Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at the Department of Mathematics, Los Angeles Valley College, CA. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: **problems4ssma@gmail.com**

To propose solutions, email them to: **solutions4ssma@gmail.com**

Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before July 1, 2023.

• **5727** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

If $f : (0, \infty) \to (0, \infty)$ is a continuous function and $\displaystyle\int_a^b f(x)dx = 5(b-a)$ where $0 < a \leqslant b$, then

$$\int_a^b \left(\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11} \right) dx \leqslant 9(b-a).$$

• **5728** *Proposed by Florică Anastase, "Alexandru Odobescu" high school, Lehliu-Gară, Călăraşi, Romania.*

Define the sequences $(a_n)_{n\geqslant 1}$, $(b_n)_{n\geqslant 1}$ as follows:

$$a_n = \int_1^n \left\lfloor \frac{n^2}{x} \right\rfloor dx \quad \text{and} \quad b_1 > 1,\ b_{n+1} = 1 + \log(b_n)$$

where $\lfloor \cdot \rfloor$ denotes greatest integer (i.e., floor) function. Find the limit

$$\Omega = \lim_{n\to\infty} \frac{a_n \cdot \log \sqrt[n]{b_n}}{\log n}.$$

• **5729** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $0 < a < b$ be real numbers. Prove the following inequality

$$(a+b)^{a+b}(b-a)^{b-a} < \left(\frac{a^2+b^2}{b}\right)^{2b}.$$

• **5730** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $\forall a,b,c\in\mathbb{N}:\ f(a)^{f(b)^{f(c)}} + a^{b^{f(c)}} = f(a)^{b^c} + a^{f(b)^c}$.

• **5731** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve for real x: $\sqrt{6-11x+6x^2-x^3}+\sqrt{12-19x+8x^2-x^3}=\sqrt{15-23x+9x^2-x^3}$.

• **5732** *Proposed by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia..*

If $0 < s < 1$, prove

$$\int_1^{\infty}\frac{t\,dt}{(t^2+1)(t-1)^s}=\frac{\pi}{2^{s/2}}\csc(\pi s)\cos\left(\frac{\pi s}{4}\right).$$

Solutions

To Formerly Published Problems

• **5703** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve for real x:

$$x^2+(x-6)\sqrt{x-7}+12=13x.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

The continuous function $f:[7,\infty)\to\mathbb{R}$ defined by $f(x)=x^2+(x-6)\sqrt{x-7}+12-13x$

is monotonically strictly increasing for $x > 7$, since

$$f'(x) = 2x + \sqrt{x-7} + \frac{x-6}{2\sqrt{x-7}} - 13 > 0.$$

In addition $f\left(7^+\right) \approx -30$, $f(16) = 90$. So, by continuity of f and Intermediate Value Theorem, there is a unique real value of x at which f assumes the value 0, and we easily verify that $f(11) = 0$, so $x = 11$ is the only real solution.

Solution 2 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

We will show that $x = 11$ is the only real solution to this equation. Make the substitution $u = \sqrt{x-7}$. Then $x = u^2 + 7$, and since we need $x \geqslant 7$, each real solution of the given equation corresponds to a nonnegative solution of

$$\begin{aligned}(u^2+7)^2 + (u^2+1)u + 12 = 13(u^2+7) &\Rightarrow u^4 + 14u^2 + 49 + u^3 + u + 12 = 13u^2 + 91 \\ &\Rightarrow u^4 + u^3 + u^2 + u = 30.\end{aligned}$$

It is easy to verify that this equation has a solution of $u = 2$. Moreover, because the function $f(u) = u^4 + u^3 + u^2 + u$ is strictly increasing over $[0, \infty)$, the transformed equation has a unique solution. Therefore, the original equation has a unique real solution of $x = 2^2 + 7 = 11$.

Solution 3 by Trey Smith, Angelo State University, San Angelo, TX.

Let $y = \sqrt{x-7}$. Then $x = y^2 + 7$. Substituting into the original equation and simplifying, we obtain

$$y^4 + y^3 + y^2 + y - 30 = 0.$$

Using the rational root theorem, we have that $y = 2$ is a solution. Dividing $y^4 + y^3 + y^2 + y - 30$ by $y - 2$, and setting the quotient equal to 0, we have

$$x^3 + 3x^2 + 7x + 15 = 0.$$

The only possible real roots for this equation must be negative, but $y = \sqrt{x-7} \geqslant 0$. Hence, there are no other real solutions. So $x = 11$ is the only real solution for the original equation.

Solution 4 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

The only real solution is $x = 11$. If we let $f(x) = (x-1)(x-12) + (x-6)\sqrt{x-7}$, then the solutions to the given equation correspond to the zeros of f. Notice that $f(11) = 0$, so that $x = 11$ is a solution to the equation. If $x > 12$, then $f(x) > 0$, so all real solutions must be less than or equal 12, and greater than or equal to 7. If $f(x) = 0$, then

$$(x^2 - 13x + 12)^2 = \left((6-x)\sqrt{x-7}\right)^2$$

$$x^4 - 26x^3 + 193x^2 - 312x + 144 = x^3 - 19x^2 + 120x - 252$$
$$x^4 - 27x^3 + 212x^2 - 432x + 396 = 0$$
$$(x-11)(x^3 - 16x^2 + 36x - 36) = 0$$

so either $x = 11$ or $x^3 - 16x^2 + 36x - 36 = 0$. However, the maximum value of $g(x) = x^3 - 16x^2 + 36x - 36$ on the interval $[7, 12]$ is $g(12) = -180 < 0$, so the only solution to the equation is $x = 11$.

Solution 5 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

 # First Solution: By the given equation,

$$(x-1)(x-12) = (6-x)\sqrt{x-7} \ldots\ldots\ldots (*)$$

Let y be a solution of $(*)$. It is clear that $y \notin \{1,\ 6,\ 7,\ 12\}$. Since $y-1$, $y-12$, and $6-y$ are non-zero real numbers, then so is $\sqrt{y-7}$, such that $y > 7$. If $y > 12$, then in $(*)$, $LHS > 0 > RHS$, contradiction. Consequently, $y \in (7,\ 12)$. Notice that for $x \in (7,\ 12)$,

$$\frac{d\left((x-1)(x-12)\right)}{dx} = 2x - 13 > 0$$

and

$$\frac{d\left((6-x)\sqrt{x-7}\right)}{dx} = \frac{10 - \frac{3x}{2}}{\sqrt{x-7}} < 0$$

such that $(x-1)(x-12)$ and $(6-x)\sqrt{x-7}$ are respectively strictly increasing and strictly decreasing over x in the interval $(7,\ 12)$. So, the number of solutions satisfying $(*)$ is at most 1. It is easy to check that $x = 11$ works for $(*)$. Thus, $x = 11$ is the only solution.

 # Second Solution: By squaring both sides of the equation $(*)$, then

$$(x-1)^2(x-12)^2 = (x-6)^2(x-7)$$
$$\Longrightarrow$$
$$(x-11)\left(x\left((x-8)^2 - 28\right) - 36\right) = 0$$

Obviously, $x = 11$ works. If $x \neq 11$, then $x\left((x-8)^2 - 28\right) = 36$. As we have mentioned in the first solution, we should own $7 < x < 12$, but it implies that $x\left((x-8)^2 - 28\right) < 0$. Hence, $x = 11$ is the only solution.

Solution 6 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since x is real, it follows that $x \geqslant 7$. By substitution $y = x - 7$, the equation becomes

$$\begin{aligned}(y+7)^2 + (y+1)\sqrt{y} + 12 &= 13(y+7)\\ y^2 + y\sqrt{y} + y + \sqrt{y} &= 30.\end{aligned}$$

Now, because $y \geqslant 0$, we may let $y = z^2$, so the equation becomes $z^4 + z^3 + z^2 + z = 30$, or equivalently $z(z+1)(z^2+1) = 30$. Since $30 = 2 \cdot 3 \cdot 5$, it follows that $z = 2$ is the only real solution, because function $f(z) = z^4 + z^3 + z^2 + z$ is strictly increasing. Therefore, $y = 4$, and $x = 11$, and the problem is done.

Solution 7 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We write the given equation in the form

$$(x-6)\sqrt{x-7} = -x^2 + 13x - 12$$

and square both sides to obtain

$$(x^2 - 12x + 36)(x-7) = x^4 - 26x^3 + 193x^2 - 312x + 144,$$

which is in turn equivalent to

$$(x-11)(x^3 - 16x^2 + 36x - 36) = 0.$$

Then to show that $x = 11$ is the unique real solution of the original equation, we let

$$f(x) = x^2 + (x-6)\sqrt{x-7} - 13x + 12$$

for each real number $x \geqslant 7$, and we calculate

$$f'(x) = 2x - 13 + \sqrt{x-7} + \frac{x-6}{2\sqrt{x-7}}.$$

Since $f'(x) > 0$ for every real $x > 7$, $f(x)$ is increasing on $[7, \infty)$ and thus has at most one real zero. Since $f(11) = 0$, we conclude that the only real solution of the original equation is $x = 11$.

Solution 8 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Rewrite

$$x^2 + (x-6)\sqrt{x-7} + 12 = 13x$$

as

$$(x-1)(x-12) = (6-x)\sqrt{x-7},$$

and then let $f(x) = (x-1)(x-12)$ and $g(x) = (6-x)\sqrt{x-7}$. Because

$$f(x) \geqslant 0 > g(x)$$

for $x \geqslant 12$ and the domain of g is the set $\{x : x \geqslant 7\}$, if the equation $f(x) = g(x)$ has any real solutions, they must lie on the interval $[7, 12)$. Now, $f(7) = -30$, $f(12) = 0$, and

$$f'(x) = 2x - 13 > 0$$

for $x \geqslant 7$, while $g(7) = 0$, $g(12) = -6\sqrt{5}$, and

$$g'(x) = -\sqrt{x-7} + \frac{6-x}{2\sqrt{x-7}} < 0$$

for $x > 7$. Thus, the equation $f(x) = g(x)$ has a unique solution on the interval $[7, 12)$. By inspection,

$$f(11) = 10 = g(11);$$

hence the equation

$$x^2 + (x-6)\sqrt{x-7} + 12 = 13x$$

has the unique real solution $x = 11$.

Solution 9 by David A. Huckaby, Angelo State University, San Angelo, TX.

