrt{2}+1\right) - \frac{1}{8}\sqrt{2}.$$

Then

$$\int_0^1 \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}}{\sqrt{x^2+1}}\,dx$$
$$= \frac{5}{32}\ln\left(\sqrt{2}+1\right) + \frac{193}{288}\sqrt{2} - \frac{7}{9} \cong 0.30766 > \frac{1}{2}\log\left(1+\sqrt{2}\right)\ln(2) \cong 0.30546,$$

which completes the proof.

Solution 4 by Perfetti Paolo, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

I need the following result

$$f(x) \doteq \ln(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) \geqslant 0, \quad 0 \leqslant x \leqslant 1$$

The proof is straightforward.

$$f'(x) = \frac{1}{1+x} - 1 + x - x^2 + x^3, \quad f''(x) = \frac{-1}{(1+x)^2} + 1 - 2x + 3x^2,$$
$$f'''(x) = \frac{2}{(1+x)^3} - x + 6x, \quad f^{(iv)}(x) = \frac{-6}{(1+x)^4} + 6, \quad f^{(v)}(x) = \frac{24}{(1+x)^5} > 0$$

and $f'(0) = f''(0) = f'''(0) = f^{(iv)}(0) = 0$. It follows $f(x) > 0$ for any $x > 0$.

$$\int_0^e \frac{\ln(e+x)}{\sqrt{x^2+e}}dx \underbrace{=}_{x=et} \int_0^1 \frac{1+\ln(1+t)}{e\sqrt{t^2+1}}e\,dt \geqslant \ln(1+\sqrt{2}) + \frac{\ln 2}{2}\ln(1+\sqrt{2})$$

$$\int_0^1 \frac{1}{\sqrt{t^2+1}}dt = \ln(t+\sqrt{t^2+1})|_{t=1} = \ln(1+\sqrt{2})$$

thus we are left with proving

$$\int_0^1 \frac{\ln(1+t)}{\sqrt{t^2+1}}dt \geqslant \frac{\ln 2}{2}\ln(1+\sqrt{2})$$

$$\int_0^1 \frac{\ln(1+t)}{\sqrt{t^2+1}}dt \geqslant \int_0^1 \frac{1}{\sqrt{t^2+1}}(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4})dt \tag{1}$$

$$\int_0^1 \frac{t}{\sqrt{t^2+1}}dt = (1+t^2)^{\frac{1}{2}}\Big|_0^1 = \sqrt{2} - 1$$

$$\int_0^1 \frac{t^2}{\sqrt{t^2+1}}dt = t(1+t^2)^{\frac{1}{2}}\Big|_0^1 - \int_0^1 (1+t^2)^{\frac{1}{2}}dt = \sqrt{2} - \int_0^1 \frac{1+t^2}{(1+t^2)^{\frac{1}{2}}}dt$$

whence

$$\int_0^1 \frac{t^2}{\sqrt{t^2+1}}dt = \frac{\sqrt{2}}{2} - \frac{1}{2}\int_0^1 \frac{1}{(1+t^2)^{\frac{1}{2}}}dt = \frac{\sqrt{2}}{2} - \frac{\ln(1+\sqrt{2})}{2}$$

$$\int_0^1 \frac{t^3}{\sqrt{t^2+1}}dt = t^2(1+t^2)^{\frac{1}{2}}\Big|_0^1 - 2\int_0^1 t(1+t^2)^{\frac{1}{2}}dt = \sqrt{2} - 2\int_0^1 \frac{t^3+t}{\sqrt{1+t^2}}dt$$

whence

$$\int_0^1 \frac{t^3}{\sqrt{t^2+1}}dt = \frac{\sqrt{2}}{3} - \frac{2}{3}(\sqrt{2}-1) = \frac{2}{3} - \frac{\sqrt{2}}{3}$$

$$\int_0^1 \frac{t^4}{\sqrt{t^2+1}}dt = t^3(1+t^2)^{\frac{1}{2}}\Big|_0^1 - 3\int_0^1 t^2(1+t^2)^{\frac{1}{2}}dt = \sqrt{2} - 3\int_0^1 \frac{t^4+t^2}{\sqrt{1+t^2}}dt$$

whence

$$\int_0^1 \frac{t^4}{\sqrt{t^2+1}}dt = \frac{\sqrt{2}}{4} - \frac{3}{4}\left(\frac{\sqrt{2}}{2} - \frac{\ln(1+\sqrt{2})}{2}\right) = \frac{3}{8}\ln(1+\sqrt{2}) - \frac{\sqrt{3}}{8}$$

It follows that the r.h.s. of (1) is equal to $\dfrac{193\sqrt{2}}{288} - \dfrac{7}{9} + \dfrac{5\ln(1+\sqrt{2})}{32}$ and

$$\frac{193\sqrt{2}}{288} - \frac{7}{9} + \frac{5\ln(1+\sqrt{2})}{32} - \frac{\ln 2 \ln(1+\sqrt{2})}{2} \geqslant 0.002$$

so concluding the proof.

Solution 5 by Péter Fülöp, Gyömrő, Hungary.

Perform the substitution in the integral of LHS: $x = et$ we get:

$$LHS = \int_0^1 \frac{1+\ln(1+t)}{\sqrt{t^2+1}}dt = \underbrace{\int_0^1 \frac{1}{\sqrt{t^2+1}}}_{\sinh^{-1}(1)=\ln(1+\sqrt{2})} dt + \underbrace{\int_0^1 \frac{\ln(t+1)}{\sqrt{t^2+1}}dt}_{I_{L1}}$$

Let's denote the second integral by I_{L1} and integrate it by parts.

$$u' = \frac{1}{\sqrt{1+t^2}} \text{ and } v = \ln(1+t),$$
then
$$u = \sinh^{-1}(t) \text{ and } v' = \frac{1}{t+1}$$

$$I_{L1} = \Big[\sinh^{-1}(t)\ln(t+1)\Big]_0^1 - \int_0^1 \frac{\sinh^{-1}(t)}{t+1}dt$$

Regarding LHS we get:

$$LHS = \ln(1+\sqrt{2}) + \ln(1+\sqrt{2})2\ln\sqrt{2} - \int_0^1 \frac{\sinh^{-1}(t)}{t+1}dt =$$
$$\ln(1+\sqrt{2})(1+\ln\sqrt{2}) + \ln(1+\sqrt{2})\ln\sqrt{2} - \underbrace{\int_0^1 \frac{\sinh^{-1}(t)}{t+1}dt}_{I_{L2}}$$

Let's do the $t = \sinh(z)$ substitution in the integral I_{L2}:

$$I_{L2} = \int_0^{\ln(1+\sqrt{2})} \frac{z\cosh(z)}{1+\sinh(z)}dz$$

Integration by parts of I_{L2} we get:

$$u = z \text{ and } v' = \frac{\cosh(z)}{1+\sinh(z)},$$
then
$$u' = 1 \text{ and } v = \ln(1+\sinh(z))$$

$$I_{L2} = \Big[z\ln(1+\sinh(z))\Big]_0^{\ln(1+\sqrt{2})} - \int\limits_0^{\ln(1+\sqrt{2})} \ln(1+\sinh(z))dz =$$

$$\ln(1+\sqrt{2})\ln 2 - \int\limits_0^{\ln(1+\sqrt{2})} \ln(1+\sinh(z))dz$$

Regarding the unequality we get:

$$\ln(1+\sqrt{2})(1+\ln\sqrt{2}) + \ln(1+\sqrt{2})\ln\sqrt{2} - \ln(1+\sqrt{2})\ln 2 +$$
$$\int\limits_0^{\ln(1+\sqrt{2})} \ln(1+\sinh(z))dz \geqslant \ln(1+\sqrt{2})(1+\ln\sqrt{2})$$

After the cancellations:

$$\int\limits_0^{\ln(1+\sqrt{2})} \ln(1+\sinh(z))dz > 0$$

Since $\ln(1+\sinh(z))$ function is positive in the range $[0, \ln(1+\sqrt{2})]$ then the integral of it will also be positive. But it won't be zero.

which completes the proof.

Solution 6 by Toyesh Prakash Sharma, Agra College, Agra, India.

As
$$\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}\text{dx} = \int_0^e \frac{\sqrt{x^2+e^2}}{x^2+e^2}\log(e+x)\,\text{dx}$$
Using Weighted Chebyshev's Integral Inequality

$$\int_0^e \frac{1}{x^2+e^2}\text{dx}\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}\text{dx} \geqslant \int_0^e \frac{\sqrt{x^2+e^2}}{x^2+e^2}\text{dx}\int_0^e \frac{\log(e+x)}{x^2+e^2}\text{dx}$$

$$\left[\frac{1}{e}\arctan\frac{x}{e}\right]_0^e \int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}dx \geqslant \left[\log\left(x+\sqrt{x^2+e^2}\right)\right]_0^e \int_0^e \frac{\log(e+x)}{x^2+e^2}\text{dx}$$

$$\Rightarrow \frac{\pi}{4e}\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}dx \geqslant \log\left(1+\sqrt{2}\right)\int_0^e \frac{\log(e+x)}{x^2+e^2}\text{dx}$$

Let $I=\int_0^e \frac{\log(e+x)}{x^2+e^2}\text{dx}$. Then

$$I=\int_0^e \frac{\log(e+x)}{x^2+e^2}\text{dx}=\int_0^e \frac{\log e+\log\left(1+\frac{x}{e}\right)}{x^2+e^2}\text{dx}$$

$$=\int_0^e \frac{\log e}{x^2+e^2}\text{dx}+\frac{1}{e^2}\int_0^e \frac{\log\left(1+\frac{x}{e}\right)}{1+\left(\frac{x}{e}\right)^2}\text{dx}=\log e\left[\frac{1}{e}\arctan\frac{x}{e}\right]_0^e+\frac{1}{e}\int_0^1 \frac{\log(1+t)}{1+t^2}dt$$

$$=\frac{\pi}{4e}\log e+\frac{1}{e}\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)\,d\theta=\frac{\pi}{4e}\log e+\frac{1}{e}\frac{\pi}{4}\log\sqrt{2}=\frac{\pi}{4e}\log\left(e\sqrt{2}\right)$$

