Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at the Department of Mathematics, Los Angeles Valley College, CA. Please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. Thank you!

To propose problems, email them to: **problems4ssma@gmail.com**

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Please follow the instructions for submission of problems and solutions provided at the end of this document. Your adherence to all submission requirements is of the greatest help in running this Section successfully. Thank you!

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted before August 1, 2023.

• **5733** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all strictly increasing function(s) $f:\mathbb{N}\to\mathbb{N}$ so that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{f(i)^8}=\left(\frac{f(6)-f(5)}{5}+\frac{f(7)-6}{7}\right)\cdot\zeta(2)^2\cdot\zeta(4)$$

where ζ is Riemann zeta function.

• **5734** *Proposed by Narendra, Bhandari, Bajura, Nepal.*

Prove

$$\int_0^{\frac{\pi}{4}}\frac{\log\left(2\tanh^{-1}(\tan x)\right)\tanh^{-1}(\tan x)}{\tan 2x}\,dx=\frac{\pi^2}{96}\log\left(\frac{2e^3\pi^3}{A^{36}}\right),$$

where A is Glaisher- Kinkelin constant and e is Euler's number.

• **5735** *Proposed by Mihaly Bencze Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve the system of equations for real x, y and z:

$$\sqrt{3x+1}=z^2+1,\ \sqrt{3z+1}=y^2+1,\ \sqrt{3y+1}=x^2+1.$$

• **5736** *Proposed by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

If m is a non-negative integer, evaluate

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{k+2}{m+n}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2n+2i+1}\right]^{n}\right\}.$$

• **5737** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Find the limit

$$\ell=\lim_{n\to\infty}\left[\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n}.$$

• **5738** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1,\ldots,x_n>0$ be real numbers and $s=\sum_{i=1}^{n}x_i$. Prove

$$\prod_{i=1}^{n}x_i^{x_i}\geqslant\left(\frac{s}{n+s}\right)^{s}\prod_{i=1}^{n}(1+x_i)^{x_i}.$$

When does equality occur?

Solutions

To Formerly Published Problems

• **5709** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1,\ldots,x_n>0$ be real numbers such that $x_1+x_2+\ldots+x_n=1$. Prove the following inequality

$$1+\sum_{i=1}^{n}x_i^2\geqslant\prod_{i=1}^{n}(1+x_i)^{x_i}.$$

When does equality occur?

Solution 1 by Toyesh Prakash Sharma, Agra College, Agra, India.

Consider a function $f(x)=x\ln(1+x)$ then $f''(x)=(x+2)/(x+1)^2>0\forall x\geqslant 0$ so, we can say that considered function is convex function then using Weighted Jensen's Inequality for convex function

$$f\left(\frac{\sum_{i=1}^{n}\lambda_i x_i}{\sum_{i=1}^{n}\lambda_i}\right) \geqslant \frac{\sum_{i=1}^{n}\lambda_i f(x_i)}{\sum_{i=1}^{n}\lambda_i}$$

$$\Rightarrow \left(\frac{\sum_{i=1}^{n}\lambda_i x_i}{\sum_{i=1}^{n}\lambda_i}\right)\ln\left(1+\frac{\sum_{i=1}^{n}\lambda_i x_i}{\sum_{i=1}^{n}\lambda_i}\right) \geqslant \frac{\sum_{i=1}^{n}\lambda_i \ln(1+x_i)}{\sum_{i=1}^{n}\lambda_i}$$

Assuming $\lambda_i = x_i$ where $i \in \mathbb{N}$. Then

$$\left(\frac{\sum_{i=1}^{n}{x_i}^2}{\sum_{i=1}^{n}x_i}\right)\ln\left(1+\frac{\sum_{i=1}^{n}{x_i}^2}{\sum_{i=1}^{n}x_i}\right) \geqslant \frac{\sum_{i=1}^{n}\ln(1+x_i)^{x_i}}{\sum_{i=1}^{n}x_i}$$

Since $\sum_{i=1}^{n} x_i = 1$.

$$\left(\sum_{i=1}^{n}{x_i}^2\right)\ln\left(1+\sum_{i=1}^{n}{x_i}^2\right) \geqslant \ln\left(\prod_{i=1}^{n}(1+x_i)^{x_i}\right)$$

For $0 < x_1, x_2, \cdots x_n \leqslant 1$ we can say that $\sum_{i=1}^{n}{x_i}^2 \leqslant \sum_{i=1}^{n}x_i$ using this

$$\ln\left(1+\sum_{i=1}^{n}{x_i}^2\right) \geqslant \frac{1}{\sum_{i=1}^{n}{x_i}^2}\ln\left(\prod_{i=1}^{n}(1+x_i)^{x_i}\right) \geqslant \frac{1}{\sum_{i=1}^{n}x_i}\ln\left(\prod_{i=1}^{n}(1+x_i)^{x_i}\right) = \ln\left(\prod_{i=1}^{n}(1+x_i)^{x_i}\right)$$

$$\Rightarrow 1+\sum_{i=1}^{n}{x_i}^2 \geqslant \prod_{i=1}^{n}(1+x_i)^{x_i}$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

The function x→log(1+x) is concave, since $\frac{d^2}{dx^2}\log(1+x) = -\frac{1}{(1+x)^2} < 0$. Hence, by Jensen's inequality,

$$\sum_{i=1}^{n}x_i\log(1+x_i) \leqslant \log\left(1+\sum_{i=1}^{n}x_i^2\right)$$

with equality if and only if $x_1=x_2=\ldots=x_n=1/n$. The claimed inequality follows by taking exponents on both sides.

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since $x_1 + x_2 + \ldots + x_n = 1$, the inequality to prove may be written as

$$\sum_{i=1}^{n}x_i(1+x_i) \geqslant \prod_{i=1}^{n}(1+x_i)^{x_i}$$

which follows immediately by the weighted AM-GM inequality. In addition, the equality occur if and only if $x_1 = x_2 = \ldots = x_n$.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $x_1, x_2, \ldots, x_n > 0$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Consider the function

$$f(x) = \ln(1 + x).$$

Then

$$f'' = -\frac{1}{(1+x)^2} < 0,$$

so by Jensen's inequality

$$\sum_{i=1}^{n} x_i \ln(1 + x_i) = \sum_{i=1}^{n} x_i f(x_i) \leqslant f\left(\sum_{i=1}^{n} x_i^2\right) = \ln\left(1 + \sum_{i=1}^{n} x_i^2\right).$$

Equality holds when $x_i = \frac{1}{n}$ for each i. Exponentiation yields

$$\prod_{i=1}^{n}(1 + x_i)^{x_i} \leqslant 1 + \sum_{i=1}^{n} x_i^2.$$

If the condition $x_1, x_2, \ldots, x_n > 0$ is replaced by $x_1, x_2, \ldots, x_n \geqslant 0$, the indicated inequality holds with equality when k of the variables have value $\frac{1}{k}$ and the remaining $n-k$ variables are zero, for each $k = 1, 2, \ldots, n$.

Solution 5 by Samuel Aguilar (student) and the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

We use the weighted Arithmetic Mean - Geometric Mean Inequality with terms $1 + x_i$ and weights x_i. Since the sum of the weights is 1, then

$$\begin{aligned}
\sum_{i=1}^{n} x_i(1 + x_i) &\geqslant \prod_{i=1}^{n} (1 + x_i)^{x_i} \\
\sum_{i=1}^{n} x_i + x_i^2 &\geqslant \prod_{i=1}^{n} (1 + x_i)^{x_i} \\
1 + \sum_{i=1}^{n} x_i^2 &\geqslant \prod_{i=1}^{n} (1 + x_i)^{x_i}.
\end{aligned}$$

Equality occurs if and only if

$$1 + x_1 = 1 + x_2 = \cdots = 1 + x_n;$$

that is, when

$$x_1 = x_2 = \cdots x_n = \frac{1}{n}.$$

Solution 6 Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

We apply the weighted AGM inequality

$$\left(\prod_{i=1}^{n} a_i^{w_i}\right)^{1/\sum_{i=1}^{n} w_i} \leqslant \frac{\sum_{i=1}^{n} w_i a_i}{\sum_{i=1}^{n} w_i}$$

with $a_i = 1 + x_i$ and $w_i = x_i$. Then we have

$$\prod_{i=1}^{n}(1+x_i)^{x_i} \leqslant \frac{\sum_{i=1}^{n} x_i(1+x_i)}{\sum_{i=1}^{n} x_i} = 1 + \sum_{i=1}^{n} x_i^2.$$

Solution 7 Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Since $x_1 + x_2 + ... + x_n = 1$, note that

$$1 + \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i(1+x_i) \quad ....(1)$$

By weighted AM-GM inequality, note that

$$\sum_{i=1}^{n} x_i(1+x_i) \geqslant (x_1 + x_2 + ... + x_n)\left(\prod_{i=1}^{n}(1+x_i)^{x_i}\right)^{\frac{1}{x_1+x_2+...+x_n}} = \prod_{i=1}^{n}(1+x_i)^{x_i} \quad ....(2)$$

By (1) and (2), we get the desired inequality. Equality holds if and only if $x_1 = x_2 = ... = x_n = \frac{1}{n}(x_1 + x_2 + ... + x_n) = \frac{1}{n}$.