The given equation can be rewritten as

$$\begin{aligned} x^2 - 13x + 12 + (x-6)\sqrt{x-7} &= 0 \\ (x-12)(x-1) + (x-6)\sqrt{x-7} &= 0 \\ (x-7)(x-1) - 5(x-1) + (x-7)\sqrt{x-7} + \sqrt{x-7} &= 0 \\ (x-7)(x-7) + 6(x-7) - 5(x-1) + (x-7)\sqrt{x-7} + \sqrt{x-7} &= 0 \\ (x-7)^2 + (x-7)\sqrt{x-7} + x - 37 + \sqrt{x-7} &= 0 \\ (x-7)^2 + (x-7)\sqrt{x-7} + x - 7 + \sqrt{x-7} - 30 &= 0 \end{aligned}$$

This is a quartic equation in $\sqrt{x-7}$, with coefficients 1, 1, 1, 1, and -30. Since the leading coefficient is 1, the only possible rational roots are ±1, ±2, ±3, ±5, ±6, ±10, and ±30. Checking each candidate yields 2 as the only rational root. So $\sqrt{x-7} = 2$, whence $x = 11$ is a real root of the original equation. Synthetic division then yields

$$[\sqrt{x-7} - 2][(x-7)\sqrt{x-7} + 3(x-7) + 7\sqrt{x-7} + 15] = 0.$$

The second factor is a cubic in $\sqrt{x-7}$ with coefficients $a = 1$, $b = 3$, $c = 7$, and $d = 15$. We will use the general formula for cubic equations:

$$\sqrt{x-7} = (q + [q^2 + (r-p^2)^3]^{1/2})^{1/3} + (q - [q^2 + (r-p^2)^3]^{1/2})^{1/3} + p,$$

where $p = -\dfrac{b}{3a}$, $q = p^3 + \dfrac{bc - 3ad}{6a^2}$, and $r = \dfrac{c}{3a}$. Now $p = -\dfrac{3}{3(1)} = -1$, $q = (-1)^3 + \dfrac{3(7) - 3(1)(15)}{6(1)^2} = -5$, and $r = \dfrac{7}{3(1)} = \dfrac{7}{3}$. So

$$\sqrt{x-7} = (-5 + [25 + (4/3)^3]^{1/2})^{1/3} + (-5 - [25 + (4/3)^3]^{1/2})^{1/3} - 1.$$

Note that since $-5 - [25 + (4/3)^3]^{1/2} < 0$, the second term gives rise to one real root and two non-real complex roots. (Note further that since neither complex root is purely imaginary, neither root when squared would yield a real number.) Taking the real cube root from the second term, we have

$$x = \left[(-5 + [25 + (4/3)^3]^{1/2})^{1/3} + (-5 - [25 + (4/3)^3]^{1/2})^{1/3} - 1\right]^2 + 7 \approx 13.53709.$$

This irrational root, along with $x = 11$, are the two real roots of the original equation.

Solution 10 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX.

To begin, re-write the given equation in the form

$$x^2 - 13x + 12 = (6 - x)\sqrt{x - 7}. \tag{1}$$

Since x is real, any solution of (1) must have $(6 - x)\sqrt{x - 7}$ real as well. If $x < 7$, then $\sqrt{x - 7}$ is complex while $x^2 - 13x + 12$ is real. It follows that we must have $6 - x = 0$, i.e., $x = 6$. However, it is easily seen that $x = 6$ is not a solution of (1). Hence, we may assume that $x \geqslant 7$ and it follows that $\sqrt{x - 7}$ is real and non-negative.

If we substitute $y = \sqrt{x - 7}$, then $x = y^2 + 7$ and (1) becomes

$$\left(y^2 + 7\right)^2 - 13\left(y^2 + 7\right) + 12 = \left(6 - y^2 - 7\right)y$$

which simplifies to

$$y^4 + y^3 + y^2 + y - 30 = 0. \tag{2}$$

By trial and error, we find that $y = 2$ is one solution of (2) and long division establishes that

$$\begin{aligned}(y - 2)\left(y^3 + 3y^2 + 7y + 15\right) &= y^4 + y^3 + y^2 + y - 30\\ &= 0.\end{aligned}$$

It follows that all other solutions of (2) come from

$$y^3 + 3y^2 + 7y + 15 = 0.$$

However, because $y = \sqrt{x - 7} \geqslant 0$, this implies that there are no additional solutions of (2). Thus, the condition $2 = y = \sqrt{x - 7}$ yields $x = 11$ as the only solution of (1).

Solution 11 by by G. C. Greubel, Newport News, VA.

Writing the equation as

$$x^2 + (x - 6)\sqrt{x - 7} = 13\,x - 12$$

and square both sides to obtain

$$x^4 + x^3 - 188\,x^2 + 396\,x - 396 + 2\,x^2\,(x - 6)\sqrt{x - 7} = 0.$$

Using the original equation to replace $\sqrt{x - 7}$ leads to the equation

$$x^4 - 27\,x^3 + 212\,x^2 - 432\,x + 396 = 0$$

which can be factored into the form

$$(x - 11)(x^3 - 16\,x^2 + 36\,x - 36) = 0.$$

The cubic equation has one real valued solution, which is

$$x_1 = \frac{1}{3}\left(16 + \sqrt[3]{2\,(995 + 9\,\sqrt{2217})} + \sqrt[3]{2\,(995 - 9\,\sqrt{2217})}\right),$$

and the other solution is $x = 11$. By comparing these possible values to the graph of the equation leads to the one solution which is $x = 11$.

Solution 12 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Setting $a = x - 6$ and $b = \sqrt{x-7}$, we see that $a^2 - b^2 = x^2 - 13x + 43$. Thus $a^2 - b^2 - 43 + ab = -12$, or

$$a^2 - b^2 - 31 + ab = 0. \qquad (1)$$

Substituting $b^2 + 1$ for a in (1), we get $(b^2+1)^2 - b^2 - 31 + b(b^2+1) = 0$, or $b^4 + b^3 + b^2 + b - 31 = 0$, which can be written as

$$(b-2)(b^3 + 3b^2 + 7b + 15) = 0. \qquad (2)$$

Now we have $b \geqslant 0$ and $b^3 + 3b^2 + 7b + 15 > 0$, so that $b = 2$ is the only solution of (2). This, in turn, implies that $x = 11$ is the only solution of the given equation.

Solution 13 by Michel Bataille, Rouen, France.

Let x be a solution. Then, $x \geqslant 7$ and $X = \sqrt{x-7}$ satisfies $X \geqslant 0$ and

$$(X^2 + 7)^2 + X(X^2 + 1) + 12 = 13(X^2 + 7),$$

that is,

$$X^4 + X^3 + X^2 + X - 30 = 0$$

or

$$(X-2)(X^3 + 3X^2 + 7X + 15) = 0.$$

Since $X^3 + 3X^2 + 7X + 15 > 15$, we must have $X = 2$. It follows that $x = 11$ is the only possible solution. Conversely, it is readily checked that 11 satisfies the proposed equation. In conclusion, $x = 11$ is the unique solution.

Solution 14 by Péter Fülöp, Gyömrő, Hungary.

We have to exclude the $x < 7$ real numbers from the solutions because of the quantity under the root cannot be a negative number.

$$x^2 - 7x + (x-6)\sqrt{x-7} = 6x - 12$$
$$\sqrt{x-7}\Big[x\sqrt{x-7} + x - 6\Big] = 6(x-7) + 30$$
$$(x - 7 + 1)\sqrt{x-7}\Big[\sqrt{x-7} + 1\Big] = 30$$

Introducing $z = \sqrt{x-7}$

$$(z^2+1)(z+1)z = 30$$

$$z^4 + z^3 + z^2 + z = 30$$

It can be seen that $z = 2$ is root of the equation. Regarding the original equation $x = 11$ satisfies that.

Let's devide the polynomial $(z^4 + z^3 + z^2 + z - 30)$ by $(z-2)$, the result is the following polynomial of the third degree:

$$z^3 + 3z^2 + 7z + 15 = 0$$

After some rearrangements we get the suitable form of the equation so that the roots can be easily determined.

$$(z+1)^3 + 4(z+1) + 10 = 0$$

Known that $z^3 + 3pz + 2q = 0$ (the reduced form of the third degree equation) and $D = q^2 + p^3 > 0$ then it has one real and two complex roots.

In our case $D = \dfrac{19}{3} > 0$

Based on the Cardano-Tartaglia form the real root is:

$$z_0 + 1 = \left[\sqrt[3]{-5 + \sqrt{\frac{739}{27}}}\right] - \left[\sqrt[3]{5 + \sqrt{\frac{739}{27}}}\right] \approx -1,55677..$$

$z \approx -2,55677....$ could not be equal to $\sqrt{x-7}$ in $\Re$, so it is a false root.

The real solution is $x = 11$.

Solution 15 by Paolo Perfetti, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

Let $f(x) = x^2 + (x-6)\sqrt{x-7} + 12 - 13x$

$$f'(x) = 2x - 13 + \sqrt{x-7} + \frac{x-6}{2\sqrt{x-7}} \geqslant 14 - 13 + 2\left(\sqrt{x-7}\frac{x-6}{2\sqrt{x-7}}\right)^{\frac{1}{2}} \geqslant 1 + \sqrt{2}$$

It follows that $x = 11$ is the unique solution.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Daniel Văcaru, Pitești, Romania; and the proposer.

• **5704** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Let a and k be positive integers that are relatively prime and of different parity. Further assume that k is not a perfect square. Let u_n and v_n be integers such that

$$\left(a+\sqrt{k}\right)^n = u_n + v_n\sqrt{k}, \quad n = 1,\ 2,\ \ldots$$

Prove that u_n and v_n are relatively prime for all natural numbers n.

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

For all positive integers n,

$$\left(a+\sqrt{k}\right)^{n+1} = \left(u_n + v_n\sqrt{k}\right)\left(a+\sqrt{k}\right) = (au_n + kv_n) + (u_n + av_n)\sqrt{k},$$

so that

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = M \begin{bmatrix} u_n \\ v_n \end{bmatrix},$$

where $M = \begin{bmatrix} a & k \\ 1 & a \end{bmatrix}$ and $(u_1, v_1) = (a, 1)$. Since k is not a perfect square, then $\det(M) = a^2 - k \neq 0$.

Lemma 1 *Let p be a prime number that does not divide $a^2 - k$. Then for all natural numbers n, $(u_n, v_n) \not\equiv (0,0) \pmod{p}$.*

Since p does not divide $a^2 - k$, then p and $\det(M)$ are relatively prime, so there exists an integer s such that $s\det(M) \equiv 1 \pmod{p}$. Thus,

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \equiv \begin{bmatrix} a & -k \\ -1 & a \end{bmatrix} \begin{bmatrix} su_{n+1} \\ sv_{n+1} \end{bmatrix} \pmod{p},$$

so that $(u_{n+1}, v_{n+1}) \equiv (0,0) \pmod{p}$ if and only if $(u_n, v_n) \equiv (0,0) \pmod{p}$. Since $(u_1, v_1) = (a, 1) \not\equiv (0,0) \pmod{p}$, then by induction $(u_n, v_n) \not\equiv (0,0) \pmod{p}$ for all natural numbers n. Suppose n is a natural number for which u_n and v_n are not relatively prime. Then there is a prime number p that is a common divisor of u_n and v_n; by the lemma, $p \mid \left(a^2 - k\right)$, so that $k \equiv a^2 \pmod{p}$. Since a and k are relatively prime, then $a \not\equiv 0 \pmod{p}$. In addition, since a and k have opposite parity, then $a^2 - k$ must be odd, and $p \neq 2$. We show by induction that

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \equiv \begin{bmatrix} 2^{n-1}a^n \\ 2^{n-1}a^{n-1} \end{bmatrix} \pmod{p}$$

for all natural numbers n. This is true for $n = 1$, since $(u_1, v_1) = (a, 1)$. For the inductive step,

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} \equiv \begin{bmatrix} a & a^2 \\ 1 & a \end{bmatrix} \begin{bmatrix} 2^{n-1}a^n \\ 2^{n-1}a^{n-1} \end{bmatrix} \equiv \begin{bmatrix} 2^n a^{n+1} \\ 2^n a^n \end{bmatrix} \pmod{p},$$

so the statement is true for all natural numbers n. Since p does not divide 2 or a, then

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} \equiv \begin{bmatrix} 2^{n-1}a^n \\ 2^{n-1}a^{n-1} \end{bmatrix} \not\equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix} \pmod{p},$$

which is a contradiction. Therefore, u_n and v_n are relatively prime for all natural numbers n.

Solution 2 by Michel Bataille, Rouen, France.