Now, using this

$$\Rightarrow \frac{\pi}{4e}\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}dx\geqslant\log\left(1+\sqrt{2}\right)\frac{\pi}{4e}\log\left(e\sqrt{2}\right)$$

$$\int_0^e \frac{\log(e+x)}{\sqrt{x^2+e^2}}dx\geqslant\log\left(1+\sqrt{2}\right)\log\left(e\sqrt{2}\right)$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

• **5699** *Proposed by Narendra Bhandari, Bajura, Nepal.*

Prove

$$\sum_{n=1}^{\infty}\binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{64^n(2n-1)^2}=\frac{\sinh^{-1}(1)}{8\pi}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We note that

$$\sinh^{-1}(1)\ =\log\left(1+\sqrt{2}\right).$$

We have

$$\binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{(2n-1)^2}=\frac{2\,(4n-4)!}{(n-1)!\,(n-1)!\,(2n-1)!}=\frac{1}{(2n-1)}\binom{4n-4}{2n-2}\binom{2n-2}{n-1}.$$

By Stirling's asymptotic formula,

$$\frac{(4n)!}{n!n!\,(2n+1)!64^n}=O\left(\frac{(4n)^{4n+\frac{1}{2}}e^{-4n}}{n^{n+\frac{1}{2}}e^{-n}n^{n+\frac{1}{2}}e^{-n}n(2n)^{2n+\frac{1}{2}}e^{-2n}64^n}\right)=O\left(\frac{1}{n^2}\right).$$

So the given sum converges absolutely. We have

$$\frac{1}{\sqrt{1-z}}=\sum_{n=0}^{\infty}\frac{1}{4^n}\binom{2n}{n}z^n,\quad |z|<1,$$

which implies

$$\frac{1}{2\sqrt{1-z}}+\frac{1}{2\sqrt{1+z}}=\sum_{n=0}^{\infty}\frac{1}{16^n}\binom{4n}{2n}z^{2n},\quad |z|<1,$$

and

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,4^n}\binom{2n}{n}z^{2n}=\frac{1}{z}\int_0^z\frac{1}{\sqrt{1-t^2}}dt=\frac{\arcsin(z)}{z},\quad |z|<1.$$

Therefore, by Parseval's theorem (see for instance https://en.wikipedia.org/wiki/Parseval%27s_theorem),

$$S:=\sum_{n=1}^{\infty}\binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{64^n(2n-1)^2}=\frac{1}{32}\sum_{n=1}^{\infty}\frac{1}{64^n\,(2n+1)}\binom{4n}{2n}\binom{2n}{n}=$$

$$=\frac{1}{32}\sum_{n=1}^{\infty}\frac{1}{16^n}\binom{4n}{2n}\bullet\frac{1}{(2n+1)\,4^n}\binom{2n}{n}=\frac{1}{64\pi}\int_0^{\pi}\left(\frac{1}{\sqrt{1-e^{-ix}}}+\frac{1}{\sqrt{1+e^{-ix}}}\right)\frac{\arcsin\left(e^{ix}\right)}{e^{ix}}dx=$$

$$=\frac{1}{64\pi}\int_C\left(\frac{1}{\sqrt{1-\frac{1}{z}}}+\frac{1}{\sqrt{1+\frac{1}{z}}}\right)\frac{\arcsin(z)}{z}\frac{dz}{iz},$$

where C denotes the halfcircle with center 0 and radius 1 that starts at z=1 and that is run through in the positive direction. In this complex integral we take the main branch of the square root, defined by

$$\sqrt{Re^{it}}=\sqrt{R}e^{\frac{it}{2}},\quad -\pi\leqslant t<\pi.$$

We deform the halfcircle to the segment [-1,1]. Then, by Cauchy's theorem and taking into account that $\frac{\arcsin(z)}{z}$ is an analytic function in |z|<1,

$$S=-\frac{1}{64\pi}\int_{-1}^{1}\left(\frac{1}{\sqrt{1-\frac{1}{z}}}+\frac{1}{\sqrt{1+\frac{1}{z}}}\right)\frac{\arcsin(z)}{z}\frac{dz}{iz}=$$

$$=-\frac{1}{64i\pi}\int_{-1}^{0}\left(\frac{1}{\sqrt{1-\frac{1}{z}}}+\frac{i}{\sqrt{-1-\frac{1}{z}}}\right)\frac{\arcsin(z)}{z}\frac{dz}{z}-\frac{1}{64i\pi}\int_0^1\left(\frac{-i}{\sqrt{-1+\frac{1}{z}}}+\frac{1}{\sqrt{1+\frac{1}{z}}}\right)\frac{\arcsin(z)}{z}\frac{dz}{z}=$$

$$=\frac{1}{64i\pi}\int_0^1\left(\frac{1}{\sqrt{1+\frac{1}{z}}}+\frac{i}{\sqrt{-1+\frac{1}{z}}}\right)\frac{\arcsin(z)}{z}\frac{dz}{z}-\frac{1}{64i\pi}\int_0^1\left(\frac{-i}{\sqrt{-1+\frac{1}{z}}}+\frac{1}{\sqrt{1+\frac{1}{z}}}\right)\frac{\arcsin(z)}{z}\frac{dz}{z}=$$

$$= \frac{1}{32\pi} \int_0^1 \frac{1}{\sqrt{z(1-z)}} \frac{\arcsin(z)}{z} dz.$$

We note that

$$\int \frac{1}{z\sqrt{z(1-z)}} dz = -\frac{2\sqrt{1-z}}{\sqrt{z}} + C.$$

Integration by parts therefore gives

$$S = \frac{1}{32\pi} \int_0^1 \frac{1}{\sqrt{z(1-z)}} \frac{\arcsin(z)}{z} dz = \frac{1}{32\pi} \int_0^1 \frac{2\sqrt{1-z}}{\sqrt{z}} \frac{1}{\sqrt{1-z^2}} dz = \frac{1}{16\pi} \int_0^1 \frac{1}{\sqrt{z(1+z)}} dz =$$

$$= \frac{1}{8\pi} \log\left(\sqrt{1+z} + \sqrt{z}\right) \Big|_{z=0}^{z=1} = \frac{1}{8\pi} \log\left(1 + \sqrt{2}\right).$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Starting from the generating function for the central binomial coefficients,

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

replace x by $x/4$, then replace x by x^2, transpose the $n = 0$ term to the right side, and divide by x^2 to obtain

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{2n-2}}{4} = \frac{1}{x^2\sqrt{1-x^2}} - \frac{1}{x^2}.$$

Now, integrate both sides from 0 to x:

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{2n-1}}{4^n(2n-1)} = \frac{1-\sqrt{1-x^2}}{x};$$

divide by x and again integrate from 0 to x:

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{x^{2n-1}}{4^n(2n-1)^2} = \frac{\sqrt{1-x^2} + x\sin^{-1} x - 1}{x};$$

multiply by x and differentiate:

$$\sum_{n=1}^{\infty} \frac{n}{4^n(2n-1)^2} \binom{2n}{n} x^{2n-1} = \frac{\sin^{-1} x}{2};$$

and replace x by $\sin^2\theta$ and divide by $\sin^2\theta$:

$$\sum_{n=1}^{\infty} \frac{n}{4^n(2n-1)^2} \binom{2n}{n} \sin^{4n-4}\theta = \frac{\sin^{-1}(\sin^2\theta)}{2\sin^2\theta}.$$

Integration from $\theta = 0$ to $\theta = \pi/2$ and using Wallis' formula,

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2}\binom{2n}{n}\frac{1}{4^n},$$

yields

$$\sum_{n=1}^{\infty} \frac{n}{4^n(2n-1)^2}\binom{2n}{n} \cdot \frac{\pi}{2}\binom{4n-4}{2n-2}\frac{1}{4^{2n-2}} = \int_0^{\pi/2} \frac{\sin^{-1}(\sin^2\theta)}{2\sin^2\theta}\, d\theta,$$

or

$$\sum_{n=1}^{\infty} \binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{64^n(2n-1)^2} = \frac{1}{16\pi}\int_0^{\pi/2} \frac{\sin^{-1}(\sin^2\theta)}{\sin^2\theta}\, d\theta.$$

Integration by parts applied to the integral on the right side followed by the substitution $u = \sin\theta$ yields

$$\begin{aligned}
\int_0^{\pi/2} \frac{\sin^{-1}(\sin^2\theta)}{\sin^2\theta}\, d\theta &= 2\int_0^{\pi/2} \frac{\cos^2\theta}{\sqrt{1-\sin^4\theta}}\, d\theta \\
&= 2\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{1+\sin^2\theta}}\, d\theta = 2\int_0^1 \frac{du}{\sqrt{1+u^2}} \\
&= 2\sinh^{-1}(1).
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{64^n(2n-1)^2} = \frac{\sinh^{-1}(1)}{8\pi}.$$

For more infinite series similar to this, see the article by the proposer of this problem: "Infinite Series Associated with the Ratio and Product of Central Binomial Coefficients," *Journal of Integer Sequences*, volume 25 (2022), 22.6.5.