Equality holds if and only if $x_1 = x_2 = \cdots = x_n$—that is, if and only if $x_i = 1/n$, $i = 1, 2, \ldots, n$.

Solution 8 Michel Bataille, Rouen, France.

The function defined by $f(x) = \ln(1+x)$ being strictly concave on $(0, \infty)$, we have

$$\sum_{i=1}^{n} \alpha_i f(x_i) \leqslant f\left(\sum_{i=1}^{n} \alpha_i x_i\right)$$

whenever $\alpha_i \geqslant 0$ ($i = 1, \ldots, n$) and $\alpha_1 + \cdots + \alpha_n = 1$, with equality if and only if $x_1 = x_2 = \cdots = x_n$. Taking $\alpha_i = x_i$, we obtain

$$\sum_{i=1}^{n} x_i \ln(1+x_i) \leqslant \ln(1 + \sum_{i=1}^{n} x_i^2).$$

The resquested inequality follows by exponentiation (since $(1+x_i)^{x_i} = \exp(x_i \ln(1+x_i))$). Equality occurs if and only if $x_1 = x_2 = \cdots = x_n = \dfrac{1}{n}$.

Solution 9 Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.

By the AM-GM inequality, if $x_1, \ldots, x_n > 0$ and $a_1, \ldots, a_n \geqslant 0$, then

$$x_1 a_1 + \cdots + x_n a_n \geqslant a_1^{x_1} \cdots a_n^{x_n},$$

with equality if and only if $a_1 = \cdots = a_n$. Setting $a_1 = 1 + x_1, \ldots, a_n = 1 + x_n$, we get

$$x_1(1 + x_1) + \cdots + x_n(1 + x_n) \geqslant (1 + x_1)^{x_1} \cdots (1 + x_n)^{x_n},$$

which is equivalent to the desired inequality. The equality occurs if and only if $x_1 = \cdots = x_n$.

Also solved by Moti Levy, Rehovot, Israel; Paolo Perfetti, dipartimento di matematica, Universit'a di "Tor Vergata", Roma, Italyand the proposer.

• **5710** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania..*

Define the sequences $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ by $a_n = \sum_{k=1}^{n} \arctan\left(\dfrac{1}{k^2 - k + 1}\right)$ and $\lim_{n\to\infty} \dfrac{b_{n+1}}{n b_n} = \pi$. Compute $\lim_{n\to\infty} \left(\dfrac{\pi}{2} - a_n\right) \sqrt[n]{b_n}$.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

$$\frac{1}{k^2 - k + 1} = \frac{k - (k-1)}{k(k-1) + 1} = \frac{\frac{1}{k-1} - \frac{1}{k}}{1 + \frac{1}{k(k-1)}},$$

so

$$\arctan\left(\frac{1}{k^2 - k + 1}\right) = \arctan\left(\frac{1}{k-1}\right) - \arctan\left(\frac{1}{k}\right),$$

and

$$a_n = \frac{\pi}{4} + \sum_{k=2}^{n} \arctan\left(\frac{1}{k^2 - k + 1}\right) = \frac{\pi}{2} - \arctan\left(\frac{1}{n}\right).$$

Thus,

$$\lim_{n\to\infty} \left(\frac{\pi}{2} - a_n\right) \sqrt[n]{b_n} = \lim_{n\to\infty} \arctan\left(\frac{1}{n}\right) \sqrt[n]{b_n} = \lim_{n\to\infty} \frac{\arctan\left(\frac{1}{n}\right)}{\frac{1}{n}} \cdot \frac{\sqrt[n]{b_n}}{n}.$$

By L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{\arctan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+1/n^2}\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = 1;$$

moreover,

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} &= \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} = \lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}\\ &= \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{\pi}{e}.\end{aligned}$$

Finally,

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n} = \frac{\pi}{e}.$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Using the fact that $\arctan(u)-\arctan(v) = \arctan\left(\frac{u-v}{1+uv}\right)$ and taking $u=k$, and $v=k-1$, we have

$$\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan(k)-\arctan(k-1)$$

so, $a_n = \sum_{k=1}^{n}\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan(n)$, and the limit to be found becomes

$$L = \lim_{n\to\infty}\left(\frac{\pi}{2}-\arctan(n)\right)\sqrt[n]{b_n}.$$

$$\left(\frac{\pi}{2}-\arctan(n)\right)\sqrt[n]{b_n} = \frac{\sqrt[n]{b_n}}{n}\cdot n\cdot\left(\frac{\pi}{2}-\arctan(n)\right) = \frac{\sqrt[n]{b_n}}{n}\cdot\frac{\frac{\pi}{2}-\arctan(n)}{\frac{1}{n}},\ \forall n\geqslant 1.$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}} = \lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n} = \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{\pi}{e}.$$

$$\lim_{n\to\infty}\frac{\frac{\pi}{2}-\arctan(n)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{-1}{1+n^2}}{\frac{-1}{n^2}} = 1.$$

$$L = \lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n} = \frac{\pi}{e}\cdot 1 = \frac{\pi}{e}.$$

Solution 3 Albert Stadler, Herrliberg, Switzerland.

We assume in addition that $(\mathrm{b}_n)_{n\geqslant 1}$ is a sequence of positive numbers. We have

$$a_n = \sum_{k=1}^{n}\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan(1) + \sum_{k=2}^{n}\left(\arctan\left(\frac{1}{k-1}\right) - \arctan\left(\frac{1}{k}\right)\right) =$$

$$= \frac{\pi}{2}-\arctan\left(\frac{1}{n}\right)$$

and

$$b_n = b_1 \prod_{k=2}^{n} \frac{(k-1)\, b_k}{(k-1)\, b_{k-1}} = b_1\,(n-1)! \prod_{k=2}^{n} \pi e^{o(1)} = b_1\,(n-1)!\pi^{n-1} e^{no(1)},$$

where o(**??**) is the Landau symbol and denotes a function of n that tends to 0 as n tends to infinity. Stirling's asympototic formula for the factorials then gives

$$\sqrt[n]{b_n} = \sqrt[n]{b_1}\left(\frac{2\pi}{n}\right)^{\frac{1}{2n}} \left(\frac{n}{e}\right) \pi^{1-\frac{1}{n}} e^{o(1)}.$$

Finally,

$$\lim_{n\to\infty}\left(\frac{\pi}{2} - a_n\right)\sqrt[n]{b_n} = \lim_{n\to\infty} \arctan\left(\frac{1}{n}\right) \sqrt[n]{b_1}\left(\frac{2\pi}{n}\right)^{\frac{1}{2n}} \left(\frac{n}{e}\right) \pi^{1-\frac{1}{n}} e^{o(1)} = \frac{\pi}{e}.$$

Solution 4 David Huckaby, Angelo State University, San Angelo, TX.

Making use of the identity $\arctan\alpha - \arctan\beta = \arctan\dfrac{\alpha-\beta}{1+\alpha\beta}$, we have

$$\begin{aligned} a_n = \sum_{k=1}^{n} \arctan\left(\frac{1}{k^2-k+1}\right) &= \sum_{k=1}^{n} \arctan\left(\frac{k-(k-1)}{1+k(k-1)}\right) \\ &= \sum_{k=1}^{n} \big[\arctan k - \arctan(k-1)\big] = \arctan n - \arctan 0 = \arctan n. \end{aligned}$$

Note that the limit $\lim_{n\to\infty} n\left(\dfrac{\pi}{2} - \arctan n\right)$ yields the indeterminate form $\infty \cdot 0$. Rearranging to obtain the indeterminate form $\dfrac{0}{0}$ and then applying L'Hôpital's Rule gives

$$\begin{aligned} \lim_{n\to\infty} n\left(\frac{\pi}{2} - a_n\right) &= \lim_{n\to\infty} n\left(\frac{\pi}{2} - \arctan n\right) = \lim_{n\to\infty} \frac{\frac{\pi}{2} - \arctan n}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{-\frac{1}{1+n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{1+n^2} = 1. \end{aligned} \tag{1}$$

The following result follows from, for example, the Stolz-Cesàro Theorem: If for a sequence $(y_n)_{n\geqslant 1}$ of positive real numbers, the limit $\lim_{n\to\infty} \dfrac{y_{n+1}}{y_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{y_n} = \lim_{n\to\infty} \dfrac{y_{n+1}}{y_n}$. (See `https://en.wikipedia.org/wiki/StolzâĂŞCesaro_theorem`.)