Since k is not a perfect square, the equality $u + v\sqrt{k} = x + y\sqrt{k}$ where u, v, x, y are integers implies $u = x, v = y$ (a well-known result). Thus, from $u_{n+1} + v_{n+1}\sqrt{k} = (a + \sqrt{k})^{n+1} = (a + \sqrt{k})(u_n + v_n\sqrt{k})$, we deduce that

$$u_{n+1} = au_n + kv_n, \qquad v_{n+1} = u_n + av_n. \tag{1}$$

This holds for $n = 1, 2, \ldots$ and can be extended to the case $n = 0$ by setting $u_0 = 1, v_0 = 0$ (in accordance with $(a + \sqrt{k})^0 = 1$). We use induction to prove that u_n, v_n are coprime for all nonnegative integers n. Clearly, u_0, v_0 are coprime and it is also the case of $u_1 = a$ and $v_1 = 1$. Assume that n is a positive integer and that u_n, v_n are coprime. For the purpose of a contradiction, suppose that a prime p divides both u_{n+1} and v_{n+1}. Then, p divides $u_{n+1} - av_{n+1} = (k - a^2)v_n$ and $au_{n+1} - kv_{n+1} = (a^2 - k)u_n$. It follows that p divides $|k - a^2|$, which is their gcd (since u_n, v_n are coprime). Since k, a are of opposite parity, we see that p must be odd. From (1), we readily obtain that for all $n \geqslant 1$:

$$u_{n+1} = 2au_n - (a^2 - k)u_{n-1}, \qquad v_{n+1} = 2av_n - (a^2 - k)v_{n-1}.$$

For example, $u_{n+1} = au_n + k(u_{n-1} + av_{n-1}) = au_n + ku_{n-1} + a(u_n - au_{n-1}) = 2au_n - (a^2 - k)u_{n-1}$. Since p divides u_{n+1}, v_{n+1} and $a^2 - k$, we deduce that p divides $2au_n$ and $2av_n$, hence their gcd $2a$, hence a (since p is odd) and therefore p divides $k = a^2 - (a^2 - k)$. Finally, we obtain that p divides a and k, in contradiction with the hypothesis. This completes the induction step and the proof.

Solution 3 by Trey Smith, Angelo State University, San Angelo, TX.

We start by observing that if the m divides both u_n and v_n, then m divides u_{n+1} and v_{n+1}. To see this, notice that

$$\begin{aligned} &(a + \sqrt{k})^{n+1} \\ &= (a + \sqrt{k})^n (a + \sqrt{k}) \\ &= (u_n + v_n\sqrt{k})(a + \sqrt{k}) \\ &= (au_n + kv_n) + (u_n + av_n)\sqrt{k}. \end{aligned}$$

Now $m|u_n$ and $m|v_n$. So $m|(au_n + kv_n) = u_{n+1}$, and $m|(u_n + av_n) = v_{n+1}$. Using the above observation, we have that if for some n, u_n and v_n are not relatively prime, then for all $k \geqslant n$, u_k and v_k will not be relatively prime. We say that the pair (u_n, v_n) is *good* if u_n is odd, v_n is even, u_n and v_n are relatively prime, and u_n and k are relatively prime. We start by demonstrating that (u_2, v_2) is good. To see this, observe that

$$(a + \sqrt{k})^2 = (a^2 + k) + 2a\sqrt{k}.$$

Since a and k have different parities, we have $u_2 = a^2 + k$ is odd. Clearly $v_2 = 2a$ is even. Since u_2 is odd, 2 does not divide u_2. If an odd prime p were to divide v_2, then p divides a which implies p does not divide k, so p does not divide u_2. Hence u_2 and v_2 are relatively prime. Finally, if p divides k then p does not divide a which implies p does not divide u_2. Hence k and u_2 are relatively prime. Now we show that if (u_n, v_n) is good, so is (u_{2n}, v_{2n}). Notice that

$$\begin{aligned} & u_{2n} + v_{2n}\sqrt{k} \\ &= (u_n + v_n\sqrt{k})^2 \\ &= (u_n^2 + v_n^2 k) + (2u_n v_n)\sqrt{k}. \end{aligned}$$

Since v_n is even and u_n is odd, we have that $u_{2n} = u_n^2 + v_n^2 k$ is odd. Clearly $v_{2n} = 2u_n v_n$ is even. Let p be a prime that divides v_{2n}. If $p = 2$ then p does not divide u_{2n} since u_{2n} is odd. If p divides k then p does not divide u_n but p does divide $v_n k$ which means p does not divide u_{2n}. If p divides u_n then p does not divide k and p does not divide v_n, so p does not divide u_{2n}. Thus u_{2n} and v_{2n} are relatively prime. Similarly, if p divides k it does not divide u_n so it does not divide u_{2n}. Thus u_{2n} and k are relatively prime. Hence, (u_{2n}, v_{2n}) is good. Using both observations about good pairs, we have that (u_{2^r}, v_{2^r}) is good for $r = 1, 2, \ldots$. The fact that u_n and v_n are relatively prime for all n follows by assuming that there is some n that fails. Then by our first observation, u_k and v_k are not relatively prime for any $k \geqslant n$. But, we know there is some r such that $2^r > n$, and u_{2^r} and v_{2^r} are relatively prime. Hence there is no n that fails.

Also solved by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia; and the proposer.

• **5705** *Proposed by Rafael Jakimczuk, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Prove the series $\sum_{n=1}^{\infty} a_n$ converges where the the sequence $(a_n)_{n\geqslant 1}$ is recursively defined as follows:

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{3!} + \frac{a_{n-1}}{5!} + \frac{a_{n-2}}{7!} + \cdots + \frac{a_1}{(2n+1)!} \quad (n \geqslant 1).$$

Solution 1 by G. C. Greubel, Newport News, VA.

Writing the first few terms of a_n gives

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{3!} \\ a_3 &= \frac{26}{6!} \\ a_4 &= \frac{2760}{9!} \end{aligned}$$

leads to a form of $(3\,n-3)!\,a_n = b_n$, with

$$b_n \in \{1, 1, 26, 2760, 768504, 442554840, 457050442176, 769348154736000, ...\}_{n\geqslant 1}$$

and $b_n < (3\,n-3)!$ for $n \geqslant 2$. By comparison it can be noticed that

$$0 \leqslant a_n \leqslant \left(\frac{1}{4}\right)^{n-1},$$

which is in essence the direct comparison test for convergence, and leads to

$$0 \leqslant \sum_{j=1}^{n} a_j \leqslant \frac{4}{3}\left(1 - \frac{1}{4^n}\right).$$

In the limit as $n \to \infty$ this gives

$$0 \leqslant \sum_{j=1}^{\infty} a_j \leqslant \frac{4}{3}.$$

This demonstrates that the series for a_n converges.

A second case, which also uses the direct comparison test, may be seen as follows:

Let

$$\phi_n := \left\{ \begin{array}{cc} 1 & \text{if} \quad n = 1 \\ \dfrac{1}{6} + \displaystyle\sum_{j=1}^{n} \frac{7 - 3\,(-1)^j}{10^j} & \text{if} \quad n \geqslant 2 \end{array} \right\} = \left\{ \begin{array}{cc} 1 & \text{if} \quad n = 1 \\ \dfrac{241}{198} - \dfrac{77 + 27\,(-1)^n}{99\,\cdot\,10^n} & \text{if} \quad n \geqslant 2 \end{array} \right\}.$$

Then

$$0 \leqslant \sum_{j=1}^{n} a_j \leqslant \phi_n$$

and as $n \to \infty$, we have

$$0 \leqslant \sum_{n=1}^{\infty} a_n \leqslant \frac{241}{198}.$$

Solution 2 by Moti Levy, Rehovot, Israel.

The sequence $\{a_n\}_{n\geqslant 1}$ is recursively defined as

$$a_1 = 1, \quad a_{n+1} = \sum_{k=1}^{n} \frac{a_k}{\big(2\,(n-k)+3\big)!}. \tag{3}$$

We prove that the series $\sum_{n=1}^{\infty} a_n$ converges by evaluating it.
Let $F(z)$ be the generating function of the sequence $\{a_n\}_{n\geqslant 1}$

$$F(z) := \sum_{n=1}^{\infty} a_n z^n,$$

and let $B(z)$ be the generating function of the sequence

$$B(z) := \sum_{n=1}^{\infty} \frac{z^n}{(2n+1)!}.$$

Substituting (3) in $\sum_{n=1}^{\infty} a_{n+1} z^n$ we get,

$$\begin{aligned}\sum_{n=1}^{\infty} a_{n+1} z^n &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_k \frac{1}{\big(2\,(n-k)+3\big)!}\right) z^n \\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} a_k \frac{1}{\big(2\,(n-k+1)+1\big)!}\right) z^n.\end{aligned}$$

Now $\sum_{k=1}^{n} a_k \frac{1}{\big(2\,(n-k+1)+1\big)!}$ is the convolution of $\{a_n\}_{n\geqslant 1}$ with $\left\{\frac{1}{(2n+1)!}\right\}_{n\geqslant 1}$, hence

$$\sum_{n=1}^{\infty} a_{n+1} z^n = F(z)\,B(z). \tag{4}$$

$$\sum_{n=1}^{\infty} a_{n+1} z^n = \frac{1}{z}\sum_{n=1}^{\infty} a_{n+1} z^{n+1} = \frac{1}{z}\sum_{n=0}^{\infty} a_{n+1} z^{n+1} - 1 = \frac{1}{z} F(z) - 1. \tag{5}$$

By (4) and (5) we get $\frac{1}{z} F(z) - 1 = F(z)\,B(z)$ and solving for $F(z)$, we get

$$F(z) = \frac{z}{1 - zB(z)}.$$

The generating function $B(z)$ can be evaluated using the Taylor series of $\sinh(z)$,

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}.$$

$$B(z) = \sum_{k=1}^{\infty} \frac{z^k}{(2k+1)!} = -1 + \frac{\sinh\left(\sqrt{z}\right)}{\sqrt{z}}.$$

$$F(z) = \frac{z}{1 - z\left(-1 + \frac{\sinh\left(\sqrt{z}\right)}{\sqrt{z}}\right)} = \frac{z}{1 + z - \sqrt{z}\sinh\left(\sqrt{z}\right)}.$$

Now $\sum_{n=1}^{\infty} a_n = F(1)$, hence

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2 - \sinh(1)} \cong 1.2124.$$

Solution 3 by Paolo Perfetti, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

The general term a_n of the series is positive for any n thus if $\sum_{n=1}^{N} a_n$ is bounded by a constant M for any N, the series converges.

$$a_{n+1} = \sum_{k=0}^{n-1} \frac{a_{n-k}}{(2k+3)!}, \ n \geqslant 1$$

$$\sum_{n=1}^{N} a_{n+1} = \sum_{n=1}^{N}\sum_{k=0}^{n-1} \frac{a_{n-k}}{(2k+3)!} = \sum_{k=0}^{N-1} \frac{1}{(2k+3)!} \sum_{n=k+1}^{N} a_{n-k} = \sum_{k=0}^{N-1} \frac{1}{(2k+3)!} \sum_{p=1}^{N-k} a_p$$

hence using the positivity of a_n for any n

$$\sum_{n=2}^{N+1} a_n = \sum_{k=0}^{N-1} \frac{1}{(2k+3)!} \sum_{p=1}^{N-k} a_p \leqslant \sum_{k=0}^{N-1} \frac{1}{(2k+3)!} \sum_{p=1}^{N+1} a_p$$

if and only if

$$\sum_{n=1}^{N+1} a_n \leqslant 1 + \sum_{k=0}^{N-1} \frac{1}{(2k+3)!} \sum_{p=1}^{N+1} a_p$$

This implies

$$\sum_{n=1}^{N+1} a_n \leqslant \frac{c}{1-c}, \qquad c = \sum_{k=0}^{N-1} \frac{1}{(2k+3)!}$$

and

$$\sum_{k=0}^{N-1} \frac{1}{(2k+3)!} < \sum_{k=0}^{\infty} \frac{1}{(2k+3)!} < \sum_{k=0}^{\infty} \frac{1}{6}\frac{1}{(2k)!} = \frac{\cosh(1)}{6} < \frac{1}{2}$$

I use $(2k+3)! = (2k)!(2k+1)(2k+2)(2k+3) \geqslant (2k)! \cdot 6$.