Solution 3 by Moti Levy, Rehovot, Israel.

$$S := \sum_{n=1}^{\infty} \binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{(2n-1)^2}\frac{1}{64^n} = \frac{1}{32}\sum_{n=0}^{\infty}\binom{2n}{n}\binom{4n}{2n}\frac{1}{(2n+1)}\frac{1}{64^n} \tag{2}$$

Evaluation of combinatorial sum by Egorychev's method:

$$\binom{4n}{2n} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{(1+z)^{4n}}{z^{2n+1}} dz. \tag{3}$$

Plugging (3) into (2) and changing the order of summation and integration, we get

$$S = \frac{1}{64\pi i}\oint \sum_{n=0}^{\infty}\left(\frac{(1+z)^{4n}}{z^{2n+1}}\binom{2n}{n}\frac{1}{(2n+1)}\frac{1}{64^n}\right) dz. \tag{4}$$

The Taylor series of arcsin (x) is

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n(2n+1)} x^{2n+1}, \tag{5}$$

hence

$$\sum_{n=0}^{\infty} \left(\frac{(1+z)^{4n}}{z^{2n+1}} \binom{2n}{n} \frac{1}{(2n+1)} \frac{1}{64^n} \right) = 4 \frac{\arcsin\left(\frac{(1+z)^2}{4z}\right)}{(1+z)^2}. \tag{6}$$

Plugging (6) into (4) , we obtain

$$\begin{aligned}
S &= \frac{1}{16\pi i} \oint\limits_{|z|=1} \frac{\arcsin\left(\frac{(1+z)^2}{4z}\right)}{(1+z)^2} dz = \frac{1}{16\pi} \int_0^{2\pi} \frac{\arcsin\left(\frac{\left(1+e^{it}\right)^2}{4e^{it}}\right)}{\left(1+e^{it}\right)^2} e^{it} dt \\
&= \frac{1}{16\pi} \int_{\frac{1}{4}}^{\infty} \frac{\arcsin\left(\frac{1}{4u}\right)}{\sqrt{1-4u}} du = \frac{1}{32\pi} \int_1^{\infty} \frac{\arcsin\frac{1}{v}}{\sqrt{v-1}} dv = \frac{1}{16\pi} \int_1^{\infty} \frac{1}{v\sqrt{v+1}} dv \\
&= \frac{1}{16\pi} \int_0^1 \frac{1}{\sqrt{w(w+1)}} dw = \frac{1}{8\pi} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx \\
&= \frac{\sinh^{-1}(1)}{8\pi} = \frac{\ln\left(1+\sqrt{2}\right)}{8\pi} \cong 0.0035069.
\end{aligned}$$

Solution 4 by Perfetti Paolo, Universitą di "Tor Vergata", Roma, Italy.

$$\int_0^{\pi/2} (\sin x)^{2n} dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \binom{2n}{n} \frac{n}{(2n-1)^2} x^n = \sqrt{x} \arcsin(2\sqrt{x})$$

yields

$$\binom{4n-4}{2n-2} = \frac{2^{4n-3}}{\pi} \int_0^{\pi/2} (\sin x)^{4n-4} dx$$

whence (use the positivity of each term to invert the integral with the series)

$$\begin{aligned}
&\int_0^{\pi/2} \left[\sum_{n=1}^{\infty} \binom{2n}{n} \frac{n}{64^n(2n-1)^2} \frac{2^{4n}}{8\pi} (\sin x)^{4n-4} \right] dx = \\
&\frac{1}{8\pi} \int_0^{\pi/2} \frac{1}{(\sin x)^4} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{n}{(2n-1)^2} \left(\frac{(\sin x)^4}{4} \right)^n dx = \\
&= \frac{1}{8\pi} \int_0^{\pi/2} \frac{1}{(\sin x)^4} \frac{(\sin x)^2}{2} \arcsin((\sin x)^2) dx = \frac{1}{16\pi} \int_0^{\pi/2} \frac{\arcsin((\sin x)^2)}{(\sin x)^2} dx
\end{aligned}$$

$x = \arctan t$ yields

$$\frac{1}{16\pi}\int_0^\infty \frac{1}{t^2}\arcsin\frac{t^2}{1+t^2}dt \underbrace{=}_{y=1/t} \frac{1}{16\pi}\int_0^\infty \arcsin\frac{1}{1+y^2}dy$$

Now let's integrate by parts

$$\frac{1}{16\pi}y\arcsin\frac{1}{1+y^2}\Big|_0^\infty + \frac{1}{16\pi}\int_0^\infty \frac{2y^2\,dy}{(1+y^2)\sqrt{2+y^2}} = \frac{1}{16\pi}\int_0^\infty \frac{2y^2\,dy}{(1+y^2)\sqrt{2+y^2}}$$

$u = \sqrt{y^2+2}$ yields

$$\frac{1}{16\pi}2\int_{\sqrt{2}}^\infty \frac{du}{u^2-1} = \frac{1}{16\pi}\int_{\sqrt{2}}^\infty \left[\frac{1}{u-1} - \frac{1}{u+1}\right]du = \frac{1}{16\pi}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{\ln(1+\sqrt{2})^2}{16\pi} = \frac{(\sinh)^{-1}(1)}{8\pi}$$

Solution 5 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\because \quad \int_0^{\frac{\pi}{2}} \sin^{2n}(t)\mathrm{d}t = \frac{\pi}{2^{2n+1}}\binom{2n}{n}, \qquad \binom{4n-4}{2n-2} = \frac{2^{4n-3}}{\pi}\int_0^{\frac{\pi}{2}} \sin^{4n-4}(t)\mathrm{d}t$$

$$\therefore \quad S = \sum_{n=1}^{\infty}\binom{2n}{n}\binom{4n-4}{2n-2}\frac{n}{64^n(2n-1)^2} = \sum_{n=0}^{\infty}\frac{16^n}{8\pi}\binom{2n}{n}\frac{n}{64^n(2n-2)^2}\int_0^{\frac{\pi}{2}} \sin^{4n-4}(t)\mathrm{d}t$$

$$\because \quad \frac{1}{(2n-1)^2} = -\int_0^1 \ln(x)x^{2n-2}\mathrm{d}x$$

$$\therefore \quad S = \frac{1}{8\pi}\int_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^{n-1}(x^2)^{n-1}\ln(x)n\int_0^{\frac{\pi}{2}}(\sin^4 t)^{n-1}\mathrm{d}t\mathrm{d}x$$

in which, let $y = -x^2\sin^4 t$

$$\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x^2\sin^4 t)^{n-1}\times n = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}y^{n-1}\times n = \left[\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}y^n\right]'$$

$$= \left[(1+y)^{-\frac{1}{2}}\right]' = \left[\frac{1}{\sqrt{1+y}}\right]' = \frac{-1}{2(1+y)^{\frac{3}{2}}}$$

$$\therefore \quad S = \frac{1}{8\pi}\int_0^1 \ln x\frac{-1}{2(1-x^2\sin^4 t)^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\mathrm{d}t\mathrm{d}x$$

Now prove $\displaystyle\int_0^1\int_0^{\frac{\pi}{2}}\mathrm{d}t\mathrm{d}x\frac{\ln x}{(1-x^2\sin^4 t)^{\frac{3}{2}}} = -2\ln(1+\sqrt{2}),\quad \left(\sinh^{-1}(1) = \ln(1+\sqrt{2})\right)$

$$\int_0^1 \frac{\ln x}{(1-x^2\sin^4 t)^{\frac{3}{2}}}\mathrm{d}x = \frac{2\sqrt{2}x\ln x}{\sqrt{4x^2\cos(2t)-x^2\cos(4t)-3x^2+8}} - \csc^2(t)\arcsin[x\sin^2 t]|_0^1$$
$$= -\csc^2(t)\arcsin(\sin^2 t)$$

Now prove $\int_0^{\frac{\pi}{2}} \csc^2(t)\arcsin(\sin^2 t)\mathrm{d}t = 2\ln(1+\sqrt{2})$

Let $y = \sin^2 t, \quad \mathrm{d}y = 2\sin t\cos t\mathrm{d}t = (\sin 2t)\mathrm{d}t, \quad \csc^2 t = \frac{1}{y}$

$$LHS = \int_0^1 \frac{\arcsin(y)}{y}\frac{1}{2\sqrt{y}\sqrt{1-y}}\mathrm{d}y$$
$$= \frac{1}{2}\left[\frac{-2\arcsin y\sqrt{1-y}}{y}|_0^1 + \int_0^1 \frac{2\sqrt{1-y}}{\sqrt{y}}\frac{1}{\sqrt{1-y^2}}\mathrm{d}y\right] = \int_0^1 \frac{2}{\sqrt{y}\sqrt{1+y}}\mathrm{d}y$$
$$= 2\sinh^{-1}(1) = 2\ln(1+\sqrt{2})$$

Q.E.D.

Also solved by Kaushik Mahanta, NIT Silchar, Assam, India; Péter Fülöp, Gyömrő, Hungary; and the proposer.

• **5700** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata," Rome, Italy.*

Evaluate
$$\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x\ln(\cos x)}{1+\sin x\cos x}dx.$$

Solution 1 by G. C. Greubel, Newport News, VA.