Define the sequence $(y_n)_{n\geqslant 1}$ by $y_n = \dfrac{b_n}{n^n}$. Then

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \frac{\frac{b_{n+1}}{(n+1)^{n+1}}}{\frac{b_n}{n^n}} = \lim_{n\to\infty} \left[\frac{b_{n+1}}{nb_n} \cdot \left(\frac{n}{n+1}\right)^n \left(\frac{n}{n+1}\right) \right]$$
$$= \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} \cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \lim_{n\to\infty} \frac{n}{n+1} = \pi \cdot \frac{1}{e} \cdot 1 = \frac{\pi}{e}.$$

So $\lim_{n\to\infty} \sqrt[n]{y_n} = \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{\pi}{e}$. Combining this with the result from (1), we have

$$\lim_{n\to\infty} \left(\frac{\pi}{2} - a_n\right) \sqrt[n]{b_n} = \lim_{n\to\infty} n\left(\frac{\pi}{2} - a_n\right) \cdot \lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n}$$
$$= 1 \cdot \lim_{n\to\infty} \sqrt[n]{\frac{b_n}{n^n}} = \lim_{n\to\infty} \sqrt[n]{y_n} = \frac{\pi}{e}.$$

Solution 5 by G. C. Greubel, Newport News, VA.

The inverse tangent series is

$$a_n = \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{k^2-k+1}\right) = \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{1-k(1-k)}\right)$$
$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{k-(k-1)}{1+k(k-1)}\right) = \sum_{k=1}^{n} \left(\tan^{-1}(k) - \tan^{-1}(k-1)\right)$$
$$= \tan^{-1}(n) - \tan^{-1}(0) = \tan^{-1}(n).$$

This gives

$$\frac{\pi}{2} - a_n = \frac{\pi}{2} - \tan^{-1}(n) = \cot^{-1}(n) = \tan^{-1}\left(\frac{1}{n}\right).$$

For b_n the limit it must satisfy is

$$\lim_{n\to\infty} \frac{b_{n+1}}{n\,b_n} = \pi$$

which might suggest that b_n $(\pi\, n)^n\, e^{-n}$ and leads to

$$\frac{b_{n+1}}{n\,b_n} \approx \frac{\pi}{e}\,\left(1+\frac{1}{n}\right)^{n+1}$$

and the limit of this suggested form is

$$\lim_{n\to\infty} \frac{b_{n+1}}{n\,b_n} = \frac{\pi}{e} \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \left(1+\frac{1}{n}\right)^n = \frac{\pi}{e}\, e = \pi.$$

This is the limit b_n must satisfy. The desired limit can now be seen as:

$$\begin{aligned} L &= \lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\,\sqrt[n]{b_n} \\ &= \frac{\pi}{e}\lim_{n\to\infty} n\,\tan^{-1}\left(\frac{1}{n}\right) \\ &= \frac{\pi}{e}\lim_{n\to\infty} n\,\left(\frac{1}{n}-\frac{1}{3\,n^3}+\frac{1}{5\,n^5}-O\left(\frac{1}{n^7}\right)\right) \\ &= \frac{\pi}{e}\lim_{n\to\infty}\left(1-\frac{1}{3\,n^2}+\frac{1}{5\,n^4}-O\left(\frac{1}{n^6}\right)\right) \\ &= \frac{\pi}{e}. \end{aligned}$$

This gives the desired result as

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\,\sqrt[n]{b_n}=\frac{\pi}{e}.$$

Solution 6 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Here, we denote the Euler number by e.

<u>*Lemma 1*</u>: $a_n=\arctan(n),\quad \forall n\in\mathbb{Z}^+$.
Proof:
Firstly, we use the fact that $\arctan(x)-\arctan(y)=\arctan\left(\frac{x-y}{xy+1}\right)$ over $x,y\geqslant 0$. Then, $\arctan\left(\frac{1}{k^2-k+1}\right)=\arctan(k)-\arctan(k-1),\forall k\in\mathbb{Z}^+$. Therefore,

$$a_n=\sum_{k=1}^{n}\arctan\left(\frac{1}{k^2-k+1}\right)=\sum_{k=1}^{n}(\arctan(k)-\arctan(k-1))=\arctan(n).\qquad\square$$

<u>*Lemma 2*</u>: Let $(x_n)_{n\geqslant 1}$ be a real sequence such that $x_n>0$ for all large enough n, and $\lim_{n\to\infty}\frac{x_{n+1}}{nx_n}=m>0$. Then, $\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\frac{m}{e}$.
Proof:
Let $y_n=\frac{x_n}{n^n},\forall n\in\mathbb{Z}^+$. Then,

$$m=\lim_{n\to\infty}\frac{x_{n+1}}{nx_n}=\lim_{n\to\infty}\frac{y_{n+1}}{y_n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=e\cdot\lim_{n\to\infty}\frac{y_{n+1}}{y_n}\quad\Longrightarrow\quad\lim_{n\to\infty}\frac{y_{n+1}}{y_n}=\frac{m}{e}.$$

By Cauchy second theorem on limit,

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}=\lim_{n\to\infty}\sqrt[n]{y_n}=\lim_{nto\infty}\frac{y_{n+1}}{y_n}=\frac{m}{e}.\qquad\square$$

By lemma 1 and l'Hopital rule,

$$\lim_{n\to\infty} n\left(\frac{\pi}{2}-a_n\right) = \lim_{n\to\infty} n\left(\frac{\pi}{2}-\arctan(n)\right) = \lim_{n\to 0}\frac{\arctan(n)}{n} = \lim_{n\to 0}\frac{1}{n^2+1} = 1.$$

By lemma 2, $\lim\limits_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = \frac{\pi}{e}$.
Thus, the final result is

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n} = \lim_{n\to\infty} n\left(\frac{\pi}{2}-a_n\right)\cdot\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n} = 1\cdot\frac{\pi}{e} = \frac{\pi}{e}.$$

Solution 7 by Michel Bataille, Rouen, France.

We remark that for $k \geqslant 2$,

$$\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan\left(\frac{1}{k-1}\right) - \arctan\left(\frac{1}{k}\right)$$

(since $\arctan a + \arctan b = \arctan\frac{a+b}{1-ab}$ if $ab < 1$). It follows that for $n \geqslant 2$ we have

$$\sum_{k=1}^{n}\arctan\left(\frac{1}{k^2-k+1}\right) = \arctan 1 + \sum_{k=2}^{n}\left(\arctan\frac{1}{k-1} - \arctan\frac{1}{k}\right) = \frac{\pi}{4}+\frac{\pi}{4}-\arctan\frac{1}{n}$$

and therefore

$$\frac{\pi}{2} - a_n = \arctan\frac{1}{n}. \tag{1}$$

On the other hand, if we set $c_n = \frac{b_n}{n^n}$, a simple calculation gives

$$\frac{c_{n+1}}{c_n} = \left(1+\frac{1}{n}\right)^{-n}\left(1+\frac{1}{n}\right)^{-1}\frac{b_{n+1}}{nb_n}$$

and from the hypothesis, we obtain $\lim\limits_{n\to\infty}\frac{c_{n+1}}{c_n} = \frac{1}{e}\cdot 1\cdot\pi = \frac{\pi}{e}$, hence $\lim\limits_{n\to\infty}\sqrt[n]{c_n} = \frac{\pi}{e}$ too (a well-known result). This means that

$$\sqrt[n]{b_n} \sim n\cdot\frac{\pi}{e} \text{ as } n\to\infty. \tag{2}$$

Since $\arctan\frac{1}{n} \sim \frac{1}{n}$ as $n\to\infty$, (1) and (2) yield

$$\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n} \sim \frac{1}{n}\cdot n\cdot\frac{\pi}{e}$$

as $n\to\infty$ and so

$$\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n} = \frac{\pi}{e}.$$

Solution 8 byToyesh Prakash Sharma, Agra College, Agra, India.