This means that the sequence $\left\{\sum_{k=1}^{N} a_k\right\}_{N\geqslant 1}$ is monotonically increasing and bounded; thus converges.

Solution 4 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

We prove the series converges by using the comparison test with a geometric series $\sum r^n$, where r is a real number and $0 < r < 1$. We begin with the following lemma.

Lemma 2 *If r is a real number satisfying $\frac{10}{11} \leqslant r < 1$ and n is a positive integer, then*

$$\sum_{k=1}^{n} \frac{r^{n-k}}{(2k+1)!} < r^n.$$

Using the power series expansion for e^x, we see that if x is a positive real number then

$$e^x - 1 - x = \sum_{k=2}^{\infty} \frac{x^k}{k!} > \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

In addition, if $x > 1$, then $x^{2k+1} > x^k$, so that

$$e^x - 1 - x > \sum_{k=1}^{\infty} \frac{x^{2k+1}}{(2k+1)!} > \sum_{k=1}^{\infty} \frac{x^k}{(2k+1)!} = \frac{x}{3!} + \frac{x^2}{5!} + \frac{x^3}{7!} + \cdots .$$

If $\frac{10}{11} \leqslant r < 1$, we let $R = 1/r$, so that $1 < R < 1.1$ Since $f(x) = e^x - 1 - x$ is increasing for $x > 0$, then $e^R - 1 - R < e^{1.1} - 2.1 < 1$ for $1 < R < 1.1$. Thus,

$$\sum_{k=1}^{n} \frac{r^{n-k}}{(2k+1)!} = r^n \sum_{k=1}^{n} \frac{R^k}{(2k+1)!} < r^n \left(e^R - 1 - R\right) < r^n.$$

We now let $r = \frac{10}{11}$ and prove that $0 < a_{n+1} < r^n$ for all positive integers n. It is clear from the definition that $a_{n+1} > 0$ for all positive integers n; we prove the remaining inequality using strong induction. Since $a_2 = 1/6 < 10/11 = r$, the inequality is true for $n = 1$. Suppose that n is a positive integer and that $a_{k+1} < r^k$ for all integers k with $1 \leqslant k \leqslant n$. Then

$$a_{n+2} = \sum_{k=1}^{n+1} \frac{a_{n+2-k}}{(2k+1)!} < \sum_{k=1}^{n+1} \frac{r^{n+1-k}}{(2k+1)!},$$

and by the Lemma, $a_{n+2} < r^{n+1}$; thus, $0 < a_{n+1} < r^n$ for all positive integers n. Since the geometric series $\sum_{n=1}^{\infty} r^n$ converges (to 10) for $r = 10/11$, then by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges as well.

While our argument shows that the series converges to a number less than 10, we note that this is far from sharp; numerical computation shows that the series converges to approximately 1.21241688554.

Solution 5 by Charles Burnette, Xavier University of Louisiana, New Orleans, LA.

Observe that for each positive integer n,

$$a_{n+1} \geqslant \frac{a_n}{3!} > 0 \Rightarrow \frac{a_{n+1}}{a_n} \geqslant \frac{1}{6}.$$

Repeatedly iterating this lower bound yields

$$\frac{a_{n+k}}{a_n} = \frac{a_{n+k}}{a_{n+k-1}} \cdot \frac{a_{n+k-1}}{a_{n+k-2}} \cdot \frac{a_{n+k-2}}{a_{n+k-3}} \cdots \frac{a_{n+1}}{a_n} \geqslant \frac{1}{6^k}$$

for every integer $k \geqslant 0$. Hence

$$\frac{a_{n+1}}{a_n} = \sum_{k=1}^{n} \frac{a_{n-k+1}}{(2k+1)!a_n} \leqslant \sum_{k=1}^{n} \frac{6^{k-1}}{(2k+1)!} = \frac{1}{6\sqrt{6}} \sum_{k=1}^{n} \frac{(\sqrt{6})^{2k+1}}{(2k+1)!} < \frac{1}{6\sqrt{6}} \sum_{k=1}^{\infty} \frac{(\sqrt{6})^{2k+1}}{(2k+1)!}.$$

This together with the fact that

$$\forall z \in \mathbb{C}: \quad \sinh(z) = \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m+1)!}$$

tells us that

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leqslant \frac{\sinh(\sqrt{6}) - \sqrt{6}}{6\sqrt{6}} \approx 0.224 < 1.$$

So $\sum_{n=1}^{\infty} a_n$ converges by the Ratio Test.

Solution 6 by Michel Bataille, Rouen, France.

We will need the following result: if $f(x) = 2x - \sinh(x)$, then for some $x_0 > 1$ we have $f(x_0) = 0$, $f(x) > 0$ if $0 < x < x_0$ and $f(x) < 0$ if $x > x_0$.
Indeed, f is strictly increasing on $[0, \cosh^{-1}(2)]$ and strictly decreasing on $[(\cosh^{-1}(2), \infty)$ with $f(0) = 0$. Therefore $f(\cosh^{-1}(2)) > 0$ and observing that $\lim_{x\to\infty} f(x) = -\infty$, we have $f(x_0) = 0$ for some $x_0 \in (\cosh^{-1}(2), \infty)$. Also, $f(x) > 0$ for $0 < x < x_0$, $f(x) < 0$ for $x > x_0$ and because $f(1) = 2 - \sinh(1) > 0$, we must have $x_0 > 1$.
Let T be the entire function defined by $T(z) = \sum_{n=1}^{\infty} \frac{z^n}{(2n+1)!}$ and let F be the analytic function defined by $F(z) = \frac{z}{1 - T(z)}$ on the open set $\{z : |T(z)| < 1\}$ containing 0. Then $F(z)$ has a unique expansion as a power series $\sum_{n=1}^{\infty} b_n z^n = \sum_{n=0}^{\infty} z(T(z))^n$ whose radius of convergence is the distance

from 0 to the closest singularity (which is a solution to the equation $T(z) = 1$).

Now, let $r = \dfrac{1+x_0^2}{2}$ so that $1 < r < x_0^2$. We have $T(r) = \displaystyle\sum_{n=1}^{\infty} \frac{r^n}{(2n+1)!} = \frac{\sinh(\sqrt{r})}{\sqrt{r}} - 1$. Clearly, $T(r) > 0$ and also $T(r) < 1$ (since $f(\sqrt{r}) > 0$). It follows that $\displaystyle\sum_{n=0}^{\infty} r(T(r))^n = \sum_{n=1}^{\infty} b_n r^n$ is convergent. Therefore $\rho \geqslant r > 1$ and for $|z| < \rho$, $F(z) = \displaystyle\sum_{n=1}^{\infty} b_n z^n$ writes as

$$(\sum_{n=1}^{\infty} b_n z^n)(\sum_{n=1}^{\infty} \frac{z^n}{(2n+1)!}) = (\sum_{n=1}^{\infty} b_n z^n) - z.$$

Calculating the Cauchy product on the left (the two series on the left are absolutely convergent), the uniqueness of a power series expansion shows that the sequence (b_n) must satisfy $b_1 = 1$ and $b_{n+1} = \dfrac{b_n}{3!} + \dfrac{b_{n-1}}{5!} + \dfrac{b_{n-2}}{7!} + \cdots + \dfrac{b_1}{(2n+1)!}$ for $n \geqslant 1$. We deduce that $b_n = a_n$ for all $n \geqslant 1$, so that $\displaystyle\sum_{n=1}^{\infty} a_n z^n$ is convergent when $|z| < \rho$. In particular $\displaystyle\sum_{n=1}^{\infty} a_n$ converges (since $\rho > 1$).

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

• **5706** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose $a, b, c > 0$. Prove

$$\prod_{cyc} \frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geqslant \left(1+\sqrt[3]{a^2b^2c^2}\right)^3.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We note that

$$(1+ab)(1+ac) - \left(1+a\sqrt{bc}\right)^2 = a\left(\sqrt{b}-\sqrt{c}\right)^2 \geqslant 0.$$

Hence

$$\prod_{cyc} \frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} \geqslant \prod_{cyc} \frac{(1+ab)(1+ac)}{\sqrt{(1+ab)(1+ac)}} = \prod_{cyc} \sqrt{(1+ab)(1+ac)} =$$

$$= (1+ab)(1+bc)(1+ca).$$

Finally, by the AM-GM inequality,

$$(1+ab)(1+bc)(1+ca)-\left(1+\sqrt[3]{a^2b^2c^2}\right)^3=$$

$$=\left(ab+bc+ca-3\sqrt[3]{a^2b^2c^2}\right)+\left(a^2bc+ab^2c+abc^2-3\sqrt[3]{a^4b^4c^4}\right)\geqslant 0.$$

Solution 2 by Daniel Văcaru, Pitești, Romania.

$$\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}}=\frac{1+ab+ac+a^2bc}{1+a\sqrt{bc}}\overset{AM\geqslant GM}{\geqslant}\frac{1+2a\sqrt{bc}+a^2bc}{1+a\sqrt{bc}}=1+a\sqrt{bc}.$$

We have

$$\begin{aligned}\prod_{\text{cyc}}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}}&\geqslant\left(1+a\sqrt{bc}\right)\left(1+b\sqrt{ac}\right)\left(1+c\sqrt{ab}\right)\\&=\left(1+a\sqrt{bc}+b\sqrt{ac}+abc\sqrt{ab}\right)\left(1+c\sqrt{ab}\right)\\&=1+a\sqrt{bc}+b\sqrt{ac}+abc\sqrt{ab}+c\sqrt{ab}+abc\sqrt{ac}+abc\sqrt{cb}+a^2b^2c^2\end{aligned}$$

But

$$a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ab}\overset{\text{AM}\geqslant\text{GM}}{\geqslant}3\sqrt[3]{abc}\sqrt[3]{\left(\sqrt{bc}\right)\left(\sqrt{ac}\right)\left(\sqrt{ab}\right)}=3\sqrt[3]{(abc)^2}$$

and

$$\begin{aligned}abc\sqrt{ab}+abc\sqrt{bc}+abc\sqrt{ca}&=abc\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\\&\overset{\text{AM}\geqslant\text{GM}}{\geqslant}3abc\sqrt[3]{\left(\sqrt{ab}\right)\left(\sqrt{bc}\right)\left(\sqrt{ca}\right)}\\&\geqslant 3abc\sqrt[3]{\left(\sqrt{ab}\right)\left(\sqrt{bc}\right)\left(\sqrt{ca}\right)}\\&=3abc\sqrt[3]{abc}=3\sqrt[3]{(abc)^4}\end{aligned}$$

It follows that by the above results, we have

$$\prod_{\text{cyc}}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}}\geqslant 1+3\sqrt[3]{a^2b^2c^2}+3\sqrt[3]{a^4b^4c^4}+a^2b^2c^2=\left(1+\sqrt[3]{a^2b^2c^2}\right)^3.$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

The AGM inequality gives us

$$\begin{aligned}\prod_{cyc}\frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} &= \prod_{cyc}\frac{1+a(b+c)+a^2bc}{1+a\sqrt{bc}}\\ &\geqslant \prod_{cyc}\frac{1+2a\sqrt{bc}+a^2bc}{1+a\sqrt{bc}}\\ &= \prod_{cyc}\frac{(1+a\sqrt{bc})^2}{1+a\sqrt{bc}}\\ &= \prod_{cyc}(1+a\sqrt{bc})\\ &\geqslant (1+\sqrt[3]{a^2b^2c^2})^3,\end{aligned}$$

where the last inequality is a known consequence of the AGM inequality: $\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geqslant \sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3}$. Equality holds in the original problem if and only if $a = b = c$.