This solution uses a derivative of the Beta function to obtain a result. Note that the denominator in the integral can be expanded into a series as seen by

$$I = \int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x\ \ln(\cos x)}{1+\sin x\ \cos x}\,dx$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\int_{-\pi/2}^{\pi/2} \sin^n(x)\ \cos^{n+2}(x)\ \ln(\cos x)\,dx$$
$$= \sum_{n=0}^{\infty}(-1)^{n+1}\,I_{n,2}$$

where I_n is evaluated as follows. In general,

$$\begin{aligned}
I_{n,m} &= \int_{-\pi/2}^{\pi/2} \sin^n(x)\,\cos^{n+m}(x)\,\ln(\cos x)\,dx \\
&= \partial_m \int_{-\pi/2}^{\pi/2} \sin^n(x)\,\cos^{n+m}(x)\,dx \\
&= \partial_m \left(\int_0^{\pi/2} + \int_{-\pi/2}^{0}\right) \sin^n(x)\,\cos^{n+m}(x)\,dx \\
&= \partial_m \,(1+(-1)^n)\,B\left(\frac{n+1}{2}, \frac{n+m+1}{2}\right) \\
&= (1+(-1)^n)\,B\left(\frac{n+1}{2}, \frac{n+m+1}{2}\right)\left(\psi\left(\frac{n+m+1}{2}\right) - \psi\left(n+\frac{m}{2}+1\right)\right).
\end{aligned}$$

When $m = 2$ this becomes

$$I_{n,2} = \frac{1+(-1)^n}{2}\,B\left(\frac{n+1}{2}, \frac{n+1}{2}\right)\left(\psi\left(\frac{n+3}{2}\right) - \psi(n+2)\right)$$

and leads to

$$\begin{aligned}
I &= \sum_{n=0}^{\infty} (-1)^{n+1}\, I_{n,2} \\
&= -\sum_{n=0}^{\infty} \frac{1+(-1)^n}{2}\,B\left(\frac{n+1}{2}, \frac{n+1}{2}\right)\left(\psi\left(\frac{n+3}{2}\right) - \psi(n+2)\right) \\
&= -\sum_{n=0}^{\infty} B\left(\frac{2n+1}{2}, \frac{2n+1}{2}\right)\left(\psi\left(\frac{2n+3}{2}\right) - \psi(2n+2)\right) \\
&= -\frac{\pi}{4} \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n}\left(\psi\left(n+\frac{3}{2}\right) - \psi(2n+2)\right) \\
&= -\frac{\pi}{4}\cdot\frac{1}{3}\,(\pi - 2\sqrt{3}\,\ln(2+\sqrt{3})) \\
&= \frac{\pi}{12}\,(2\sqrt{3}\,\ln(2+\sqrt{3}) - \pi).
\end{aligned}$$

This gives the result as

$$\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x\,\ln(\cos x)}{1+\sin x\,\cos x}\,dx = \frac{\pi}{12}\,(2\sqrt{3}\,\ln(2+\sqrt{3}) - \pi).$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

With

$$|\sin x\cos x| = \left|\frac{1}{2}\sin 2x\right| \leqslant \frac{1}{2},$$

the function

$$\frac{1}{1+\sin x\cos x}$$

can be expanded in a geometric series

$$\frac{1}{1+\sin x\cos x} = \sum_{n=0}^{\infty}(-1)^n \sin^n x\cos^n x$$

valid for all x. Now, for odd n,

$$\int_{-\pi/2}^{\pi/2} \sin^n x\cos^{n+2} x\ln(\cos x)\,dx = 0$$

because the integrand is an odd function; thus,

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x\ln(\cos x)}{1+\sin x\cos x}\,dx &= -\sum_{n=0}^{\infty}\int_{-\pi/2}^{\pi/2} \sin^{2n} x\cos^{2n+2} x\ln(\cos x)\,dx \\ &= -2\sum_{n=0}^{\infty}\int_{0}^{\pi/2} \sin^{2n} x\cos^{2n+2} x\ln(\cos x)\,dx.\end{aligned}$$

Next,

$$\int_0^{\pi/2} \sin^a x\cos^b x\,dx = \frac{1}{2}B\left(\frac{a+1}{2},\frac{b+1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b}{2}+1\right)},$$

where $B(x,y)$ is the beta function and $\Gamma(x)$ is the gamma function, and

$$\int_0^{\pi/2} \sin^a x\cos^b x\ln(\cos x)\,dx = \frac{1}{4}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b}{2}+1\right)}\left(\psi_0\left(\frac{b+1}{2}\right)-\psi_0\left(\frac{a+b}{2}+1\right)\right),$$

where $\psi_0(x)$ is the digamma function. It then follows that

$$\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x\ln(\cos x)}{1+\sin x\cos x}\,dx = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(2n+2)}\left(\psi_0\left(n+\frac{3}{2}\right)-\psi_0(2n+2)\right).$$

But,

$$\begin{aligned}\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(2n+2)} &= \frac{\pi}{2}\cdot\frac{1}{4^{2n}}\binom{2n}{n}, \\ \psi_0\left(n+\frac{3}{2}\right) &= -\gamma-2\ln 2+2H_{2n+2}-H_{n+1}, \text{ and} \\ \psi_0(2n+2) &= -\gamma+H_{2n+1},\end{aligned}$$

where H_n is the nth harmonic number, so

$$\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x \ln(\cos x)}{1+\sin x \cos x}\, dx = -\frac{\pi}{4}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^{2n}}\left(-2\ln 2 + \overline{H}_{2n+1}\right),$$

where

$$\overline{H}_{2n+1} = \sum_{j=1}^{2n+1}(-1)^{j-1}\frac{1}{j}.$$

The generating function for the central binomial coefficients is

$$\sum_{n=0}^{\infty}\binom{2n}{n}x^n = \frac{1}{\sqrt{1-4x}},$$

so

$$-\frac{\pi}{4}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^{2n}}(-2\ln 2) = \frac{\pi\ln 2}{\sqrt{3}}.$$

Moreover,

$$\overline{H}_{2n+1} = \int_0^1 \frac{1+x^{2n+1}}{1+x}\, dx.$$

Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \text{then} \quad \sum_{n=0}^{\infty} a_n \overline{H}_{2n+1} = \int_0^1 \frac{f(1)+xf(x^2)}{1+x}\, dx.$$

Now,

$$\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{x}{16}\right)^n = \frac{1}{\sqrt{1-\frac{x}{4}}} = \frac{2}{\sqrt{4-x}},$$

so

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{4^{2n}}\overline{H}_{2n+1} = \int_0^1 \frac{\frac{2}{\sqrt{3}}+\frac{2x}{\sqrt{4-x^2}}}{1+x}\, dx = \frac{2\ln 2}{\sqrt{3}} + \int_0^1 \frac{2x}{(1+x)\sqrt{4-x^2}}\, dx.$$

For the remaining integral, the substitution $x = 2\sin\theta$, followed by the Weierstrass substitution $t = \tan\frac{\theta}{2}$ and multiple applications of partial fractions yields

$$\begin{aligned}
\int_0^1 \frac{2x}{(1+x)\sqrt{4-x^2}}\, dx &= \int_0^{\pi/6} \frac{4\sin\theta}{1+2\sin\theta}\, d\theta \\
&= 16\int_0^{\tan\frac{\pi}{12}} \frac{t}{(1+4t+t^2)(1+t^2)}\, dt \\
&= 4\int_0^{\tan\frac{\pi}{12}} \left(\frac{1}{1+t^2} - \frac{1}{1+4t+t^2}\right) dt \\
&= \frac{\pi}{3} - \frac{2}{\sqrt{3}}\int_0^{\tan\frac{\pi}{12}} \left(\frac{1}{t+2-\sqrt{3}} - \frac{1}{t+2+\sqrt{3}}\right) dt \\
&= \frac{\pi}{3} - \frac{2}{\sqrt{3}}\ln\frac{2+\sqrt{3}}{2},
\end{aligned}$$

where we have used the fact that $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. Finally,

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \frac{-\cos^2 x \ln(\cos x)}{1+\sin x\cos x}\,dx &= \frac{\pi \ln 2}{\sqrt{3}} - \frac{\pi}{4}\left(\frac{2\ln 2}{\sqrt{3}} + \frac{\pi}{3} - \frac{2}{\sqrt{3}}\ln\frac{2+\sqrt{3}}{2}\right) \\ &= \frac{\pi\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi^2}{12}.\end{aligned}$$

Solution 3 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-\cos^2 x\ln(\cos x)}{1+\sin x\cos x}\mathrm{d}x = \int_{-\frac{\pi}{2}}^{0} \frac{-\cos^2 x\ln(\cos x)}{1+\sin x\cos x}\mathrm{d}x + \int_{0}^{\frac{\pi}{2}} \frac{-\cos^2 x\ln(\cos x)}{1+\sin x\cos x}\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} -\cos^2 x\ln(\cos x)\left(\frac{1}{1+\sin x\cos x} + \frac{1}{1-\sin x\cos x}\right)\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \ln(\cos x)\frac{2\cos^2 x}{\sin^2 x\cos^2 x - 1}\mathrm{d}x$$

Let $t = \tan x$,

$$I = \int_0^\infty \frac{\ln(1+t^2)}{1+t^2+t^4}\mathrm{d}t = \int_0^1 \frac{(1+x^2)\ln(1+x^2)}{1+x^2+x^4}\mathrm{d}x - 2\int_0^1 \frac{x^2\ln x}{1+x^2+x^4}\mathrm{d}x = I_1 - 2I_2$$

$$I_2 = \int_0^1 \frac{x^2\ln x}{1+x^2+x^4}\mathrm{d}x = \int_0^1 \frac{x^2-x^4}{1-x^6}\ln(x)\mathrm{d}x = \sum_{n=0}^\infty \int_0^1 (x^2-x^4)x^{6n}\ln x\mathrm{d}x$$

$$\because \quad \int_0^1 x^n \ln x\mathrm{d}x = -\frac{1}{(n+1)^2}$$

$$\therefore \quad I_2 = \sum_{n=0}^\infty \left[\frac{1}{(6n+5)^2} - \frac{1}{(6n+3)^2}\right] = \frac{1}{36}\sum_{n=0}^\infty \frac{1}{(n+\frac{5}{6})^2} - \frac{1}{9}\sum_{n=0}^\infty \frac{1}{(2n+1)^2}$$

$= \frac{1}{36}\psi'(\frac{5}{6}) - \frac{\pi^2}{72}$ in which ψ' is trigamma function.