Let

$$l=\lim_{n\to\infty}\left(\frac{\pi}{2}-a_n\right)\sqrt[n]{b_n}=\overbrace{\lim_{n\to\infty}n\left(\frac{\pi^2}{4}-a_n{}^2\right)}^{l_1}\overbrace{\lim_{n\to\infty}\sqrt[n]{b_n}\frac{1}{n\left(\frac{\pi}{2}+a_n\right)}}^{l_2}$$

With reference to the solution of the problem 5656, Feb. 2022 issue of SSMJ there we have already prove $a_n=\arctan n$ and $l_1=\lim_{n\to\infty}n\left(\frac{\pi^2}{4}-a_n{}^2\right)=\pi$ and also with the help of the Cauchy's second theorem on limit we have $\lim_{n\to\infty}\sqrt[n]{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{b_n}$ so,

$$l=\pi\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\lim_{n\to\infty}\frac{1}{\left(\frac{\pi}{2}+\arctan n\right)}=\pi^2\frac{1}{\left(\frac{\pi}{2}+\frac{\pi}{2}\right)}=\pi$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Yunyong Zhang, Chinaunicom, Yunnan, China; Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Roma, Italy; Moti Levy, Rehovot, Israel; Daniel Vacaru, Romania; and the proposer.

• **5711** *Proposed by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Rome, Italy.*

Let a,b,c,d be nonnegative real numbers. Prove that

$$(a^5-a+4)(b^5-b+4)(c^5-c+4)(d^5-d+4)\geqslant(a+b+c+d)^4.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Let $f(x)=x^5-x+4$ and $g(x)=x^4+3$. Then

$$f(x)-g(x)=x^5-x^4-x+1=(x+1)(x^2+1)(x-1)^2\geqslant 0$$

for $x\geqslant 0$. That is, $f(x)\geqslant g(x)$ for x nonnegative. Therefore Hölder's inequality yields

$$\begin{aligned}\left(\prod_{cyclic}(a^5-a+4)\right)^{1/4} &\geqslant\Big((a^4+3)(b^4+3)(c^4+3)(d^4+3)\Big)^{1/4}\\ &=\Big((a^4+1+1+1)(1+b^4+1+1)(1+1+c^4+1)(1+1+1+d^4)\Big)^{1/4}\\ &\geqslant(a+b+c+d),\end{aligned}$$

which proves the desired result. Equality holds if and only if $a = b = c = d = 1$.

Solution 2 by Michel Bataille, Rouen, France.

We observe that for any nonnegative real number x,

$$(x^5 - x + 4) - (x^4 + 3) = (x^4 - 1)(x - 1) = (x - 1)^2(x^3 + x^2 + x + 1) \geqslant 0$$

so that $x^5 - x + 4 \geqslant x^4 + 3 > 0$. We deduce that it is sufficient to prove that

$$(a^4 + 3)(b^4 + 3)(c^4 + 3)(d^4 + 3) \geqslant (a + b + c + d)^4. \qquad (1)$$

Now, Holder's inequality yields

$$(a^4 + 1 + 1 + 1)^{1/4}(1 + b^4 + 1 + 1)^{1/4}(1 + 1 + c^4 + 1)^{1/4}(1 + 1 + 1 + d^4)^{1/4}$$

$$\geqslant (a^4 \cdot 1 \cdot 1 \cdot 1)^{1/4} + (1 \cdot b^4 \cdot 1 \cdot 1)^{1/4} + (1 \cdot 1 \cdot c^4 \cdot 1)^{1/4} + (1 \cdot 1 \cdot 1 \cdot 1 \cdot d^4)^{1/4},$$

that is,

$$[(a^4 + 3)(b^4 + 3)(c^4 + 3)(d^4 + 3)]^{1/4} \geqslant a + b + c + d$$

and (1) follows at once.

Solution 3 by Vasile Cirtoaje, Petroleum-Gas University of Ploiesti, Romania.

Since

$$a^5 - a + 4 - (a^4 + 3) = a^5 - a^4 - a + 1 = (a - 1)(a^4 - 1) \geqslant 0$$

and, similarly, since

$$b^5 - b + 4 \geqslant b^4 + 3, \;\; c^5 - c + 4 \geqslant c^4 + 3, \;\; d^5 - d + 4 \geqslant d^4 + 3,$$

it suffices to show that

$$(a^4 + 3)(b^4 + 3)(c^4 + 3)(d^4 + 3) \geqslant (a + b + c + d)^4.$$

This inequality follows immediately from Holder's inequality

$$(a^4 + 1 + 1 + 1)(1 + b^4 + 1 + 1)(1 + 1 + c^4 + 1)(1 + 1 + 1 + d^4) \geqslant (a + b + c + d)^4.$$

The equality occurs for $a = b = c = d = 1$.

Also solved by Albert Stadler, Herrliberg, Switzerland; and the problem proposer.

• **5712** *Proposed by Syed Shahabudeen, Ernakulam, Kerala, India.*
Prove that

$$\lim_{n\to\infty} \sum_{q=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{q+4}} = \frac{\pi^2}{12} - \frac{1}{2}$$

Solution 1 by Kaushik Mahanta, NIT Silchar, Assam, India.

We let L denote the given limit and write

$$\begin{aligned}
L &= \lim_{n\to\infty}\sum_{q=0}^{n-3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\int_0^{\infty}\frac{e^{-(k_1+k_2+k_3+2)t}\,t^{q+3}\,dt}{\Gamma(q+4)}\\
&= \lim_{n\to\infty}\sum_{q=0}^{n-3}\int_0^{\infty}\frac{e^{-2t}\,t^{q+3}}{\Gamma(q+4)}\left(\sum_{k=0}^{\infty}e^{-kt}\right)^3 dt\\
&= \lim_{n\to\infty}\sum_{q=0}^{n-3}\frac{1}{\Gamma(q+4)}\int_0^{\infty}\frac{e^{-2t}\,t^{q+3}\,dt}{(1-e^{-t})^3}\\
&= \lim_{n\to\infty}\sum_{q=0}^{n-3}\frac{1}{\Gamma(q+4)}\int_0^{\infty}e^{-2t}\,t^{q+3}\sum_{r=0}^{\infty}\frac{(r+1)(r+2)}{2}\,e^{-rt}\,dt\\
&= \lim_{n\to\infty}\sum_{q=0}^{n-3}\sum_{r=0}^{\infty}\frac{(r+1)(r+2)}{2\Gamma(q+4)}\int_0^{\infty}e^{-(r+2)}\,t^{q+3}\,dt\\
&= \lim_{n\to\infty}\sum_{q=0}^{n-3}\sum_{r=0}^{\infty}\frac{(r+1)(r+2)}{2\Gamma(q+4)}\int_0^{\infty}e^{-(r+2)}\,t^{q+3}\,dt\\
&= \frac{1}{2}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{(r+1)(r+2)}{(r+2)^{q+4}} = \frac{1}{2}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{(r+1)}{(r+2)^{q+3}}\\
&= \frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{(r+2)^2} = \frac{1}{2}\big(\zeta(2)-1\big) = \frac{\pi^2}{12}-\frac{1}{2}.
\end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France.