Solution 4 by Michel Bataille, Rouen, France.

Setting $x = bc, y = ca, z = ab$, the inequality to be proved becomes

$$\frac{[(1+x)(1+y)(1+z)]^2}{(1+\sqrt{yz})(1+\sqrt{zx})(1+\sqrt{xy})} \geqslant (1+\sqrt[3]{xyz})^3. \tag{1}$$

Noticing that

$$(1+x)(1+y) = 1+x+y+xy \geqslant 1+2\sqrt{xy}+xy = (1+\sqrt{xy})^2$$

we see that

$$[(1+x)(1+y)(1+z)]^2 = (1+x)(1+y)(1+y)(1+z)(1+z)(1+x) \geqslant (1+\sqrt{xy})^2(1+\sqrt{yz})^2(1+\sqrt{zx})^2$$

so that (1) will follow if we show that

$$(1+\sqrt{yz})(1+\sqrt{zx})(1+\sqrt{xy}) \geqslant (1+\sqrt[3]{xyz})^3. \tag{2}$$

Now, for $u, v, w > 0$ we have, by AM-GM inequality,

$$\frac{1}{[(1+u)(1+v)(1+w)]^{1/3}} + \frac{(uvw)^{1/3}}{[(1+u)(1+v)(1+w)]^{1/3}}$$

$$\leqslant \frac{1}{3}\left(\frac{1}{1+u}+\frac{1}{1+v}+\frac{1}{1+w}\right)+\frac{1}{3}\left(\frac{u}{1+u}+\frac{v}{1+v}+\frac{w}{1+w}\right) = 1,$$

hence $1+(uvw)^{1/3} \leqslant [(1+u)(1+v)(1+w)]^{1/3}$. Letting $u = \sqrt{yz}, v = \sqrt{zx}, w = \sqrt{xy}$, we readily obtain inequality (2).

Solution 5 by Moti Levy, Rehovot, Israel.

The inequality is equivalent to

$$\left((1+ab)(1+bc)(1+ca)\right)^2 \geqslant \left(1+a\sqrt{bc}\right)\left(1+b\sqrt{ca}\right)\left(1+c\sqrt{ab}\right)\left(1+(abc)^{\frac{2}{3}}\right)^3.$$

Hence it suffices to prove the following two inequalities:

$$(1+ab)(1+bc)(1+ca) \geqslant \left(1+a\sqrt{bc}\right)\left(1+b\sqrt{ca}\right)\left(1+c\sqrt{ab}\right) \tag{6}$$

$$(1+ab)(1+bc)(1+ca) \geqslant \left(1+(abc)^{\frac{2}{3}}\right)^3 \tag{7}$$

Let $f(x,y,z)$ be defined as follows:

$$f(x,y,z) := (1+x)(1+y)(1+z), \quad x,y,z>0. \tag{8}$$

The inequality (6) is equivalent to

$$f(x,y,z) \geqslant f\left(\sqrt{xy}, \sqrt{yz}, \sqrt{zx}\right), \tag{9}$$

and inequality (7) is equivalent to

$$f(x,y,z) \geqslant f\left(\sqrt[3]{xyz}, \sqrt[3]{xyz}, \sqrt[3]{xyz}\right). \tag{10}$$

Proof of inequality (6):
Let $x=u^2$, $y=v^2$ and $z=w^2$, then inequality (6) is equivalent to

$$\left(1+u^2\right)\left(1+v^2\right)\left(1+w^2\right) \geqslant (1+uv)(1+vw)(1+wu).$$

After expansion and arranging terms according to their order, it follows that it is enough to prove the following two inequalities:

$$u^2+w^2+v^2 \geqslant uv+uw+vw, \tag{11}$$
$$u^2v^2+u^2w^2+v^2w^2 \geqslant u^2vw+uv^2w+uvw^2. \tag{12}$$

Inequalities (11) and (12) are true by Muirhead's Theorem.

Proof of inequality (7): Let $x=u^3$, $y=v^3$ and $z=w^3$, then inequality (7) is equivalent to

$$\left(1+u^3\right)\left(1+v^3\right)\left(1+w^3\right) \geqslant (1+uvw)^3.$$

After expansion and arranging terms according to their order, it follows that it is enough to prove the following two inequalities:

$$u^3v^3+u^3w^3+v^3w^3 \geqslant 3u^2v^2w^2, \tag{13}$$
$$u^3+v^3+w^3 \geqslant 3uvw. \tag{14}$$

Again, inequalities (13) and (14) are true by Muirhead's Theorem.

Solution 6 by Péter Fülöp, Gyömrő, Hungary.

Rearranging the left hand side (LHS) of the inequality!

$$LHS = \prod_{cyc} \frac{(1+ab)(1+ac)}{1+a\sqrt{bc}} = \prod_{cyc} \frac{(1+ab+ac+a^2bc)}{1+a\sqrt{bc}} = \prod_{cyc} \frac{(1+a(b+c)+a^2bc)}{1+a\sqrt{bc}}$$

Let's apply the AM-GM inequality for $b+c$:

$$LHS \geqslant \prod_{cyc} \frac{(1+2a\sqrt{bc}+a^2bc)}{1+a\sqrt{bc}} = \prod_{cyc} \frac{(1+a\sqrt{bc})^2}{1+a\sqrt{bc}} = \prod_{cyc}(1+a\sqrt{bc})$$

After performing the multiplication we get:

$$LHS \geqslant 1+(abc)^2+abc(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})+\sqrt{abc}(\sqrt{a}+\sqrt{b}+\sqrt{c})$$

Applying the AM-GM inequality for $\sqrt{ab}+\sqrt{bc}+\sqrt{ac}$ and $\sqrt{a}+\sqrt{b}+\sqrt{c}$:

$$LHS \geqslant 1+(abc)^2+abc(3\sqrt[3]{abc})+\sqrt{abc}(3\sqrt[3]{\sqrt{abc}}) =$$

$$1+(abc)^2+3\sqrt[3]{(abc)^4}+3\sqrt[3]{(abc)^2} = RHS$$

It just equals to the right hand side (RHS) of the inequality. The statement is proved!

Solution 7 by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.

By the Cauchy-Schwarz inequality, we have

$$(1+ab)(1+ac) \geqslant (1+a\sqrt{bc})^2,$$

$$(1+bc)(1+ba) \geqslant (1+b\sqrt{ca})^2,$$

$$(1+ca)(1+cb) \geqslant (1+c\sqrt{ab})^2.$$

In addition, by Horner's inequality, we have

$$(1+a\sqrt{bc})(1+b\sqrt{ca})(1+c\sqrt{ab}) \geqslant \left(1+\sqrt[3]{a^2b^2c^2}\right)^3.$$

Multiplying all these inequalities yields the desired inequality. The equality occurs for $a=b=c$.

Also solved by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia; Paolo Perfetti, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy; and the proposer.

• **5707** *Proposed by Narendra Bhandari, Bajura District, Nepal.*

Prove that

$$\int_0^{\frac{\pi}{2}} \Big(\sin x \cdot \operatorname{arctanh}^2(\sin x) - 2\sin x \cdot \operatorname{arctanh}(\sin x)\Big)\, dx = 4G - \pi$$

where $G := \sum_{k=1}^{\infty} (-1)^{k+1}/(2k-1)^2$ is Catalan's constant.

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let $y = \sin x$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \Big(\sin x \cdot \operatorname{arctanh}^2(\sin x) - 2\sin x \cdot \operatorname{arctanh}(\sin x)\Big)\, \mathrm{d}x \\
&= \int_0^1 y \Big(\operatorname{arctanh}^2(y) - 2\operatorname{arctanh}(y)\Big) \frac{\mathrm{d}y}{\sqrt{1-y^2}} \\
&= -\int_0^1 \frac{y\mathrm{d}y \Big(\operatorname{arctanh}^2(-y) - 2\operatorname{arctanh}(-y)\Big)}{\sqrt{1-y^2}} \\
\because \quad & \operatorname{arctanh} y = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)
\end{aligned}$$

Let $\dfrac{1+y}{1-y} = t, \quad y = \dfrac{1-t}{1+t}, \quad \mathrm{d}y = -\dfrac{2}{(1+t)^2}\mathrm{d}t$

Then $I = -\displaystyle\int_0^1 \frac{1-t}{1+t}\left(-\frac{2}{(1+t)^2}\mathrm{d}t\right)\cdot \frac{1}{\sqrt{\frac{4t}{(1+t)^2}}}\left(\frac{1}{4}\ln^2 t - \ln t\right)$

$$\begin{aligned}
&= \int_0^1 \frac{1-t}{(1+t)^2\sqrt{t}}\left(\frac{1}{4}\ln^2 t - \ln t\right)\mathrm{d}t \\
&= 2\int_0^1 \frac{1-x^2}{\big((1+x)^2\big)^2}\Big(\ln^2 x - 2\ln x\Big)\,\mathrm{d}x = I_1 - I_2
\end{aligned}$$

in which

$$I_1 = 2\int_0^1 \frac{(1-x^2)\ln^2 x}{\big((1+x)^2\big)^2}\mathrm{d}x = 4G$$

$$I_2 = 4\int_0^1 \frac{(1-x^2)\ln x}{\left((1+x)^2\right)^2}\mathrm{d}x = \pi$$

$$\therefore \quad I = 4G - \pi$$

NOTE 1:

$$\because \quad \int_0^1 \frac{(1-x^2)\ln^2 x}{\left((1+x)^2\right)^2}\mathrm{d}x$$

$$= -i\left[Li_2(ix) - Li_2(-ix) + \ln x\left(\frac{ix\ln x}{1+x^2} + \ln(1-ix) - \ln(1+ix)\right)\right]\Big|_0^1$$

$$= -i\left[Li(i) - Li_2(-i)\right]$$

$$= -i\left[\left(-\frac{\pi^2}{48} + iG\right) - \left(-\frac{\pi^2}{48} - iG\right)\right]$$

$$= -i(2iG) = 2G$$

$$\therefore \quad I_1 = 2 \times 2G = 4G$$

NOTE 2:

$$\because \quad \int_0^1 \frac{(1-x^2)\ln x}{\left((1+x)^2\right)^2}\mathrm{d}x$$

$$= \left[\frac{x\ln x}{x^2+1} + \frac{1}{2}i\ln(i-x) - \frac{1}{2}i\ln(i+x)\right]\Big|_0^1$$

$$= -\frac{1}{2}i\left[\ln\left(\frac{i-1}{i+1}\right)\right] = \frac{-i}{2} \times \left(\frac{i\pi}{2}\right) = \frac{\pi}{4}$$

$$\therefore \quad I_2 = 4 \times \frac{\pi}{4} = \pi$$

Solution 2 by Péter Fülöp, Gyömrő, Hungary.