Now evaluate $I_1 = \int_0^1 \frac{(1+x^2)\ln(1+x^2)}{1+x^2+x^4}\mathrm{d}x = \frac{1}{2}\left[\int_0^1 \frac{\ln(1+x^2)}{1+x+x^2}\mathrm{d}x + \int_0^1 \frac{\ln(1+x^2)}{1-x+x^2}\mathrm{d}x\right]$

Let $J(a) = \int_0^1 \frac{\ln(1+ax^2)}{1+x+x^2}\mathrm{d}x$

$$J'(a) = \int_0^1 \frac{x^2}{(1+ax^2)(1+x+x^2)}\mathrm{d}x$$

$$= \frac{1}{6\sqrt{a}(a^2-a+1)}\left\{\sqrt{a}\left[3\ln(1+a) + 2\sqrt{3}(2a-1)\frac{\pi}{6} - 3\ln 3\right] - 6(a-1)\arctan(\sqrt{a})\right\}$$

$\because \quad J(0) = 0$

$$\therefore \quad J(1) = \int_0^1 \frac{\ln(1+x^2)}{1+x+x^2}\mathrm{d}x = \int_0^1 J'(a)\mathrm{d}a = \int_0^1 J'(x)\mathrm{d}x$$

$$= \int_0^1 \frac{3\ln(1+x) + \frac{\sqrt{3}\pi}{3}(2x-1) - 3\ln 3}{6(x^2-x+1)}\mathrm{d}x - \int_0^1 \frac{(x-1)\arctan(\sqrt{x})}{\sqrt{x}(x^2-x+1)}\mathrm{d}x$$

Also, let $K(a) = \int_0^1 \frac{\ln(1+ax^2)}{1-x+x^2}\mathrm{d}x$

$$K'(a) = \frac{-3\ln(1+a) + 2\sqrt{3}(2a-1)\frac{\pi}{3}}{6(a^2-a+1)} - \frac{(a-1)\arctan(\sqrt{a})}{\sqrt{a}(a^2-a+1)}$$

$K(0) = 0$

$$K(1) = \int_0^1 \frac{\ln(1+x^2)}{1-x+x^2}\mathrm{d}x = \int_0^1 \frac{-3\ln(1+x) + \frac{2\sqrt{3}\pi}{3}(2x-1)}{6(x^2-x+1)} - \frac{(x-1)\arctan(\sqrt{x})}{\sqrt{x}(x^2-x+1)}\mathrm{d}x$$

$$I_1 = \frac{1}{2}\left[-\frac{\ln 3}{2}\frac{2\pi}{3\sqrt{3}} - 2\int_0^1 \frac{(x-1)\arctan(\sqrt{x})}{\sqrt{x}(x^2-x+1)}\mathrm{d}x\right]$$

$$= -\frac{\pi\ln 3}{6\sqrt{3}} - \int_0^1 \frac{(x-1)\arctan(\sqrt{x})}{\sqrt{x}(x^2-x+1)}\mathrm{d}x = -\frac{\pi\ln 3}{6\sqrt{3}} - 2\int_0^1 \frac{(x^2-1)}{x^4-x^2+1}\arctan x\mathrm{d}x$$

In which let $M = 2\int_0^1 \frac{(x^2-1)}{x^4-x^2+1}\arctan x\mathrm{d}x$

Then $M = \frac{1}{\sqrt{3}}[\ln(-x^2+\sqrt{3}x-1) - \ln(x^2+\sqrt{3}x+1)]\arctan x|_0^1 - \frac{1}{\sqrt{3}}\int_0^1 \frac{\ln(-x^2+\sqrt{3}x-1) - \ln(x^2+\sqrt{3}x}{1+x^2}$

$$= \frac{\pi}{4\sqrt{3}}[i\pi + 2\ln(2-\sqrt{3})] - \frac{1}{\sqrt{3}}\left[\int_0^1\left(\frac{i\pi}{1+x^2} + \frac{\ln(\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1})}{1+x^2}\right)\mathrm{d}x\right]$$

$$= \frac{\pi}{4\sqrt{3}}[i\pi + 2\ln(2-\sqrt{3})] - \frac{\pi i\pi}{4\sqrt{3}} - \frac{1}{\sqrt{3}}\int_0^1 \frac{\ln(\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1})}{1+x^2}\mathrm{d}x$$

$$= \frac{\pi}{2\sqrt{3}}\ln(2-\sqrt{3}) + \frac{1}{\sqrt{3}}\int_0^1 \frac{\ln(x^2+\sqrt{3}x+1) - \ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x$$

Now evaluate $S = S_1 - S_2 = \int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x - \int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x$

First evaluate $S_1' = \int_0^\infty \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x$

$$= \int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + \int_1^\infty \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + \int_1^0 \frac{\ln(\frac{1}{x^2}+\frac{\sqrt{3}}{x}+1)}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln(x^2+\sqrt{3}x+1) - 2\ln x}{1+x^2}\mathrm{d}x$$

$$\because \quad \int_0^1 \frac{\ln x}{1+x^2}\mathrm{d}x = G \text{ is Catalan's constant}$$

$$\therefore \quad S_1' = 2\int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + 2G$$

$$\text{i.e. } 2S_1 = 2\int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x - 2G$$

$$\text{Also, } \int_0^\infty \frac{\ln(x^2+2x\sin a+1)}{1+x^2}\mathrm{d}x = \pi\ln\left|2\cos\frac{a}{2}\right| + a\ln\left|\tan\frac{a}{2}\right| - 2S\,gn(a)\int_0^{\frac{|a|}{2}} \ln(\tan x)\mathrm{d}x$$

$$\text{When } \sin a = \frac{\sqrt{3}}{a}, a = \frac{\pi}{3}$$

$$\int_0^\infty \frac{\ln(x^2+2x\sin a+1)}{1+x^2}\mathrm{d}x = \pi\ln\left|2\cos\frac{\pi}{6}\right| + a\ln\left|\tan\frac{\pi}{6}\right| - 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$= \pi\ln(\sqrt{3}) + \frac{\pi}{3}\ln(\frac{1}{\sqrt{3}}) - 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$= \frac{\pi}{2}\ln 3 - \frac{\pi}{6}\ln 3 - 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$\text{So, } 2S_1 = \int_0^1 \frac{\ln(x^2+\sqrt{3}x+1)}{1+x^2}\mathrm{d}x = \frac{\pi}{3}\ln 3 - 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x - 2G$$

$$\text{Then evaluate } S_2' = \int_0^\infty \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + \int_1^\infty \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + \int_1^0 \frac{\ln(\frac{1}{x^2}-\frac{\sqrt{3}}{x}+1)}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + \int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2} - 2\int_0^1 \frac{\ln x}{1+x^2}\mathrm{d}x$$

$$= \int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x + 2G$$

$$\therefore \quad 2S_2 = 2\int_0^1 \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x = \int_0^\infty \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x - 2G$$

$$\text{Also, } \int_0^\infty \frac{\ln(x^2-\sqrt{3}x+1)}{1+x^2}\mathrm{d}x = \pi\ln 2 + \pi\ln\sqrt{3} + (-\frac{\pi}{3})\ln(-\frac{\sqrt{3}}{3}) + 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$= \frac{\pi}{2}\ln 3 + \frac{\pi}{6}\ln 3 + 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$= \frac{2\pi}{3}\ln 3 + 2\int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$\therefore \quad 2S_2 = \int_0^1 \frac{\ln(x^2 - \sqrt{3}x + 1)}{1 + x^2} \mathrm{d}x = \frac{2\pi}{3} \ln 3 + 2 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x - 2G$$

$$\therefore \quad 2(S_1 - S_2) = \frac{\pi}{3} \ln 3 - 2 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x - 2G - \left[\frac{2\pi}{3} \ln 3 + 2 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x - 2G \right]$$

$$= -\frac{\pi}{3} \ln 3 - 4 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x$$

$$\therefore \quad S = S_1 - S_2 = -\frac{\pi}{6} \ln 3 - 2 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x$$

$$\therefore \quad I = -\frac{\pi \ln 3}{6\sqrt{3}} + \frac{\pi^2}{36} - \frac{\psi'(\frac{5}{6})}{18} - \frac{\pi}{2\sqrt{3}} \ln(2 - \sqrt{3}) - \frac{1}{2\sqrt{3}} \left(-\frac{\pi}{6} \ln 3 - 2 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x \right)$$

in which

$$-\frac{\pi}{6} \ln 3 - 2 \int_0^{\frac{\pi}{6}} \ln(\tan x) \mathrm{d}x$$

$$= -\frac{\pi}{6} \ln 3 - 2 \times \left\{ -\frac{\pi}{12} \ln 3 + \frac{i}{2} \left[Li_2(\frac{i}{\sqrt{3}}) - Li_2(-\frac{i}{\sqrt{3}}) \right] \right\}$$

$$= i \left[Li_2(-\frac{i}{\sqrt{3}}) - Li_2(\frac{i}{\sqrt{3}}) \right]$$

$$\therefore \quad I = I = -\frac{\pi \ln 3}{6\sqrt{3}} + \frac{\pi^2}{36} - \frac{\psi'(\frac{5}{6})}{18} - \frac{\pi}{2\sqrt{3}} \ln(2 - \sqrt{3}) - \frac{i}{2\sqrt{3}} \left[Li_2(-\frac{i}{\sqrt{3}}) - Li_2(\frac{i}{\sqrt{3}}) \right]$$

Appendix 1

$$I(a) = \int_0^\infty \frac{\ln(1 + 2x \sin a + x^2)}{1 + x^2} \mathrm{d}x$$

where $a \in \left(0, \frac{\pi}{2}\right)$

Differentiating $I(a)$ w.r.t a yields

$$I'(a) = \int_0^\infty \frac{2x \cos a}{(1 + x^2)(1 + 2x \sin a + x^2)} \mathrm{d}x$$

$$= \cot a \int_0^\infty \left(\frac{1}{1 + x^2} - \frac{1}{1 + 2x \sin a + x^2} \right) \mathrm{d}x$$

$$= \cot a \left[\tan^{-1} x - \frac{1}{\cos a} \tan^{-1} \left(\frac{x + \sin a}{\cos a} \right) \right]_0^\infty$$

$$= \cot a \left[\frac{\pi}{2} - \frac{1}{\cos a}\left(\frac{\pi}{2} - a\right)\right]$$