We remark that if m is a non-negative integer, then there are $\dfrac{(m+1)(m+2)}{2}$ triples of non-negative integers (a,b,c) such that $a+b+c=m$ (indeed, if s is an integer such that $0\leqslant s\leqslant m$, there are $s+1$ pairs (a,b) with $a,b\geqslant 0$ and $a+b=s$, hence there are $\sum_{s=0}^{m}(s+1)=\dfrac{(m+1)(m+2)}{2}$ triples (a,b,c) such that $a,b,c\geqslant 0$ and $a+b+c=m$). We deduce that

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}} = \sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2}\cdot\frac{1}{(m+2)^{q+4}} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{m}{(m+1)^{q+3}}.$$

Now, for $n\geqslant 3$, we have

$$\sum_{q=0}^{n-3}\sum_{m=1}^{\infty}\frac{m}{(m+1)^{q+3}} = \sum_{m=1}^{\infty}\frac{m}{(m+1)^3}\sum_{q=0}^{n-3}\left(\frac{1}{m+1}\right)^q = \sum_{m=1}^{\infty}\frac{m}{(m+1)^3}\frac{1-\frac{1}{(m+1)^{n-2}}}{1-\frac{1}{m+1}}$$

and therefore, the sum $S_n = \sum_{q=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{q+4}}$ satisfies

$$2S_n = \sum_{m=1}^{\infty} \frac{1}{(m+1)^2} - \sum_{m=1}^{\infty} \frac{1}{(m+1)^n} = \frac{\pi^2}{6} - 1 - \sum_{m=1}^{\infty} \frac{1}{(m+1)^n}.$$

From

$$0 \leqslant \sum_{m=1}^{\infty} \frac{1}{(m+1)^n} \leqslant \int_1^{\infty} \frac{dx}{x^n} = \frac{1}{n-1}$$

we see that $\lim_{n\to\infty} \sum_{m=1}^{\infty} \frac{1}{(m+1)^n} = 0$ and it follows that the required limit is

$$\lim_{n\to\infty} S_n = \frac{1}{2}\left(\frac{\pi^2}{6} - 1\right) = \frac{\pi^2}{12} - \frac{1}{2}.$$

Solution 3 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA. We begin by reindexing the inner three sums. Set $k = k_1 + k_2 + k_3 + 2$. Then the sum $k_1 + k_2 + k_3$ corresponds to a weak composition of $k - 2$ into three parts. There are $\binom{k}{2}$ such compositions. We have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{q=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{q+4}} &= \lim_{n\to\infty} \sum_{q=0}^{n-3} \sum_{k\geqslant 2} \binom{k}{2} \frac{1}{k^{q+4}} \\
&= \lim_{n\to\infty} \sum_{q=0}^{n-3} \sum_{k\geqslant 2} \frac{k(k-1)}{2k^{q+4}} \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{k\geqslant 2} \sum_{q=0}^{n-3} (k-1) \left(\frac{1}{k}\right)^{q+3} \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{k\geqslant 2} \sum_{i=3}^{n} (k-1) \left(\frac{1}{k}\right)^{i}.
\end{aligned}$$

The inner sum is a partial sum of a geometric series. Therefore,

$$\begin{aligned}
\frac{1}{2}\lim_{n\to\infty}\sum_{k\geqslant 2}\sum_{i=3}^{n}(k-1)\left(\frac{1}{k}\right)^{i} &= \frac{1}{2}\lim_{n\to\infty}\sum_{k\geqslant 2}(k-1)\left(\frac{1-\left(\frac{1}{k}\right)^{n+1}}{1-\frac{1}{k}}-1-\frac{1}{k}-\left(\frac{1}{k}\right)^{2}\right)\\
&= \frac{1}{2}\lim_{n\to\infty}\sum_{k\geqslant 2}(k-1)\left(\frac{\frac{k^{n+1}-1}{k^{n+1}}}{\frac{k-1}{k}}-\frac{k^2+k+1}{k^2}\right)\\
&= \frac{1}{2}\lim_{n\to\infty}\sum_{k\geqslant 2}\frac{k^{n+1}-1}{k^n}-\frac{(k-1)(k^2+k+1)}{k^2}\\
&= \frac{1}{2}\sum_{k\geqslant 2}\lim_{n\to\infty}\left(k-\frac{1}{k^n}-\frac{k^3-1}{k^2}\right)\\
&= \frac{1}{2}\sum_{k\geqslant 2}k-k+\frac{1}{k^2}=\frac{1}{2}\sum_{k\geqslant 2}\frac{1}{k^2}
\end{aligned}$$

In 1734, Euler solved what has come to be known as the Basel problem, proving $\sum_{k\geqslant 1}\frac{1}{k^2}=\frac{\pi^2}{6}$. We may complete the proof with

$$\begin{aligned}
\frac{1}{2}\sum_{k\geqslant 2}\frac{1}{k^2} &= \frac{1}{2}\left(\sum_{k\geqslant 1}\frac{1}{k^2}-1\right)\\
&= \frac{1}{2}\left(\frac{\pi^2}{6}-1\right)=\frac{\pi^2}{12}-\frac{1}{2}.
\end{aligned}$$

Solution 4 by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Rome, Italy.

$$\int_0^\infty e^{-pt}t^r\,dt=\frac{r!}{p^{r+1}} \implies \frac{1}{(k_1+k_2+k_3+2)^{q+4}}=\frac{1}{(q+3)!}\int_0^\infty e^{-t(k_1+k_2+k_3+2)}t^{q+3}dt$$

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}}=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(q+3)!}\int_0^\infty e^{-t(k_1+k_2+k_3+2)}t^{q+3}dt=$$
$$=\frac{1}{(q+3)!}\int_0^\infty\frac{t^{q+3}e^{-2t}}{(1-e^{-t})^3}dt=\frac{1}{(q+3)!}\int_0^\infty\frac{t^{q+3}e^{t}}{(e^t-1)^3}dt$$

Now let's integrate by parts twice

$$
\begin{aligned}
\int_0^\infty \frac{t^{q+3}e^{-2t}}{(1-e^{-t})^3}dt &= \frac{-1}{2}\frac{t^{q+3}}{(e^t-1)^2}\Big|_0^\infty + \frac{1}{2}\int_0^\infty \frac{(q+3)t^{q+2}\,dt}{(e^t-1)^2} = \\
&= \frac{1}{2}\int_0^\infty \frac{(q+3)t^{q+2}e^{-t}e^t\,dt}{(e^t-1)^2} = \frac{-(q+3)}{2}\frac{t^{q+2}e^{-t}}{e^t-1}\Big|_0^\infty + \\
&+ \frac{q+3}{2}\int_0^\infty \left((q+2)t^{q+1}-t^{q+2}\right)e^{-2t}(1-e^{-t})^{-1}dt = \\
&= \frac{q+3}{2}\int_0^\infty \sum_{k=0}^\infty \left((q+2)t^{q+1}-t^{q+2}\right)e^{-t(k+2)}dt = \\
&= \frac{q+3}{2}\sum_{k=0}^\infty \int_0^\infty \left((q+2)t^{q+1}-t^{q+2}\right)e^{-t(k+2)}dt = \\
&= \frac{q+3}{2}\sum_{k=0}^\infty \left(\frac{(q+2)!}{(k+2)^{q+2}} - \frac{(q+2)!}{(k+2)^{q+3}}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{q=0}^{n-3}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty\sum_{k_3=0}^\infty \frac{1}{(k_1+k_2+k_3+2)^{q+4}} &= \frac{1}{2}\sum_{q=0}^{n-3}\sum_{k=0}^\infty\left(\frac{1}{(k+2)^{q+2}} - \frac{1}{(k+2)^{q+3}}\right) = \\
= \frac{1}{2}\sum_{k=0}^\infty \frac{1}{(k+2)^2}\frac{1-\frac{1}{(k+2)^{n-2}}}{1-\frac{1}{k+2}}\left(1-\frac{1}{(k+2)}\right) &= \frac{1}{2}\sum_{k=0}^\infty \frac{1}{(k+2)^2}\left(1-\frac{1}{(k+2)^{n-2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{q=0}^{n-3}\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty\sum_{k_3=0}^\infty \frac{1}{(k_1+k_2+k_3+2)^{q+4}} &= \frac{1}{2}\sum_{k=0}^\infty\frac{1}{(k+2)^2} = \frac{1}{2}\sum_{k=2}^\infty \frac{1}{k^2} = \\
= \frac{1}{2}\left(\frac{\pi^2}{6}-\frac{1}{2}\right) = \frac{\pi^2}{12}-\frac{1}{2}
\end{aligned}
$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

We have

$$
\begin{aligned}
\sum_{k_1=0}^\infty\sum_{k_2=0}^\infty\sum_{k_3=0}^\infty \frac{1}{(k_1+k_2+k_3+2)^{q+4}} &= \sum_{k=0}^\infty \frac{1}{(k+2)^{q+4}} \sum_{k_1+k_2+k_3=k,\quad k_1,k_2,k_3\geqslant 0} 1 \\
&= \sum_{k=0}^\infty \frac{1}{(k+2)^{q+4}}\cdot\frac{(k+1)(k+2)}{2}
\end{aligned}
$$

and

$$\sum_{q=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}} = \frac{1}{2}\sum_{q=0}^{\infty}\sum_{k=0}^{\infty}\frac{(k+1)}{(k+2)^{q+3}}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(k+1)}{(k+2)^{3}}\cdot\frac{1}{1-\frac{1}{k+2}}$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+2)^{2}} = \frac{\pi^2}{12}-\frac{1}{2}.$$