Let's perform the following substitution: $t = \sin(x)$.

$$I = \int_0^1 \frac{t(\tanh^{-1}(t))^2}{\sqrt{1-t^2}} - \frac{2t\tanh^{-1}(t)}{\sqrt{1-t^2}}dt$$

Using that $\tanh^{-1}(t) = \frac{1}{2}\ln(\frac{1+t}{1-t})$ and performing the integration by parts we get:

$$I = \int_0^1 \underbrace{\frac{1}{2}\frac{t}{\sqrt{1-t^2}}}_{u'} \underbrace{[\frac{1}{2}\ln^2\left(\frac{1+t}{1-t}\right) - 2\ln\left(\frac{1+t}{1-t}\right)]}_{v} dt$$

The u', v are given as indicated in the above integral, follows that:

$$u = -\frac{1}{2}\sqrt{1-t^2} \text{ and } v' = \frac{2}{1-t^2}(\ln\left(\frac{1+t}{1-t}\right) - 2)$$

$$I = \underbrace{-\frac{1}{2}\sqrt{1-t^2}\Big[\frac{1}{2}\ln^2\left(\frac{1+t}{1-t}\right) - 2\ln\left(\frac{1+t}{1-t}\right)\Big]\Big|_{t=0}^{1}}_{0} + \int_0^1 \frac{1}{\sqrt{1-t^2}}(\ln\left(\frac{1+t}{1-t}\right) - 2)dt$$

$$I = \int_0^1 \frac{1}{\sqrt{1-t^2}}\ln\left(\frac{1+t}{1-t}\right) - 2\frac{1}{\sqrt{1-t^2}}dt$$

Using the $t = \sin(r)$ substitution we get the followings:

$$I = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin(r)}{1-\sin(r)}\right) - 2dr = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin(r)}{1-\sin(r)}\right)dr - \pi$$

Because of the double-angle formulae we can write that $\sin(r) = 2\sin(\frac{r}{2})\cos(\frac{r}{2})$, we get:

$$\frac{1+\sin(r)}{1-\sin(r)} = \frac{(\cos(\frac{r}{2}) + \sin(\frac{r}{2}))^2}{(\cos(\frac{r}{2}) - \sin(\frac{r}{2}))^2} = \frac{\sin^2(u)}{\cos^2(u)}$$

where $u = \frac{\pi}{4} + \frac{r}{2}$, take it account in the integral we have:

$$I = 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\tan(u))du - \pi$$

Catalan's constant equals to $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\tan(u))du$, can be proved as follows:

By the substitution of $x = \tan(u)$, the integral becomes $\int_1^{\infty} \frac{\ln(x)}{1+x^2}dx$. Then, furthermore, by the substitution $x = \frac{1}{y}$, the integral becomes -$\int_0^1 \frac{\ln(y)}{1+y^2}dy$.

Using the fact $\ln(y) = \frac{d(y^a)}{dy}\Big|_{a=0}$ and $\frac{1}{1+y^2} = \sum_{k=0}^{\infty}(-y^2)^k$ we have:

$$-\frac{d}{da}\int_0^1 \sum_{k=0}^{\infty}(-1)^k y^{2k+a}dy\Big|_{a=0} = -\frac{d}{da}\sum_{k=0}^{\infty}(-1)^k \int_0^1 y^{2k+a}dy\Big|_{a=0}$$

After performing the integration,

$$-\sum_{k=0}^{\infty}\frac{d}{da}\frac{(-1)^k}{(2k+a+1)}\Bigg|_{a=0} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = G$$

The original integral equals to $I = 4G - \pi$. The statement is proved!

Solution 3 by Paolo Perfetti, dipartimento di matematica, Università di "Tor Vergata", Roma, Italy.

Let's integrate by parts

$$\int_0^{\frac{\pi}{2}} \sin x \cdot \operatorname{arctanh}^2(\sin x)dx =$$
$$= -\cos x \cdot \operatorname{arctanh}^2(\sin x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot 2\operatorname{arctanh}(\sin x)\cdot\frac{1}{\cos x}dx$$

$$\arctan(\sin x) = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x}, \quad \lim_{x\to 0}\cos x\cdot\operatorname{arctanh}(\sin x) = 1\cdot 0 = 0$$

$$\lim_{x\to\frac{\pi}{2}}\cos x\cdot\operatorname{arctanh}(\sin x) = \frac{1}{2}\lim_{x\to\frac{\pi}{2}}\cos x(\ln(1+\sin x) - \ln(1-\sin x)) =$$
$$= \frac{-1}{2}\lim_{x\to\frac{\pi}{2}}\sin(\frac{\pi}{2}-x)\ln\left(1-\cos(\frac{\pi}{2}-x)\right) = \frac{-1}{2}\lim_{y\to 0}\sin y\ln(1-\cos y) = 0$$

$$\int_0^{\frac{\pi}{2}} \sin x\cdot\operatorname{arctanh}^2(\sin x)dx = \int_0^{\frac{\pi}{2}} 2\operatorname{arctanh}(\sin x)dx \underbrace{=}_{\sin x = \tanh t} \int_0^{\infty}\frac{2y}{\cosh y}dy$$

$$\int_0^{\infty} \frac{x}{\cosh x}dx = \lim_{R\to\infty} \left(2x\arctan(e^x)\Big|_0^R - 2\int_0^R \arctan(e^x)dx \right) =$$

$$= \lim_{R\to\infty} \left(2R\frac{\pi}{2} - 2\int_1^{e^R} \frac{\arctan y}{y}dy \right) =$$

$$= \lim_{R\to\infty} \left(2R\frac{\pi}{2} - 2\int_0^{e^R} \frac{\arctan y}{y}dy \right) + 2\int_0^1 \frac{\arctan y}{y}dy =$$

$$= \lim_{R\to\infty} \left(2R\frac{\pi}{2} - 2\left(\ln y \arctan(e^y)\Big|_0^{e^R} - \int_0^{e^R} \frac{\ln y}{1+y^2}dy \right) \right) + 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} =$$

$$= \lim_{R\to\infty} \pi R - 2R\frac{\pi}{2} - \int_0^{\infty} \frac{\ln y}{1+y^2}dy + 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

because

$$\int_0^{\infty} \frac{\ln y}{1+y^2}dy \underbrace{=}_{y=1/x} \int_0^{\infty} \frac{-\ln x}{1+x^2}dx = 0$$

By the way through $\text{arctanh} x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$, $|x| < 1$, we could have written also

$$\int_0^{\frac{\pi}{2}} 2\text{arctanh}(\sin x)dx = \int_0^{\frac{\pi}{2}} 2\sum_{k=0}^{\infty} \frac{(\sin x)^{2k+1}}{2k+1}dx = \sum_{k=0}^{\infty} \int_0^{\frac{\pi}{2}} 2\frac{(\sin x)^{2k+1}}{2k+1}dx$$

This is allowed by

$$\int_0^{\frac{\pi}{2}} 2\sum_{k=0}^{\infty} \frac{(\sin x)^{2k+1}}{2k+1}dx \leqslant \int_0^{\frac{\pi}{2}} 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}dx = \pi\sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+2)} < \infty$$

and the positivity of all the terms.

$$\sum_{k=0}^{\infty} \int_0^{\frac{\pi}{2}} 2\frac{(\sin x)^{2k+1}}{2k+1}dx \underbrace{=}_{\sin x = t} \sum_{k=0}^{\infty} \frac{2}{2k+1} \int_0^1 \frac{t^{2k+1}\,dt}{\sqrt{1-t^2}}$$

$$\underbrace{=}_{t=\sqrt{y}} \sum_{k=0}^{\infty} \frac{1}{2k+1} \int_0^1 y^k(1-y)^{-1/2}dy = \sum_{k=0}^{\infty} \frac{1}{2k+1}\beta(k+1, \frac{1}{2}) =$$

$$= \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{\Gamma(k+1)\Gamma(\frac{1}{2})}{\Gamma(k+\frac{3}{2})} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{k!\sqrt{\pi}}{(k+\frac{1}{2})\Gamma(k+\frac{1}{2})} = \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2} \frac{(k!)^2 4^k \sqrt{\pi}}{2\sqrt{\pi}(2k)!} =$$

$$= 4G$$

whence $G = \sum_{k=0}^{\infty} \frac{(k!)^2 4^k}{2(2k+1)^2(2k)!}$ (a relation that can be found in https://viterbi-web.usc.edu/ adamchik/articles/catalan/catalan.htm where many other representations of G can be found)

$$\int_0^{\frac{p}{i}2} 2\sin x \cdot \operatorname{arctanh}(\sin x)dx \underbrace{=}_{\sin x=\tanh t} \int_0^\infty \frac{2t\sinh t\,dt}{\cosh^2 t} = \frac{-2t}{\cosh t}\Big|_0^\infty + \int_0^\infty \frac{2dt}{\cosh t} =$$

$$= 4\arctan e^t\Big|_0^\infty = 2\pi - \pi = \pi$$

By employing the power series of $\operatorname{arctanh}(x)$ we would obtain

$$\sum_{k=0}^\infty \frac{1}{2k+1}\int_0^1 y^{k+1/2}(1-y)^{-1/2}dy = \sum_{k=0}^\infty \frac{1}{2k+1}\beta(k+\frac{3}{2},\frac{1}{2}) =$$

$$= \sum_{k=0}^\infty \frac{1}{2k+1}\frac{\Gamma(k+\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(k+2)} = \sum_{k=0}^\infty \frac{1}{2k+1}\frac{(k+\frac{1}{2})\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2})}{(k+1)!} =$$

$$= \sum_{k=0}^\infty \frac{\pi(2k)!}{2(k+1)!4^k k!} = \sum_{k=0}^\infty \frac{\pi(2k)!}{2(k+1)!4^k k!} = \pi$$

In fact

$$\sum_{k=0}^\infty \binom{2k}{k}\frac{1}{4^k(k+1)} = 4\int_0^{1/4}\sum_{k=0}^\infty \binom{2k}{k} y^k dy = \int_0^1 \frac{4dy}{\sqrt{1-4y}} = -2\sqrt{1-4y}\Big|_0^{1/4} = 2$$

Solution 4 by Moti Levy, Rehovot, Israel.

Let

$$I := \int_0^{\frac{\pi}{2}} \left(\sin(x)\left(\tanh^{-1}\left(\sin(x)\right)\right)^2 - 2\sin(x)\tanh^{-1}\left(\sin(x)\right)\right)dx.$$

Integrating by parts gives,

$$\int_0^{\frac{\pi}{2}} \sin(x)\left(\tanh^{-1}\left(\sin(x)\right)\right)^2 dx = -\cos(x)\left(\tanh^{-1}\left(\sin(x)\right)\right)^2\Big]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} \tanh^{-1}\left(\sin(x)\right)dx$$

$$= 2\int_0^{\frac{\pi}{2}} \tanh^{-1}\left(\sin(x)\right)dx.$$

Hence

$$I = 2\int_0^{\frac{\pi}{2}} \tanh^{-1}\left(\sin(x)\right)dx - 2\int_0^{\frac{\pi}{2}} \sin(x)\tanh^{-1}\left(\sin(x)\right)dx$$

Now the first integral is

$$2\int_0^{\frac{\pi}{2}} \tanh^{-1}\left(\sin(x)\right)dx = 2\int_0^{\frac{\pi}{2}} \tanh^{-1}\left(\cos(x)\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\cos(x)}{1-\cos(x)}\right)dx = -2\int_0^{\frac{\pi}{2}} \ln\left(\tan\left(\frac{t}{2}\right)\right)dt$$

$$= -4\int_0^{\frac{\pi}{4}} \ln\left(\tan(y)\right)dy.$$

The following Fourier series expansions are well known:

$$-\ln\left(\sin(x)\right) = \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} + \ln(2),$$
$$-\ln\left(\cos(x)\right) = \sum_{k=1}^{\infty} (-1)^k \frac{\cos(2kx)}{k} + \ln(2).$$

Thus the Fourier series expansion of $\ln\left(\tan(x)\right)$ is

$$\begin{aligned}\ln\left(\tan(x)\right) &= -\sum_{k=1}^{\infty} \frac{\cos(2kx)}{k} + \sum_{k=1}^{\infty} (-1)^k \frac{\cos(2kx)}{k} \\ &= \sum_{k=1}^{\infty} \frac{-1+(-1)^k}{k} \cos(2kx) = -\sum_{m=0}^{\infty} \frac{2}{2m+1} \cos\left(2(2m+1)x\right).\end{aligned}$$

Changing the order of summation and integration we get,

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln\left(\tan(y)\right) dy &= -\sum_{m=0}^{\infty} \frac{2}{2m+1} \int_0^{\frac{\pi}{4}} \cos\left(2(2m+1)x\right) dx \\ &= -\sum_{m=0}^{\infty} \frac{2}{2m+1} \frac{(-1)^m}{2(2m+1)} = -\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = -\mathbf{G}.\end{aligned}$$

It follows that the value of the first integral is $4\mathbf{G}$.