Integrating $I'(a)$ back to $I(a)$, we have

$$I(a) - I(0) = \int_0^a \cot x \left[\frac{\pi}{2} - \frac{1}{\cos x}\left(\frac{\pi}{2} - x\right)\right] \mathrm{d}x$$

$$= \underbrace{\frac{\pi}{2}\int_0^a \left(\cot x - \frac{1}{\sin x}\right)\mathrm{d}x}_{=2\ln\left(\cos\frac{\pi}{2}\right)} + \underbrace{\int_0^a \frac{x}{\sin x}\mathrm{d}x}_{K}$$

$$K = \int_0^a \frac{x}{\sin x}\mathrm{d}x = \int_0^a x\mathrm{d}\left[\ln\left(\tan\frac{\pi}{2}\right)\right]$$

$$= \left[x\ln\left(\tan\frac{\pi}{2}\right)\right]_0^a - \int_0^a \ln\left(\tan\frac{x}{2}\right)\mathrm{d}x$$

$$= a\ln\left(\tan\frac{a}{2}\right) - 2\int_0^{\frac{a}{2}} \ln(\tan x)\mathrm{d}x$$

Now we can conclude that, for any $a \in \left(0, \frac{\pi}{2}\right]$

$$I = I(a) = \pi\ln\left(\cos\frac{a}{2}\right) + a\ln\left(\tan\frac{a}{2}\right) - 2\int_0^{\frac{a}{2}} \ln(\tan x)\mathrm{d}x$$

Similarly, for any $a \in \left(-\frac{\pi}{2}, 0\right)$

$$I = I(a) = \pi\ln\left(\cos\frac{a}{2}\right) - a\ln\left(\tan\frac{-a}{2}\right) + 2\int_0^{\frac{-a}{2}} \ln(\tan x)\mathrm{d}x$$

Appendix 2

Evaluate $I = \int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$

Let $y = \sqrt{3}\tan x, \quad \tan x = \frac{y}{\sqrt{3}}$

$$\mathrm{d}(1 + \tan^2 x) = \frac{\mathrm{d}y}{\sqrt{3}} = \left(1 + \frac{y^2}{3}\right)\mathrm{d}x$$

$$\mathrm{d}x = \frac{\sqrt{3}}{y^2 + 3}\mathrm{d}y$$

$$I = \int_0^1 \ln\left(\frac{y}{\sqrt{3}}\right)\frac{\sqrt{3}}{y^{+}3}\mathrm{d}y = \frac{1}{2i}\left[\int_0^1 \frac{\ln y}{y - \sqrt{3}i}\mathrm{d}y - \int_0^1 \frac{\ln y}{y + \sqrt{3}i}\mathrm{d}y\right] - \frac{\pi}{12}\ln 3$$

$$J_1 = \int_0^1 \frac{\ln y}{y - \sqrt{3}i}\mathrm{d}y = -\frac{1}{\sqrt{3}i}\int_0^1 \frac{\ln y}{1 - \frac{y}{\sqrt{3}i}}\mathrm{d}y$$

$$J_2 = \int_0^1 \frac{\ln y}{y + \sqrt{3}i}\mathrm{d}y = -\int_0^1 \frac{\ln y}{-\sqrt{3}i - y}\mathrm{d}y = \frac{1}{\sqrt{3}i}\int_0^1 \frac{\ln y}{1 - \frac{y}{-\sqrt{3}i}}\mathrm{d}y = Li_2\left(\frac{i}{\sqrt{3}}\right)$$

$$\because \quad \int_0^1 \frac{\ln x}{1 - zx}\mathrm{d}x = \int_0^1 \left[\sum_{n=0}^{\infty}(xz)^n \ln x\right]\mathrm{d}x = \sum_{n=0}^{\infty} z^n \left[\int_0^1 x^n \ln x \mathrm{d}x\right]$$

$$\therefore \quad -\sum_{n=0}^{\infty}\frac{z^n}{(n+1)^2} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{z^{n+1}}{(n+1)^2} = -\frac{1}{z}Li_2(z)$$

$$\therefore \quad J_1 = Li_2\left(\frac{i}{-\sqrt{3}}\right)$$

$$J_2 = Li_2\left(\frac{i}{\sqrt{3}}\right)$$

$$I = \int_0^{\frac{\pi}{6}} \ln(\tan x)\mathrm{d}x$$

$$= \frac{1}{2i}\left[Li_2\left(\frac{i}{-\sqrt{3}}\right) - Li_2\left(\frac{i}{\sqrt{3}}\right)\right] - \frac{\pi}{12}\ln 3$$

$$= -\frac{\pi}{12}\ln 3 + \frac{i}{2}\left[Li_2(\frac{i}{\sqrt{3}}) - Li_2(-\frac{i}{\sqrt{3}})\right]$$

Solution 4 by Michel Bataille, Rouen, France.

The change of variables $x = -u$ shows that

$$\int_{-\pi/2}^{0} \frac{-\cos^2 x \ln(\cos x)}{1 + \sin x \cos x}\,dx = \int_0^{\pi/2} \frac{-\cos^2 u \ln(\cos u)}{1 - \sin u \cos u}\,du,$$

hence the required integral I satisfies

$$I = \int_0^{\pi/2} (-\cos^2 x \ln(\cos x)\left(\frac{1}{1 + \sin x \cos x} + \frac{1}{1 - \sin x \cos x}\right)\,dx = 2\int_0^{\pi/2} \frac{-\cos^2 x \ln(\cos x)}{1 - \sin^2 x \cos^2 x}\,dx$$

and therefore

$$I = \int_0^{\pi/2} \frac{\ln(1 + \tan^2 x)}{1 + \tan^2 x + \tan^4 x}(1 + \tan^2 x)\,dx = \int_0^{\infty} \frac{\ln(1 + x^2)}{1 + x^2 + x^4}\,dx.$$

Let $f(z) = \dfrac{\log(i + z)}{1 + z^2 + z^4}$ where log denotes the principal determination of the logarithm and for $R > 1$, let C_R denote the contour formed by the line segment $\Re(z) \in [-R, R]$ and the semicircle Γ_R

of the half-plane $\Im(z) \geqslant 0$ with center O and radius R. We have

$$\int_{C_R} f(z)\,dz = 2\pi i \cdot S$$

where S is the sum of the residues of f at $\omega = \exp(2\pi i/3)$ and at $-\omega^2$. These residues are

$$Res(f,\omega) = \frac{\log(i+\omega)}{4\omega^3 + 2\omega} = \frac{\log(2\cos(\pi/12)e^{\frac{7\pi i}{12}})}{4+2\omega} = \frac{\ln(2\cos(\pi/12)) + i(7\pi/12)}{3 + i\sqrt{3}}$$

and

$$Res(f,-\omega^2) = \frac{\log(i-\omega^2)}{-4-2\omega^2} = \frac{\ln(2\cos(\pi/12)) + i(5\pi/12)}{-3 + i\sqrt{3}}$$

so that, after a simple calculation,

$$2\pi i \cdot S = \frac{\pi^2\sqrt{3}}{6}\,i + \frac{\pi\sqrt{3}}{3}\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right) - \frac{\pi^2}{12}. \tag{1}$$

On the other hand, setting $J(R) = \int_{\Gamma_R} f(z)\,dz$, we have

$$\int_{C_R} f(z)\,dz = \int_{-R}^{0} f(x)\,dx + \int_0^R f(x)\,dx + J(R) = \int_0^R \frac{\log(i-x)}{1+x^2+x^4}\,dx + \int_0^R \frac{\log(i+x)}{1+x^2+x^4}\,dx + J(R)$$

$$= \int_0^R \frac{\log(i-x)+\log(i+x)}{1+x^2+x^4}\,dx + J(R) = \int_0^R \frac{\ln(1+x^2) + i\pi}{1+x^2+x^4}\,dx + J(R). \tag{2}$$

We show at the end that $\lim_{R\to\infty} J(R) = 0$ and that $\int_0^\infty \frac{dx}{1+x^2+x^4} = \frac{\pi\sqrt{3}}{6}$. From (1) and (2), we now deduce that

$$I + i\pi \cdot \frac{\pi\sqrt{3}}{6} = \frac{\pi^2\sqrt{3}}{6}\,i + \frac{\pi\sqrt{3}}{3}\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right) - \frac{\pi^2}{12}$$

and finally

$$I = \frac{\pi\sqrt{3}}{3}\ln\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right) - \frac{\pi^2}{12}.$$

Proof of $\lim_{R\to\infty} J(R) = 0$
We have

$$|J(R)| = \left|\int_0^\pi \frac{\log(i+Re^{i\theta})}{1+R^2e^{2i\theta}+R^4e^{4i\theta}}\,(iRe^{i\theta})\,d\theta\right| \leqslant R\int_0^\pi \frac{|\log(i+Re^{i\theta})|}{|1+R^2e^{2i\theta}+R^4e^{4i\theta}|}\,d\theta$$

with $|\log(i+Re^{i\theta})| \leqslant |\ln(|i+Re^{i\theta}|| + |\arg(i+Re^{i\theta})| \leqslant \ln(1+R)+\pi$ and $|1+R^2e^{2i\theta}+R^4e^{4i\theta}| \sim R^4$ as $R\to\infty$. It follows that for sufficiently large R, we have $|1+R^2e^{2i\theta}+R^4e^{4i\theta}| \geqslant \frac{R^4}{2}$ and

$$|J(R)| \leqslant \cdot\frac{2(\ln(1+R)+\pi)}{R^3}$$

and the result follows since $\lim_{R\to\infty} \frac{2(\ln(1+R)+\pi)}{R^3} = 0.$

Proof of $\int_0^\infty \frac{dx}{1+x^2+x^4} = \frac{\pi\sqrt{3}}{6}$

Let $g(x) = \frac{1}{1+x^2+x^4}$. For $X > 0$, we have

$$\begin{aligned}
\int_0^X g(x)\,dx &= \frac{1}{2}\left(\int_0^X \frac{x+1}{x^2+x+1}\,dx - \int_0^X \frac{x-1}{x^2-x+1}\,dx\right) \\
&= \frac{1}{2}\left(\int_0^X \frac{x+1}{x^2+x+1}\,dx - \int_0^{-X} \frac{x+1}{x^2+x+1}\,dx\right) \\
&= \frac{1}{4}\int_{-X}^X \frac{2x+1+1}{x^2+x+1}\,dx \\
&= \frac{1}{4}\left(\ln\frac{X^2+X+1}{X^2-X+1} + \frac{2}{\sqrt{3}}\left(\arctan\left(\frac{1+2X}{\sqrt{3}}\right) - \arctan\left(\frac{1-2X}{\sqrt{3}}\right)\right)\right).
\end{aligned}$$

Letting $X\to\infty$, we see that $\int_0^\infty g(x)\,dx = \frac{\pi\sqrt{3}}{6}$.