Solution 6 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. The sums over k_1, k_2, and k_3 can be expressed as

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}} = \frac{1}{2}\sum_{j=0}^{\infty}\frac{(j+1)(j+2)}{(j+2)^{q+4}}$$

$$= \frac{1}{2}\sum_{j=2}^{\infty}\frac{j-1}{j^{q+3}}$$

$$= \frac{1}{2}\sum_{j=2}^{\infty}\left(\frac{1}{j^{q+2}}-\frac{1}{j^{q-3}}\right)$$

$$= \frac{1}{2}\left(\zeta(q+2)-\zeta(q+3)\right),$$

where $\zeta(z)$ is the Riemann zeta function. Thus,

$$\sum_{q=0}^{n-3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}} = \frac{1}{2}\sum_{q=0}^{n-3}\left(\zeta(q+2)-\zeta(q+3)\right)$$

$$= \frac{1}{2}\left(\zeta(2)-\zeta(n)\right)$$

$$= \frac{1}{2}\left(\frac{\pi^2}{6}-\zeta(n)\right).$$

Finally,

$$\lim_{n\to\infty}\zeta(n) = 1,$$

so

$$\lim_{n\to\infty}\sum_{q=0}^{n-3}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{1}{(k_1+k_2+k_3+2)^{q+4}} = \frac{\pi^2}{12}-\frac{1}{2}.$$

Solution 7 by by G. C. Greubel, Newport News, VA.

Let

$$I = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{j=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{j+4}}.$$

Using

$$\frac{1}{x^{j+4}} = \frac{1}{\Gamma(j+4)} \int_0^\infty e^{-x\,t}\, t^{j+3}\, dt$$

then

$$\begin{aligned}
S_n &= \sum_{j=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{j+4}} \\
&= \sum_{j=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(j+3)!} \int_0^\infty e^{-(k_1+k_2+k_3+2)t}\, t^{j+3}\, dt \\
&= \int_0^\infty e^{-2t} \left(\sum_{j=0}^{n-3} \frac{t^{j+3}}{(j+3)!} \right) \left(\sum_{k=0}^{\infty} e^{-kt} \right)^3 dt \\
&= \int_0^\infty \left(\sum_{j=3}^{n} \frac{t^j}{j!} \right) \frac{e^{-2t}\, dt}{(1-e^{-t})^3}.
\end{aligned}$$

Taking the desired limit leads to

$$I = \int_0^\infty \left(e^t - 1 - t - \frac{t^2}{2} \right) \frac{e^{-2t}\, dt}{(1-e^{-t})^3}.$$

By making the change of variable $u = e^{-t}$ the integral becomes

$$I = \int_0^1 \left(1 - u + u\, \ln(u) - \frac{u\, \ln^2(u)}{2} \right) \frac{du}{(1-u)^3}$$

which leads to

$$\begin{aligned}
I &= \left[\frac{2\,(1-u) + 2\,u\, \ln(u) - u^2\, \ln^2(u)}{4\,(1-u)^2} - \frac{\mathrm{Li}_2(1-u)}{2} \right]_0^1 \\
&= \frac{1}{2} \left(\mathrm{Li}_2(1) - 1 \right) + \lim_{u \to 1} \frac{2\,(1-u) + 2\,u\, \ln(u) - u^2\, \ln^2(u)}{4\,(1-u)^2} \\
&= \frac{\zeta(2) - 1}{2},
\end{aligned}$$

which is the desired result. This gives

$$\lim_{n\to\infty} \sum_{j=0}^{n-3} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{j+4}} = \frac{\zeta(2) - 1}{2}.$$

Solution 8 by by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Observe that whenever $k_1 + k_2 + k_3 + 2 > 1$, we have

$$\sum_{q=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{q+4}} = \frac{\frac{1}{(k_1+k_2+k_3+2)^4}}{1 - \frac{1}{k_1+k_2+k_3+2}} = \frac{1}{(k_1 + k_2 + k_3 + 2)^3 (k_1 + k_2 + k_3 + 1)}.$$

Thus, the initial limit in the problem equals

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^3 (k_1 + k_2 + k_3 + 1)}$$

and this expression is equivalent with

$$L = \sum_{a=0}^{\infty} \sum_{\substack{k_1 + k_2 + k_3 = a \\ k_1, k_2, k_3 \in \mathbb{N}_0}} \frac{1}{(a+2)^3 (a+1)}.$$

After that, we note that (by Stars and Bars theorem) the number of triples (k_1, k_2, k_3) of non-negative integers satisfying $k_1 + k_2 + k_3 = a$ where $a \in \mathbb{N}_0$ is $\binom{a+2}{2}$. Thus, L can be rewritten as follows.

$$L = \sum_{a=0}^{\infty} \frac{\binom{a+2}{2}}{(a+2)^3 (a+1)} = \frac{1}{2} \sum_{a=0}^{\infty} \frac{1}{(a+2)^2} = \frac{1}{2}(\zeta(2) - 1) = \frac{\pi^2}{12} - \frac{1}{2},$$

as desired.

Solution 9 by Péter Fülöp, Gyömrő, Hungary.

Let's start the first sum from 3:

$$S = \lim_{n\to\infty} \sum_{q=3}^{n} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \underbrace{\sum_{k_3=0}^{\infty} \frac{1}{(k_1 + k_2 + k_3 + 2)^{q+1}}}_{\Psi(1,q+1,k_1+k_2+2)}$$

1. Introducing the Lerch transcendent and its integral represetation we get:

$$\lim_{n\to\infty} \sum_{q=3}^{n} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{1}{\Gamma(q+1)} \int_0^{\infty} \frac{t^q e^{k_1+k_2+2}}{1 - e^{-t}} dt$$

Exchanging order of the summation and integration first in case of k_2 then k_1:

$$\lim_{n\to\infty}\sum_{q=3}^{n}\sum_{k_1=0}^{\infty}\int_0^{\infty}\frac{t^q}{q!}\frac{e^{-t(k_1+2)}}{1-e^{-t}}\underbrace{\sum_{k_2=0}^{\infty}(e^{-t})^{k_2}}_{\frac{1}{1-e^{-t}}}dt$$

After performing the summation for k_2, k_1 we get the following integral:

$$\lim_{n\to\infty}\int_0^{\infty}\sum_{q=3}^{n}\frac{t^q}{q!}\frac{e^{-2t}}{(1-e^{-t})^3}dt$$

Let's change the order of the summation and integration:

2. Using that $\lim_{n\to\infty}\sum_{q=3}^{n}\frac{t^q}{q!} = (e^t - 1 - t - \frac{t^2}{2})$ we get the following intergral without sums:

$$S = \int_0^{\infty}(e^t - 1 - t - \frac{t^2}{2})\frac{e^{-2t}}{(1-e^{-t})^3}dt$$

Let's substitute $t = e^{-x}$:

$$\int_0^1 \frac{1}{(1-x)^2} + \frac{x\ln(x)}{(1-x)^3} - \frac{x\ln^2(x)}{2(1-x)^3}dx$$

Integration by parts:

$$\int_0^1 \underbrace{\frac{1}{(1-x)^3}}_{u'}\underbrace{\left(1 - x + x\ln(x) - \frac{x}{2}\ln^2(x)\right)}_{v}dx$$

Following that $u = \frac{1}{2(1-x)^2}$ and $v' = -\frac{1}{2}\ln^2(x)$

$$S = \left[1 - x + x\ln(x) - \frac{x}{2}\ln^2(x)\right]\frac{1}{2(1-x)^2}\Big|_0^1 + \frac{1}{4}\int_0^1 \frac{\ln^2(x)}{(1-x)^2}dx$$

3. In order to determine the value of S first we have to check the following limits:

Applying the L'Hospital rule twice, we get this result:

$$\lim_{x\to 1}\left(\left[1 - x + x\ln(x) - \frac{x}{2}\ln^2(x)\right]\frac{1}{2(1-x)^2}\right) = 0$$

We can also applying the L'Hospital rule twice, we get the value of the limit:

$$\lim_{x\to 0}\left(\left[1 - x + x\ln(x) - \frac{x}{2}\ln^2(x)\right]\frac{1}{2(1-x)^2}\right) = \frac{1}{2}$$

Now we have a simple expression:

$$\underline{S = -\frac{1}{2} + \frac{1}{4}\int_0^1 \frac{\ln^2(x)}{(1-x)^2}dx}$$

4. It is trivial that $\ln^2(x) = \frac{d^2 x^{(a+b)}}{dadb}_{|a=b=0}$. Let's use this fact in the integral part of S.

$$\frac{1}{4}\frac{d^2}{dadb}\int_0^1 \frac{(x)^{a+b}}{(1-x)^2}_{|a=b=0} dx$$

Known β function and its sumation form $\beta(1, a, b) = \sum_{k=0}^{\infty}\frac{(1-b)_k}{k!(k+a)}$, where $(1-b)_k$ is the Pochhammer symbole: $\frac{\Gamma(k+1-b)}{\Gamma(1-b)}$.

$$\frac{1}{4}\frac{d^2}{dadb}\beta(a+b+1,-1)_{|a=b=0} = \frac{1}{4}\frac{d^2}{dadb}\left(\sum_{k=0}^{\infty}\frac{(2)_k}{k!(k+a+b+1)}\right)_{|a=b=0}$$

After perform the derivations we get:

$$\sum_{k=0}^{\infty} 2\frac{k+1}{(k+1)^3} = 2\zeta(2) = \frac{\pi^2}{3}$$

Finally we get: $S = \underline{\frac{\pi^2}{12} - \frac{1}{2}}$, the statement is proved.