Now for the second integral, we use the orthogonality property

$$\int_0^{\frac{\pi}{4}} \cos(2t)\cos\left(2(2m+1)t\right) dt = \begin{cases} \frac{\pi}{8} & if \quad m=0, \\ 0 & otherwise. \end{cases}$$

$$\begin{aligned}2\int_0^{\frac{\pi}{2}} \sin(x)\tanh^{-1}\left(\sin(x)\right) dx &= \int_0^{\frac{\pi}{2}} \cos(x) \ln\left(\frac{1+\cos(x)}{1-\cos(x)}\right) dx \\ &= -2\int_0^{\frac{\pi}{2}} \cos(t) \ln\left(\tan\left(\frac{t}{2}\right)\right) dt \\ &= -4\int_0^{\frac{\pi}{4}} \cos(2t) \ln\left(\tan(t)\right) dt \\ &= 8\int_0^{\frac{\pi}{4}} \cos(2t) \sum_{m=0}^{\infty} \frac{1}{2m+1} \cos\left(2(2m+1)t\right) dt \\ &= 8\sum_{m=0}^{\infty} \frac{1}{2m+1} \int_0^{\frac{\pi}{4}} \cos(2t)\cos\left(2(2m+1)t\right) dt \\ &= \pi.\end{aligned}$$

We conclude that $I = 4\mathbf{G} - \pi$.

Solution 5 by Michel Bataille, Rouen, France.

Let $a \in (0, \frac{\pi}{2})$ and let

$$I(a) = \int_0^a \Big(\sin x \cdot \operatorname{arctanh}^2(\sin x) - 2\sin x \cdot \operatorname{arctanh}(\sin x)\Big)\, dx.$$

Integration by parts gives

$$\begin{aligned}\int_0^a (-\sin x)\operatorname{arctanh}(\sin x)\, dx &= [(\cos x)\operatorname{arctanh}(\sin x)]_0^a - \int_0^a \cos x \cdot \frac{1}{\cos x}\, dx \\ &= (\cos a)\operatorname{arctanh}(\sin a) - a\end{aligned}$$

and

$$\int_0^a (\sin x)\cdot \operatorname{arctanh}^2(\sin x)\, dx = (-\cos a)\operatorname{arctanh}^2(\sin a) + 2\int_0^a \operatorname{arctanh}(\sin x)\, dx.$$

Since

$$\int_0^a \operatorname{arctanh}(\sin x)\, dx = a \operatorname{arctanh}(\sin a) - \int_0^a \frac{x}{\cos x}\, dx$$

we have

$$I(a) = (-\cos a)\operatorname{arctanh}^2(\sin a) + 2(a + \cos a)\operatorname{arctanh}(\sin a) - 2a - 2\int_0^a \frac{x}{\cos x}\, dx.$$

The change of variables $x = \frac{\pi}{2} - u$ yields

$$\begin{aligned}\int_0^a \frac{x}{\cos x}\, dx = \frac{\pi}{2}\int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \frac{du}{\sin u} - \int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \frac{u}{\sin u}\, du &= \frac{\pi}{2}[-\operatorname{arctanh}(\cos u)]_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \frac{u}{\sin u}\, du \\ &= \frac{\pi}{2}\operatorname{arctanh}(\sin a) - \int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \frac{u}{\sin u}\, du\end{aligned}$$

Gathering the results, we obtain $I(a) = J(a) + K(a)$ where

$$J(a) = (2\cos a + 2a - \pi)\operatorname{arctanh}(\sin a) - (\cos a)\operatorname{arctanh}^2(\sin a), \qquad K(a) = 2\int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \frac{u}{\sin u}\, du - 2a.$$

We have

$$\lim_{a\to\pi/2} K(a) = 2\int_0^{\pi/2} \frac{u}{\sin u}\, du - \pi = 4G - \pi$$

(since $\int_0^{\pi/2} \frac{u}{\sin u}\, du = 2\int_0^1 \frac{\arctan x}{x}\, dx = 2G$ by the change of variables $x = \tan(u/2)$). It just remains to prove that $\lim_{a\to\pi/2} J(a) = 0$.

As $a \to \frac{\pi}{2}$, we have

$$(\cos a)\operatorname{arctanh}(\sin a) = \frac{\cos a}{2}\ln(1-\sin a) + o(1) = \frac{(1+\sin a)^{1/2}}{2}\cdot(1-\sin a)^{1/2}\ln(1-\sin a) + o(1).$$

Since $\lim_{X\to 0} X^{1/2}\ln X = 0$, we see that $\lim_{a\to\frac{\pi}{2}}(\cos a)\text{arctanh}(\sin a) = 0$.
Similarly, we have $\lim_{a\to\frac{\pi}{2}}(\cos a)\text{arctanh}^2(\sin a) = 0$.
Finally, setting $b = \dfrac{\pi}{2} - a$, we have

$$(\pi - 2a)\ln(1-\sin a) = 2b\ln(1-\cos b) \;\sim\; 2\sqrt{2}(1-\cos b)^{1/2}\ln(1-\cos b) \quad \text{as } b \to 0,$$

hence $\lim_{a\to\frac{\pi}{2}}(\pi - 2a)\text{arctanh}(\sin a) = 0$. Thus $\lim_{a\to\frac{\pi}{2}} J(a) = 0$ and we are done.

Solution 6 by G. C. Greubel, Newport News, VA.

This solution makes use of

$$\begin{aligned}\tanh^{-1}(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}\\ B(x,y) &= \int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt,\end{aligned}$$

which are the inverse hyperbolic tangent and Beta function. Now consider the integral

$$\begin{aligned}I_2 &= \int_0^{\pi/2} \sin x\,\tanh^{-1}(\sin x)\,dx\\ &= \sum_{n=0}^{\infty} \frac{1}{2n+1}\int_0^{\pi/2} \sin^{2n+2}(x)\,dx\\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2n+1}\,B\left(n+\frac{3}{2},\frac{1}{2}\right)\\ &= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(\frac{3}{2}\right)}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\,(1)_n}{n!\,(2)_n}\\ &= \frac{\pi}{4}\,{}_2F_1\left(\frac{1}{2},1;2;1\right) = \frac{\pi}{2}.\end{aligned}$$

This leads to the consideration of the integral

$$I_1 = \int_0^{\pi/2} \sin x\,\text{arctanh}^2(\sin x)\,dx.$$

Let $t = \cos x$ to obtain the integral as

$$I_1 = \int_0^1 \text{arctanh}^2(\sqrt{1-t^2})\,dt.$$

Now let $t = \text{sech}\,u$ to obtain

$$I_1 = \int_0^{\infty} u^2\,\text{sech}(u)\,\tanh(u)\,du.$$

Using integration by parts this becomes

$$I_1 = 2\int_0^\infty u\ \operatorname{sech}(u)\,du = 4\,G,$$

where G is Catalan's constant. Now looking at the integral proposed, namely,

$$\begin{aligned} I &= \int_0^{\pi/2} \Big(\sin x\ \operatorname{arctanh}^2(\sin x) - 2\ \sin x\ \operatorname{arctanh}(\sin x)\Big)\ dx \\ &= I_1 - 2\,I_2 \\ &= 4\,G - \pi \end{aligned}$$

which gives the desired result.

Solution 7 by David A. Huckaby, Angelo State University, San Angelo, TX.

First note that $\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, with domain $(-1, 1)$. (See `https://en.wikipedia.org/wiki/Inverse_hyperbolic_functions`.) So the given integral is

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \Big(\sin x\cdot \operatorname{arctanh}^2(\sin x) - 2\sin x\cdot\operatorname{arctanh}(\sin x)\Big)\,dx \\ &= \lim_{b\to\frac{\pi}{2}}\int_0^b \left(\sin x\cdot\left[\frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]^2 - 2\sin x\cdot\left[\frac{1}{2}\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]\right)dx \\ &= \frac{1}{4}\lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\cdot\left[\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]^2 dx \qquad (15) \\ &\quad - \lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\cdot\left[\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right] dx \qquad (16) \end{aligned}$$

In computing both integrals in (15) and (16) we will use the following trigonometric identities:

$$\begin{aligned} \frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \left(\frac{1+\sin x}{\cos x}\right)^2 &= (\sec x + \tan x)^2 \\ &= \tan^2\left(\frac{x}{2}+\frac{\pi}{4}\right). \qquad (17) \end{aligned}$$

(The final identity is given here: `https://en.wikipedia.org/wiki/List_of_trigonometric_identities`.) Applying the first trig identity, the integral in (16) becomes

$$\begin{aligned} &\lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\cdot\left[\ln\left(\frac{1+\sin x}{\cos x}\right)^2\right] dx \\ &= 2\lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\cdot\ln(1+\sin x)\,dx - 2\lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\cdot\ln(\cos x)\,dx \qquad (18) \end{aligned}$$

Integrating by parts with $u = \ln(1+\sin x)$ and $dv = \sin x\,dx$, so that $v = -\cos x$ and $du = \dfrac{\cos x}{1+\sin x}\,dx$, the first term in (18) becomes

$$\begin{aligned}
&2\lim_{b\to\frac{\pi}{2}}\Big[-\cos x\cdot\ln(1+\sin x)\Big]_0^b + 2\lim_{b\to\frac{\pi}{2}}\int_0^b \frac{\cos^2 x}{1+\sin x}\,dx\\
&= 2(0) + 2\lim_{b\to\frac{\pi}{2}}\int_0^b \frac{1-\sin^2 x}{1+\sin x}\,dx\\
&= 2\lim_{b\to\frac{\pi}{2}}\int_0^b (1-\sin x)\,dx\\
&= 2\lim_{b\to\frac{\pi}{2}}\left[x+\cos x\right]_0^b\\
&= 2(\frac{\pi}{2}-1) = \pi - 2.
\end{aligned}$$

Integrating by parts with $u = \ln(\cos x)$ and $dv = \sin x\,dx$, so that $v = -\cos x$ and $du = -\dfrac{\sin x}{\cos x}\,dx$, the second term in (18) becomes

$$\begin{aligned}
&2\lim_{b\to\frac{\pi}{2}}\Big[-\cos x\cdot\ln(\cos x)\Big]_0^b - 2\lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\,dx\\
&= 2(0) + 2\lim_{b\to\frac{\pi}{2}}\left[\cos x\right]_0^b\,dx\\
&= 2(-1) = -2.
\end{aligned}$$

So the difference of the two integrals (18), that is, the integral in (16), is $\pi - 2 - (-2) = \pi$. Note that in the preceeding calculation, $\lim_{b\to\frac{\pi}{2}}\Big[-\cos b\cdot\ln(\cos b)\Big]$ yields the indeterminate form $0\cdot\infty$. This limit is indeed 0, as an application of L'Hôpital's Rule shows after rewriting the limit:

$$\lim_{b\to\frac{\pi}{2}}\Big[-\cos b\cdot\ln(\cos b)\Big] = \lim_{b\to\frac{\pi}{2}}\frac{\ln(\cos b)}{-\frac{1}{\cos b}} = \lim_{b\to\frac{\pi}{2}}\frac{-\frac{1}{\cos b}\sin b}{\frac{1}{\cos^2 b}\sin b} = \lim_{b\to\frac{\pi}{2}} -\cos b = 0.$$