Also solved by Moti Levy, Rehovot, Israel; Péter Fülöp, Gyömrő, Hungary; Albert Stadler, Herrliberg, Switzerland; and the proposer.

• **5701** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

Evaluate

$$\int e^{\cos^{-1}t}\cdot\left(\frac{3t^2-1}{\sqrt{t^3-t^5}}\right)dt.$$

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $x = \arccos t$, then $t = \cos x$, $\mathrm{d}t = -\sin x\mathrm{d}x$

$$I = \int e^{\cos^{-1}t}\cdot\left(\frac{3t^2-1}{\sqrt{t^3-t^5}}\right)\mathrm{d}t = \int e^x\left(\frac{1-3\cos^2 x}{\sqrt{\cos^3 x}}\right)\mathrm{d}x = I_1 - 3I_2$$

in which, $I_1 = \int e^x \cos^{-\frac{3}{2}} x\mathrm{d}x$

$$\because \quad \cos x = \frac{1}{2}\left(e^{ix}+e^{-ix}\right) = \frac{1}{2}e^{-ix}\left(1+e^{2ix}\right)$$

$$\therefore \quad I_1 = \int e^x\left(\frac{1}{2}\right)^{-\frac{3}{2}} e^{\frac{3}{2}ix}\left(1+e^{2ix}\right)^{-\frac{3}{2}}\mathrm{d}x$$

$$= \sqrt{8}\int e^{x\left(1+\frac{3}{2}i\right)}\sum_{k=0}^{\infty}\binom{-\frac{3}{2}}{k}e^{2ikx}\mathrm{d}x$$

$$= \frac{\sqrt{8}e^{x\left(1+\frac{3}{2}i\right)}}{1+\frac{3}{2}i+2ik}\sum_{k=0}^{\infty}\binom{-\frac{3}{2}}{k}e^{2ikx}.$$

Also, $I_2 = \int e^x \cos^{\frac{1}{2}} x\mathrm{d}x$

$$= \frac{1}{\sqrt{2}}\int e^{x\left(1-\frac{i}{2}\right)}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}e^{2ikx}\mathrm{d}x$$

$$= \frac{1}{\sqrt{2}}\frac{e^{x\left(1-\frac{i}{2}\right)}}{\left(1-\frac{i}{2}+2ik\right)}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}e^{2ikx}$$

$\therefore \quad I = I_1 - 3I_2$

$$= \frac{4}{\sqrt{2}}e^{x\left(1+\frac{3}{2}i\right)}\sum_{k=0}^{\infty}\binom{-\frac{3}{2}}{k}\frac{e^{2ikx}}{1+\frac{3}{2}i+2ik}$$

$$-\frac{3}{\sqrt{2}}e^{x\left(1-\frac{i}{2}\right)}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{e^{2ikx}}{1-\frac{i}{2}+2ik}$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

• **5702** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

(a) Calculate

$$\lim_{n\to\infty} n\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\mathrm{d}x\right).$$

(b) Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\mathrm{d}x\right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Partial integration gives

$$\int_0^{\infty}\frac{\sin^2 x}{x^2}dx = -\frac{\sin^2 x}{x}\Bigg|_0^{\infty}+\int_0^{\infty}\frac{2\sin x\cos x}{x}dx = \int_0^{\infty}\frac{\sin(2x)}{x}dx = \int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$$

due to the well-known value of the integral $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$.

The second mean value theorem for integrals gives the asymptotic expansion

$$\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}dx=\int_n^\infty \frac{\sin^2 x}{x^2}dx=\int_n^\infty \frac{1-\cos(2x)}{2x^2}dx=\frac{1}{2n}+O\left(\frac{1}{n^2}\right).$$

Hence

$$\lim_{n\to\infty} n\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}dx\right) = \lim_{n\to\infty} n\left(\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right) = \frac{1}{2}$$

and $\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}dx\right)$ diverges, since $\sum_{n=1}^{\infty} 1/n$ does.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

(a) By integration by parts,

$$\begin{aligned}\int_0^\infty \frac{\sin^2 x}{x^2}\,dx &= -\lim_{x\to\infty}\frac{\sin^2 x}{x}+\lim_{x\to 0}\frac{\sin^2 x}{x}+\int_0^\infty \frac{\sin 2x}{x}\,dx\\ &= \int_0^\infty \frac{\sin 2x}{x}\,dx=\int_0^\infty \frac{\sin u}{u}\,du=\frac{\pi}{2}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx &= \int_n^\infty \frac{\sin^2 x}{x^2}\,dx=\frac{1}{2}\int_n^\infty \frac{1}{x^2}\,dx-\frac{1}{2}\int_0^\infty \frac{\cos 2x}{x^2}\,dx\\ &= \frac{1}{2n}-\left.\frac{\sin 2x}{4x^2}\right|_n^\infty-\frac{1}{2}\int_n^\infty \frac{\sin 2x}{x^3}\,dx\\ &= \frac{1}{2n}+\frac{\sin 2n}{4n^2}+\left.\frac{\cos 2x}{4x^3}\right|_n^\infty+\frac{3}{4}\int_n^\infty \frac{\cos 2x}{x^4}\,dx\\ &= \frac{1}{2n}+\frac{\sin 2n}{4n^2}+O\left(\frac{1}{n^3}\right),\end{aligned}$$

and

$$n\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx\right)=\frac{1}{2}+\frac{\sin 2n}{4n}+O\left(\frac{1}{n^2}\right).$$

It follows that

$$\lim_{n\to\infty} n\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx\right)=\frac{1}{2}.$$

(b) Because

$$\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx=\int_n^\infty \frac{\sin^2 x}{x^2}\,dx\geqslant 0,$$

the series

$$\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx\right)$$

has all positive terms. Moreover,

$$\lim_{n\to\infty}\frac{\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx}{\frac{1}{n}}=\frac{1}{2}.$$

The harmonic series diverges, so the series

$$\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx\right)$$

also diverges by the limit comparison test.

Solution 3 by by G. C. Greubel, Newport News, VA.

It can be determined that

$$\begin{aligned} I &= \int_0^n \frac{\sin^2 x}{x^2}\,dx \\ &= \left[\operatorname{Si}(2x)-\frac{\sin^2(x)}{x}\right]_0^n \\ &= \operatorname{Si}(2n)-\frac{\sin^2(n)}{n}, \end{aligned}$$

where $\operatorname{Si}(x)$ is the Sine Integral. Since, for large x,

$$\operatorname{Si}(x)\approx\frac{\pi}{2}-\frac{\cos(x)}{x}-\frac{\sin(x)}{x^2}+O\left(\frac{1}{x^2}\right)$$

then

$$\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx\approx\frac{1}{2n}\left(1+\frac{\sin(2n)}{2n}\right)+O\left(\frac{1}{n^2}\right).$$

From this then the limit can be determined to be

$$\begin{aligned} L &= \lim_{n\to\infty} n\left(\frac{\pi}{2}-\int_0^n \frac{\sin^2 x}{x^2}\,dx\right) \\ &= \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{\sin(2n)}{2n}\right)+O\left(\frac{1}{n}\right) \\ &= \frac{1}{2}. \end{aligned}$$

In this case it can be shown that

$$a_n = \frac{1}{2n}\left(1+\frac{\sin(2n)}{2n}\right) \geqslant \frac{1}{2n+2}\left(1+\frac{\sin(2n+2)}{2n+2}\right) = a_{n+1}$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{2n}\left(1+\frac{\sin(2n)}{2n}\right) = 0.$$

These components suggest the use of the Alternating series test and Dirichlet's test. The series in question is not alternating and the Dirichlet test is inconclusive. Making use of the integral test then for

$$a_n = \frac{\pi}{2} - \mathrm{Si}(2n) + \frac{\sin^2(n)}{n}$$

the integral is

$$\begin{aligned}\int_1^\infty a_x\,dx &= \int_1^\infty \left(\frac{\pi}{2} - \mathrm{Si}(2x) + \frac{\sin^2(x)}{x}\right) dx \\ &= \frac{1}{2}\left[\pi x + \ln(x) - \cos(2x) - 2x\,\mathrm{Si}(2x) - \mathrm{Ci}(2x)\right]_1^\infty \\ &= \frac{1}{2}\left[\lim_{x\to\infty}\left(\ln(x) + \frac{\cos(2x)}{4x^2}\right) - (\pi - \cos(2) - 2\,\mathrm{Si}(2) - \mathrm{Ci}(2))\right] \\ &= \frac{1}{2}\left(\lim_{x\to\infty}\ln(x) - c_0\right).\end{aligned}$$

This dependency on $\ln(x)$ as x gets large suggests that the series diverges. Other tests like the root and ratio tests yield inconclusive results.

Solution 4 by by Michel Bataille, Rouen, France.