Solution 10 by Yunyong Zhang, Chinaunicom, Yunnan, China.

First we note that $\int_0^1 (\ln^m x)\, x^n dx = \frac{(-1)^m m\,!}{(n+1)^{m+1}}$.

Let $m = q+3, n = k_1 + k_2 + k_3 + 1$ then

$$\frac{1}{(k_1+k_2+k_3+2)^{q+4}} = \frac{\left[\int_0^1 \left(\ln^{q+3} x\right) x^{k_1+k_2+k_3+1} dx\right] (-1)^{q+3}}{(q+3)!}$$
$$= \frac{(-1)^{q+3}}{(q+3)!} \int_0^1 \left(\ln^{q+3} x\right) x^{k_1+k_2+k_3+1} dx.$$

$$S = \lim_{n\to\infty} \int_0^1 \sum_{q=0}^{n-3} \frac{(-1)^{q+3}}{(q+3)!} \left(\ln^{q+3} x\right) x^{\left[\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} x^{k_1+k_2+k_3}\right]} dx$$

in which

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} x^{k_1+k_2+k_3} = \left(\sum_{k_3=0}^{\infty} x^{k_1}\right)^3 = \left(\frac{1}{1-x}\right)^3.$$

So

$$S = \lim_{n\to\infty} \int_0^1 \sum_{q=0}^{n-3} \frac{(-1)^{q+3}}{(q+3)!} \frac{\left(\ln^{q+3} x\right) x}{(1-x)^3} dx = \lim_{n\to\infty} \int_0^1 \sum_{q=0}^{n-3} \frac{-x(-\ln x)^{q+3}}{(1-x)^3 (q+3)!} dx.$$

Since

$$\sum_{q=0}^{n-3} \frac{(-\ln x)^{q+3}}{(q+3)!} = \frac{\Gamma(n+1, -\ln x)}{xn!} - \frac{1}{2}\ln^2 x + \ln x - 1,$$

then

$$S = \lim_{n\to\infty} \int_0^1 \sum_{q=0}^{n-3} \frac{x}{(1-x)^3} \left[\frac{\Gamma(n+1, -\ln x)}{xn!} - \frac{1}{2}\ln^2 x + \ln x - 1\right] dx.$$

Since

$$\lim_{n\to\infty} \frac{\Gamma(n+1, -\ln x)}{n!} = 1,$$

then

$$S = \int_0^1 \sum_{q=0}^{n-3} \frac{x}{(1-x)^3} \left(\frac{1}{x} - \frac{1}{2}\ln^2 x + \ln x - 1\right) dx = \frac{\pi^2}{12} - \frac{1}{2}.$$

Also solved by Moti Levy, Rehovot, Israel; and the problem proposer.

• **5713** *Proposed by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah*

University of Science and Technology, Thuwal, Saudi Arabia.

Prove

$$\int_0^1 \frac{\log^2(x)}{1+x^2+x^4}\,dx = \frac{7}{9}\zeta(3) + \frac{\pi^3}{18\sqrt{3}}.$$

Here $\zeta(3)$ is Apéry's constant defined by $\sum_{n=1}^{\infty}\frac{1}{n^3}$.

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let I denote the given integral and write

$$I = \int_0^1 \frac{\log^2 x}{1+x^2+x^4}dx = \int_0^1 \frac{(1-x^2)\ln^2 x}{1-x^6}dx = I_1 - I_2$$

where

$$I_1 = \int_0^1 \frac{\ln^2 x}{1-x^6}dx$$

and

$$I_2 = \int_0^1 \frac{x^2\ln^2 x}{1-x^6}.$$

Since

$$\int_0^1 x^n \ln^2 dx = \frac{2}{(n+1)^3}$$

and

$$\frac{1}{1-x^6} = \sum_{k=0}^{\infty} x^{6k} \quad \text{for} \quad |x| < 1$$

then

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln^2 x}{1-x^6}dx = \sum_{k=0}^{\infty}\int_0^1 x^{6k}\ln^2 x dx \\ &= \sum_{k=0}^{\infty}\frac{2}{(6k+1)^3} = \frac{1}{108}[91\zeta(3) + 2\sqrt{3}\pi^3] \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \frac{x^2\ln^2 x}{1-x^6}\,dx = \sum_{k=0}^{\infty}\int_0^1 x^{6k+2}\ln^2 x\,dx \\ &= \sum_{k=0}^{\infty}\frac{2}{(6k+3)^3} = \frac{7\zeta(3)}{108}. \end{aligned}$$

Therefore

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \left[\frac{2}{(6k+1)^3} - \frac{2}{(6k+3)^3} \right] \\ &= \frac{1}{108} \left[84\zeta(3) + 2\sqrt{3}\pi^3 \right] \\ &= \frac{7}{9}\zeta(3) + \frac{\pi^3}{18\sqrt{3}}. \end{aligned}$$

Solution 2 by by G. C. Greubel, Newport News, VA.

Using $2a = 1 + i\sqrt{3}$ and $2b = 1 - i\sqrt{3}$, with the properties $ab = 1$, $a + b = 1$, $a - b = i\sqrt{3}$, then

$$\frac{1}{1 + x^2 + x^4} = \frac{1}{(x^2 + a)(x^2 + b)} = \frac{1}{a - b} \left(\frac{1}{x^2 + b} - \frac{1}{x^2 + a} \right)$$

then the integral in question becomes

$$\begin{aligned} I &= \int_0^1 \frac{\ln^2(x)\, dx}{1 + x^2 + x^4} \\ &= \frac{1}{a - b} \int_0^1 \left(\frac{\ln(x)}{x^2 + b} - \frac{\ln(x)}{x^2 + a} \right) dx. \end{aligned}$$

Now using the integral result

$$\begin{aligned} \int_0^1 \frac{\ln u\, du}{u^2 + a} &= -\frac{1}{\sqrt{-a}} \left(\mathrm{Li}_3\left(\frac{1}{\sqrt{-a}} \right) - \mathrm{Li}_3\left(-\frac{1}{\sqrt{-a}} \right) \right) \\ &= \frac{2}{a} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \left(\frac{-1}{a} \right)^n \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n\, b^{n+1}}{(2n+1)^3} \end{aligned}$$

then

$$I = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left(\frac{a^{n+1} - b^{n+1}}{a - b} \right).$$

It can now be seen that

$$\phi_n = \frac{a^{n+1} - b^{n+1}}{a - b} = \frac{(1 + i\sqrt{3})^{n+1} - (1 - i\sqrt{3})^{n+1}}{i\sqrt{3}\, 2^{n+1}}$$

which is sequence $A010892(n)$ of the Online Encyclopedia of Integer Sequences (OEIS) and can also be seen as the repeat of the six terms $\{1, 1, 0, -1, -1, 0\}$. With this in mind, then

$$\begin{aligned}
I &= 2\sum_{n=0}^{\infty}\frac{(-1)^n\phi_n}{(2n+1)^3}\\
&= 2\left(\sum_{n=0}^{\infty}\frac{1}{(6n+1)^3}-\sum_{n=0}^{\infty}\frac{1}{(6n+3)^3}\right)\\
&= 2\left(\sum_{n=0}^{\infty}\frac{1}{(6n+1)^3}-\frac{1}{3^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\right)\\
&= 2\left(\frac{2\sqrt{3}\pi^3+91\zeta(3)}{216}-\frac{1}{3^3}\left(1-\frac{1}{2^3}\right)\zeta(3)\right)\\
&= \frac{7\zeta(3)}{9}+\frac{\pi\zeta(2)}{3\sqrt{3}}.
\end{aligned}$$