Having found that the integral in (16) is equal to π, we turn now to the first integral (15), including the factor of $\dfrac{1}{4}$. Integrating by parts with $u = \left[\ln\dfrac{1+\sin x}{1-\sin x}\right]^2$ and $dv = \sin x\,dx$, so that $v = -\cos x$ and $du = 2\ln\left[\dfrac{1+\sin x}{1-\sin x}\right]\dfrac{1-\sin x}{1+\sin x}\dfrac{d}{dx}\left(\dfrac{1+\sin x}{1-\sin x}\right) = 2\ln\left[\dfrac{1+\sin x}{1-\sin x}\right]\dfrac{1-\sin x}{1+\sin x}\dfrac{2\cos x}{(1-\sin x)^2}\,dx = 4\ln\left[\dfrac{1+\sin x}{1-\sin x}\right]\dfrac{1}{\cos x}\,dx$, the first integral (15) becomes

$$\frac{1}{4}\lim_{b\to\frac{\pi}{2}}\int_0^b \sin x\cdot\left[\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]^2\,dx$$

$$=\frac{1}{4}\lim_{b\to\frac{\pi}{2}}\left[-\cos x\cdot\left[\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]^2\right]_0^b+\frac{1}{4}\lim_{b\to\frac{\pi}{2}}\int_0^b 4\ln\left(\frac{1+\sin x}{1-\sin x}\right)\,dx$$

$$=\frac{1}{4}\lim_{b\to\frac{\pi}{2}}\left[-\cos x\cdot\left[\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]^2\right]_0^b+\lim_{b\to\frac{\pi}{2}}\int_0^b \ln\left(\frac{1+\sin x}{1-\sin x}\right)\,dx. \qquad (19)$$

The first term in (19) evaluates to 0. Note that $\lim_{b\to\frac{\pi}{2}}\left(-\cos b\cdot\left[\ln\left(\frac{1+\sin x}{1-\sin x}\right)\right]^2\right)$ yields the indeterminate form $0\cdot\infty$. Rewriting and using L'Hôpital's Rule:

$$\lim_{b\to\frac{\pi}{2}}\frac{\left[\ln\left(\frac{1+\sin b}{1-\sin b}\right)\right]^2}{-\frac{1}{\cos b}}=\lim_{b\to\frac{\pi}{2}}\frac{4\ln\frac{1+\sin b}{1-\sin b}\cdot\frac{1}{\cos b}}{\frac{1}{\cos^2 b}}=\lim_{b\to\frac{\pi}{2}}\frac{4\ln\frac{1+\sin b}{1-\sin b}\cdot}{\frac{1}{\cos b}}$$

$$=\lim_{b\to\frac{\pi}{2}}\frac{4\cdot\frac{1-\sin b}{1+\sin b}\cdot\frac{2\cos b}{(1-\sin b)^2}}{-\frac{1}{\cos^2 b}}=\lim_{b\to\frac{\pi}{2}}-8\cos b=0.$$

To evaluate the second term in (19), we rewrite the integrand using the final trigonometric identity given in (17) above:

$$\lim_{b\to\frac{\pi}{2}}\int_0^b \ln\left(\frac{1+\sin x}{1-\sin x}\right)\,dx=\lim_{b\to\frac{\pi}{2}}\int_0^b \ln\left(\tan^2\left[\frac{x}{2}+\frac{\pi}{4}\right]\right)\,dx$$

$$=2\lim_{b\to\frac{\pi}{2}}\int_0^b \ln\left(\tan\left[\frac{x}{2}+\frac{\pi}{4}\right]\right)\,dx$$

With the substitution $u=\frac{x}{2}+\frac{\pi}{4}$, this becomes

$$2\lim_{b\to\frac{\pi}{2}}\int_{\frac{\pi}{4}}^b \ln\left(\tan u\right)\,(2\,du)=4\lim_{b\to\frac{\pi}{2}}\int_{\frac{\pi}{4}}^b \ln\left(\tan u\right)\,du$$

$$=4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\tan u\right)\,du$$

Since the final integral above is equal to Catalan's constant G (See `https://en.wikipedia.org/wiki/Catalan\'s\_constant`), the expression above is equal to $4G$. Since the expression (15) is equal to $4G$, and the integral in (16) is equal to π, the given integral is indeed equal to $4G-\pi$.

Solution 8 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

By integration by parts with $u = \operatorname{arctanh}(\sin x)$ and $dv = \sin x\, dx$,

$$\int_0^{\pi/2} \sin x \cdot \operatorname{arctanh}(\sin x)\, dx = -\cos x \cdot \operatorname{arctanh}(\sin x)\Big|_0^{\pi/2} + \int_0^{\pi/2} dx = \frac{\pi}{2};$$

by integration by parts with $u = \operatorname{arctanh}^2(\sin x)$ and $dv = \sin x\, dx$,

$$\begin{aligned}\int_0^{\pi/2} \sin x \cdot \operatorname{arctanh}^2(\sin x)\, dx &= -\cos x \cdot \operatorname{arctanh}^2(\sin x)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} \operatorname{arctanh}(\sin x)\, dx \\ &= \int_0^{\pi/2} \ln\left(\frac{1+\sin x}{1-\sin x}\right) dx \\ &= \int_0^{\pi/2} \ln\left(\frac{1+\cos x}{1-\cos x}\right) dx \\ &= 4G.\end{aligned}$$

For this last line, see [1, equation (10)] or [2]. Thus,

$$\int_0^{\pi/2} \Big(\sin x \cdot \operatorname{arctanh}^2(\sin x) - 2\sin x \operatorname{arctanh}(\sin x)\Big)\, dx = 4G - \pi.$$

Solution 9 by Albert Stadler, Herrliberg, Switzerland.

We have $\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for |x|<1. Hence $\operatorname{arctanh}\left(\frac{2y}{1+y^2}\right) = \ln\left(\frac{1+y}{1-y}\right)$ for |y|<1.
The change of variables t = sin x, dt = cos x dx gives

$$I := \int_0^{\frac{\pi}{2}} \Big(\sin x \operatorname{arctanh}^2(\sin x) - 2\sin x \operatorname{arctanh}(\sin x)\Big)\, dx =$$

$$= \int_0^1 \frac{1}{\sqrt{1-t^2}} \Big(t\operatorname{arctanh}^2(t) - 2t\operatorname{arctanh}(t)\Big)\, dt$$

We rationalize the integrand and perform the change of variables $t = \frac{2u}{1+u^2}$, $dt = \frac{2\left(1-u^2\right)}{\left(1+u^2\right)^2}du$. We get

$$I = \int_0^1 \frac{1+u^2}{1-u^2}\left(\frac{2u}{1+u^2}\operatorname{arctanh}^2\left(\frac{2u}{1+u^2}\right) - 2\frac{2u}{1+u^2}\operatorname{arctanh}\left(\frac{2u}{1+u^2}\right)\right)\frac{2\left(1-u^2\right)}{\left(1+u^2\right)^2}du =$$

$$= 4\int_0^1 \left(u\text{arctanh}^2\left(\frac{2u}{1+u^2}\right) - 2u\text{arctanh}\left(\frac{2u}{1+u^2}\right) \right) \frac{1}{(1+u^2)^2}du =$$

$$= 4\int_0^1 \left(\ln^2\left(\frac{1+u}{1-u}\right) - 2\ln\left(\frac{1+u}{1-u}\right) \right) \frac{u}{(1+u^2)^2}du.$$

Integration by parts yields

$$I = \left(\ln^2\left(\frac{1+u}{1-u}\right) - 2\ln\left(\frac{1+u}{1-u}\right) \right) \left(1 - \frac{2}{1+u^2}\right)\Bigg|_{u=0}^{u=1} + \int_0^1 \left(\frac{4\ln\left(\frac{1+u}{1-u}\right)}{1-u^2} - \frac{4}{1-u^2} \right) \frac{1-u^2}{1+u^2}du =$$

$$= \int_0^1 4\ln\left(\frac{1+u}{1-u}\right)\frac{1}{1+u^2}du - \int_0^1 \frac{4}{1+u^2}du = 4G - \pi,$$

since

$$\int_0^1 \ln\left(\frac{1+u}{1-u}\right)\frac{1}{1+u^2}du \overbrace{=}^{u=\frac{1-v}{1+v},\quad du=-\frac{2}{(1+v)^2}dv} -\int_0^1 \ln(v)\frac{1}{1+\left(\frac{1-v}{1+v}\right)^2}\frac{2}{(1+v)^2}dv =$$

$$= -\int_0^1 \ln(v)\frac{1}{1+v^2}dv = G$$

by a known integral representation of Catalan's constant; see for instance:

https://en.wikipedia.org/wiki/Catalan%27s_constant

Also solved by the proposer.

• **5708** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Solve the differential equation

$$y\sqrt{y^2+z^2}\,dz + z\sqrt{y^2+z^2}\,dy = \frac{y\,(x\,dy - y\,dx) + z\,(x\,dz - z\,dx)}{x^2+y^2+z^2}.$$

Solution 1 by Moti Levy, Rehovot, Israel.

Collecting terms, we get total differential equation:

$$\left(y\sqrt{y^2+z^2} - \frac{xz}{x^2+y^2+z^2}\right)dz + \left(z\sqrt{y^2+z^2} - \frac{xy}{x^2+y^2+z^2}\right)dy + \frac{y^2+z^2}{x^2+y^2+z^2}dx = 0.$$

After multiplication by the factor $\frac{1}{\sqrt{y^2+z^2}}$ we get an integrable total differential,

$$\left(y-\frac{xz}{\left(x^2+y^2+z^2\right)\sqrt{y^2+z^2}}\right)dz+\left(z-\frac{xy}{\left(x^2+y^2+z^2\right)\sqrt{y^2+z^2}}\right)dy+\frac{\sqrt{y^2+z^2}}{x^2+y^2+z^2}dx=0.$$

Now we use cylindrical coordinates:

$$\begin{aligned} y &= r\cos(t) \\ z &= r\sin(t) \\ x &= x \end{aligned}$$

$$\begin{aligned} dy &= \cos(t)\,dr - r\sin(t)\,dt \\ dz &= \sin(t)\,dr + r\cos(t)\,dt \\ dx &= dx \end{aligned}$$

The differential equation in cylindrical coordinates becomes:

$$\left(r\sin(2t)-\frac{x}{x^2+r^2}\right)dr+r^2\cos(2t)\,dt+\frac{r}{x^2+r^2}dx=0$$

The function $F(r,t,x)$

$$F(r,t,x)=\frac{r^2}{2}\sin(2t)+\arctan\left(\frac{x}{r}\right)$$

satisfies the following

$$\begin{aligned} \frac{\partial F(r,t,x)}{\partial r} &= r\sin(2t)-\frac{x}{x^2+r^2}, \\ \frac{\partial F(r,t,x)}{\partial t} &= r^2\cos(2t), \\ \frac{\partial F(r,t,x)}{\partial z} &= \frac{r}{x^2+r^2}. \end{aligned}$$

Hence the solution is

$$\frac{r^2}{2}\sin(2t)+\arctan\left(\frac{x}{r}\right)=0.$$

The solution in x, y, z is

$$yz+\arctan\left(\frac{x}{\sqrt{y^2+z^2}}\right)=0.$$

Also solved by the proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

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3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica ($\longleftarrow$ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