(a) It is well-known that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$. It follows that $\displaystyle\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2}\,dx = I_n$ where

$$I_n = \int_n^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Let $n \in \mathbb{N}$ and $X > 2n$. Integrating by parts, we obtain

$$\int_n^X \frac{\sin^2 x}{x^2}\,dx = \int_n^X \sin^2 x\,d(-1/x) = \frac{\sin^2 n}{n} - \frac{\sin^2 X}{X} + \int_n^X \frac{\sin 2x}{x}\,dx = \frac{\sin^2 n}{n} - \frac{\sin^2 X}{X} + \int_{2n}^{2X} \frac{\sin u}{u}\,du$$

and, letting $X \to \infty$,

$$I_n = \frac{\sin^2 n}{n} + \int_{2n}^\infty \frac{\sin u}{u}\,du.$$

In the same way, since $\displaystyle\frac{\sin u}{u}\,du = \frac{1}{u}d(-\cos u)$, integrating by parts again leads to

$$\int_{2n}^\infty \frac{\sin u}{u}\,du = \frac{\cos 2n}{2n} - \int_{2n}^\infty \frac{\cos u}{u^2}\,du = \frac{1}{2n} - \frac{\sin^2 n}{n} - \int_{2n}^\infty \frac{\cos u}{u^2}\,du$$

so that

$$I_n = \frac{1}{2n} - \int_{2n}^{\infty} \frac{\cos u}{u^2}\, du.$$

A third integration by parts gives $\displaystyle\int_{2n}^{\infty} \frac{\cos u}{u^2}\, du = -\frac{\sin 2n}{4n^2} + 2\int_{2n}^{\infty} \frac{\sin u}{u^3}\, du$ and finally

$$I_n = \frac{1}{2n} + \frac{\sin 2n}{4n^2} - 2\int_{2n}^{\infty} \frac{\sin u}{u^3}\, du.$$

Now, we have $\displaystyle\left|\frac{\sin 2n}{4n^2}\right| \leqslant \frac{1}{4n^2}$ and $\displaystyle\left|\int_{2n}^{\infty} \frac{\sin u}{u^3}\, du\right| \leqslant \int_{2n}^{\infty} \left|\frac{\sin u}{u^3}\right| du \leqslant \int_{2n}^{\infty} \frac{1}{u^3}\, du = \frac{1}{8n^2}$, hence

$$\lim_{n\to\infty} n\left(\frac{\sin 2n}{4n^2} - 2\int_{2n}^{\infty} \frac{\sin u}{u^3}\, du\right) = 0$$

and therefore

$$\lim_{n\to\infty} nI_n = \frac{1}{2}.$$

(b) From (a), we have $I_n \sim \dfrac{1}{2n}$ as $n \to \infty$, hence the series $\displaystyle\sum_{n=1}^{\infty} I_n$ is divergent.

Solution 5 by Moti Levy, Rehovot, Israel.

(a)

$$\begin{aligned} I_n &:= \int_0^n \left(\frac{\sin(x)}{x}\right)^2 dx = n\int_0^1 \left(\frac{\sin(nx)}{nx}\right)^2 dx \\ &= \operatorname{Si}(2n) - \frac{\sin^2(n)}{n}, \end{aligned}$$

where $\operatorname{Si}(y)$ is the sine integral

$$\operatorname{Si}(y) = \int_0^y \frac{\sin(t)}{t} dt.$$

The following asymptotic expansion of the sine integral is known::

$$\operatorname{Si}(y) \sim \frac{\pi}{2} - \frac{\sin(y)}{y}\sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)!}{x^{2k+1}} - \frac{\cos(y)}{y}\sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{x^{2k}}.$$

$$\begin{aligned}
&n\left(\frac{\pi}{2}-\int_0^n\left(\frac{\sin(x)}{x}\right)^2 dx\right)\\
&=n\left(\frac{\pi}{2}-\operatorname{Si}(2n)+\frac{\sin^2(n)}{n}\right)\\
&\sim n\left(\frac{\sin(2n)}{2n}\sum_{k=0}^{\infty}(-1)^k\frac{(2k+1)!}{(2n)^{2k+1}}+\frac{\cos(2n)}{2n}\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2n)^{2k}}+\frac{\sin^2(n)}{n}\right)\\
&=\frac{\sin(2n)}{2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k+1)!}{(2n)^{2k+1}}+\frac{\cos(2n)}{2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{(2n)^{2k}}+\sin^2(n)\\
&=\frac{\sin(2n)}{2}\left(\frac{1}{2n}+O\left(\frac{1}{n^3}\right)\right)+\frac{\cos(2n)}{2}\left(1-\frac{1}{2n^2}+O\left(\frac{1}{n^4}\right)\right)+\sin^2(n)\\
&=\frac{\cos(2n)}{2}+\sin^2(n)+\frac{\sin(2n)}{2}\left(\frac{1}{2n}+O\left(\frac{1}{n^3}\right)\right)+\frac{\cos(2n)}{2}\left(-\frac{1}{2n^2}+O\left(\frac{1}{n^4}\right)\right)\\
&=\frac{1}{2}+\left(\sin(2n)\left(\frac{1}{4n}+O\left(\frac{1}{n^3}\right)\right)+\cos(2n)\left(-\frac{1}{4n^2}+O\left(\frac{1}{n^4}\right)\right)\right).
\end{aligned}$$

It follows that,

$$\lim_{n\to\infty} n\left(\frac{\pi}{2}-\left(\operatorname{Si}(2n)-\frac{\sin^2(n)}{n}\right)\right)=\frac{1}{2}.$$

(b)

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\int_0^n\left(\frac{\sin(x)}{x}\right)^2 dx\right)\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}+\sum_{n=1}^{\infty}\left(\sin(2n)\left(\frac{1}{4n^2}+O\left(\frac{1}{n^4}\right)\right)+\cos(2n)\left(-\frac{1}{4n^3}+O\left(\frac{1}{n^5}\right)\right)\right).
\end{aligned}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges then the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{2}-\int_0^n\left(\frac{\sin(x)}{x}\right)^2 dx\right)$ diverges.

Solution 6 by Perfetti Paolo, dipartimento di matematica, Universitą di “Tor Vergata", Roma, Italy.

(a) It is known that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$

$$n\left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2}dx\right) = n\left(\frac{\pi}{2} - \int_0^\infty \frac{\sin^2 x}{x^2}dx + \int_n^\infty \frac{\sin^2 x}{x^2}dx\right)$$
$$= n\int_n^\infty \frac{1-\cos(2x)}{x^2}dx = n\int_n^\infty \frac{dx}{x^2} - n\int_n^\infty \frac{\cos(2x)}{x^2}dx$$
$$= 1 - n\int_n^\infty \frac{\cos(2x)}{x^2}dx = 1 - 2n\int_{2n}^\infty \frac{\cos y\,dy}{y^2} = 1 - \frac{2n\sin y}{y^2}\Big|_{2n}^\infty - 4n\int_{2n}^\infty \frac{\sin y\,dy}{y^3}$$
$$= 1 + \frac{\sin(2n)}{2n} - 4n\int_{2n}^\infty \frac{\sin y\,dy}{y^3}$$

$$n\Big|\int_{2n}^\infty \frac{\sin y dy}{y^3}\Big| \leqslant n\int_{2n}^\infty \frac{dy}{y^3} = \frac{n}{8n^2} = \frac{1}{8n}$$

It follows

$$\lim_{n\to\infty} n\left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2}dx\right) = 1$$

(b) The same computation yields

$$\sum_{n=1}^\infty \left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2}dy\right) = \sum_{n=1}^\infty \left(\frac{1}{n} + \frac{\sin(2n)}{2n^2} - 4\int_{2n}^\infty \frac{\sin y\,dy}{y^3}\right)$$

and

$$\frac{1}{n} + \frac{\sin(2n)}{2n^2} - 4\int_{2n}^\infty \frac{\sin y\,dy}{y^3} \geqslant \frac{1}{n} + \frac{\sin(2n)}{2n^2} - 4\int_{2n}^\infty \frac{dy}{y^3}\frac{1}{2n} = \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{4n^2} \geqslant \frac{1}{2n}$$

for any $n \geqslant 2$ so determining the divergence of the series.

Solution 7 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

(a) Substituting

$$\sin^2 x = \frac{1-\cos(2x)}{2} = \left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)',$$

integration by parts yields

$$\begin{aligned}\int \frac{\sin^2 x}{x^2}dx &= \frac{1}{x^2}\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right) - \int \frac{-2}{x^3}\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)dx \\ &= \frac{-1}{2x} - \frac{\sin(2x)}{4x^2} - \int \frac{\sin(2x)}{2x^3}dx.\end{aligned}$$

Using $\int_0^\infty x^{-2} \sin^2(x)\, dx = \pi/2$ it follows that

$$n\left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2} dx\right) = n\int_n^\infty \frac{\sin^2 x}{x^2} dx = \frac{1}{2} + \frac{\sin(2n)}{4n} - n\int_n^\infty \frac{\sin(2x)}{2x^3} dx.$$

Since

$$\left| n\int_n^\infty \frac{\sin(2x)}{2x^3} dx \right| \leqslant n\int_n^\infty \frac{1}{2x^3} dx = \frac{1}{4n}$$

we obtain

$$\lim_{n\to\infty} n\left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2} dx\right) = \frac{1}{2}.$$

(b) From part (a) we know that

$$\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2} dx = \frac{1}{2n} + O\left(\frac{1}{n^2}\right) \qquad (n \to \infty).$$

The divergence of the harmonic series and the convergence of $\sum_{n=1}^{\infty} n^{-2}$ imply that

$$\sum_{n=1}^{\infty}\left(\frac{\pi}{2} - \int_0^n \frac{\sin^2 x}{x^2} dx\right)$$

is divergent.

Also solved by Yunyong Zhang, Chinaunicom, Yunnan, China; and the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to LaTeX.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