This is the same as the desired result. With this result is can be stated that

$$\int_0^1\frac{\ln^2(x)\,dx}{1+x^2+x^4}=\frac{7}{9}\zeta(3)+\frac{\pi^3}{18\sqrt{3}}.$$

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

$$\begin{aligned}
\int_0^1\frac{\log^2 x}{1+x^2+x^4}\,dx &= \int_0^1\frac{(1-x^2)\log^2 x}{1-x^6}\,dx\\
&= \sum_{k=0}^{\infty}\int_0^1\left(x^{6k}-x^{6k+2}\right)\log^2 x\,dx\\
&= 2\sum_{k=0}^{\infty}\left[\frac{1}{(6k+1)^3}-\frac{1}{(6k+3)^3}\right]\\
&= -\frac{1}{216}\left[\psi_2\left(\frac{1}{6}\right)-\psi_2\left(\frac{1}{2}\right)\right],
\end{aligned}$$

where $\psi_2(x)$ is a polygamma function. It is known that

$$\psi_2\left(\frac{1}{6}\right)=-182\zeta(3)-4\sqrt{3}\pi^3$$

and

$$\psi_2\left(\frac{1}{2}\right)=-14\zeta(3),$$

so

$$\int_0^1\frac{\log^2 x}{1+x^2+x^4}\,dx=-\frac{1}{216}\left(-168\zeta(3)-\frac{12\pi^3}{\sqrt{3}}\right)=\frac{7}{9}\zeta(3)+\frac{\pi^3}{18\sqrt{3}}.$$

Solution 4 by Michel Bataille, Rouen, France.

Let I be the integral. Since $1 + x^2 + x^4 = \dfrac{1 - x^6}{1 - x^2}$, we have $I = J - K$ where

$$J = \int_0^1 \frac{\log^2(x)}{1 - x^6}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} x^{6n}\right)\log^2(x)\,dx = \sum_{n=0}^{\infty}\int_0^1 x^{6n}\log^2(x)\,dx = \sum_{n=0}^{\infty}\frac{2}{(6n+1)^3}$$

and

$$K = \int_0^1 \frac{x^2\log^2(x)}{1 - x^6}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} x^{6n+2}\right)\log^2(x)\,dx = \sum_{n=0}^{\infty}\int_0^1 x^{6n+2}\log^2(x)\,dx = \sum_{n=0}^{\infty}\frac{2}{(6n+3)^3}$$

(using the known formula $\displaystyle\int_0^1 x^k\log^m(x)\,dx = \frac{(-1)^m m!}{(k+1)^{m+1}}$ for integers $k, m \geqslant 0$, easily established with the help of integrations by parts). It follows that $I = 2S_1 - 2S_3$ where we set $S_j = \displaystyle\sum_{n=0}^{\infty}\frac{1}{(6n+j)^3}$ $(j = 1, 2, 3, 4, 5, 6)$.

Now, we have $\zeta(3) = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$, $\quad S_6 = \dfrac{1}{6^3}\zeta(3)$ and

$$S_3 = \frac{1}{27}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = \frac{1}{27}\left(\zeta(3) - \sum_{n=1}^{\infty}\frac{1}{(2n)^3}\right) = \frac{7}{8\times 27}\zeta(3)$$

$$S_2 + S_4 = \frac{1}{8}\left(\sum_{n=0}^{\infty}\frac{1}{(3n+1)^3} + \sum_{n=0}^{\infty}\frac{1}{(3n+2)^3}\right) = \frac{1}{8}\left(\zeta(3) - \frac{1}{27}\zeta(3)\right) = \frac{13}{4\times 27}\zeta(3)$$

so that $S_1 + S_5 = \dfrac{91}{108}\zeta(3)$. To calculate $S_1 - S_5$, we define the function $f : \mathbb{R} \to \mathbb{R}$ as follows: f is odd, 2π-periodic, and for $t \in [0, \pi]$, $f(t) = \pi t - t^2$. Elementary calculations and Dirichlet's theorem on Fourier series yields

$$f(t) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)t)}{(2n+1)^3} \quad (t \in \mathbb{R}).$$

With $t = \dfrac{\pi}{3}$, we obtain

$$S_1 - S_5 = \frac{2}{\sqrt{3}}\cdot\frac{\pi}{8}\cdot f\left(\frac{\pi}{3}\right) = \frac{\pi^3}{18\sqrt{3}}.$$

We deduce

$$I = (S_1 + S_5) + (S_1 - S_5) - 2S_3 = \frac{91}{108}\zeta(3) + \frac{\pi^3}{18\sqrt{3}} - \frac{7}{108}\zeta(3) = \frac{7}{9}\zeta(3) + \frac{\pi^3}{18\sqrt{3}}.$$

Solution 5 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

$$\int_0^1 \frac{\log^2 x}{1+x^2+x^4}dx = \int_0^1 \frac{(1-x^2)\log^2 x}{1-x^6}dx = \int_0^1 (1-x^2)\ln^2 x \sum_{k=0}^{\infty} x^{6k} dx =$$

$$= \int_0^1 \ln^2 x \sum_{k=0}^{\infty}(x^{6k} - x^{6k+2})dx = \sum_{k=0}^{\infty}\left(\frac{2}{(6k+1)^3} - \frac{2}{(6k+3)^3}\right) =$$

$$= 2 + \frac{2}{216}\sum_{k=1}^{\infty}\frac{1}{(k+\frac{1}{6})^3} - \frac{14\zeta(3)}{216} = 2 - \frac{7\zeta(3)}{108} + \frac{2}{216}\left[\frac{-1}{z^3} - \frac{1}{2}\psi''(z)\right]\Big|_{z=\frac{1}{6}} =$$

$$= 2 - \frac{7\zeta(3)}{108} + \frac{2}{216}\left[-216 - \frac{-182\zeta(3) - 4\sqrt{3}\pi^3}{216}\right] =$$

$$= -\frac{7\zeta(3)}{108} - \frac{-182\zeta(3) - 4\sqrt{3}\pi^3}{216} = \frac{7}{9}\zeta(3) + \frac{\pi^3}{18\sqrt{3}}$$

$\psi(z) = \big(\ln\Gamma(z)\big)'$ and the value of $\psi''(\frac{1}{6})$ may be found here (entry (31))

https://mathworld.wolfram.com/PolygammaFunction.html

Solution 6 by Péter Fülöp, Gyömrő, Hungary.

Steps of the proof:

1. Transformation of the integrand and performing the $x = t^6$ substitution.
2. Applying the fact: $\ln^2(t) = \frac{d^2 t^{(a+b)}}{dadb}\Big|_{a=b=0}$
3. Using the definition of β function and summation formulae of incomplete β function.
4. Determination of the $\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{2})^3}$
5. Determination of the $\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{6})^3}$
6. Calculating the value of trilogarithm function in $z=-1$ point $(Li_3(-1))$
7. Using polygamma function $(\psi_2(z))$

Steps:

1. Using the fact that:

$$x^4 + x^2 + 1 = \frac{x^6-1}{x^2-1}, x \neq \pm 1$$

We have the following:

$$I = \int_0^1 \frac{(x^2-1)\ln^2(x)}{x^6-1}dx$$

And introducing the $t = x^6$ substitution we get:

$$I = \frac{1}{6^3}\int_0^1 \frac{(t^{\frac{1}{3}}-1)\ln^2(t)}{t-1}t^{-\frac{5}{6}}dt = \frac{1}{6^3}\int_0^1 \frac{(t^{-\frac{1}{2}}-t^{-\frac{5}{6}})\ln^2(t)}{t-1}dt$$

2. Based on the $\ln^2(t) = \frac{d^2t^{(a+b)}}{dadb}_{|a=b=0}$ equivalence the integral can be written as follows:

$$I = \frac{1}{6^3}\frac{d^2}{dadb}\Big(\int_0^1 \frac{(t^{-\frac{1}{2}+a+b}-t^{-\frac{5}{6}+a+b})}{t-1}dt\Big)_{|a=b=0}$$

3. Known β function and its sumation form $\beta(1,a,b) = \sum_{k=0}^{\infty}\frac{(1-b)_k}{k!(k+a)}$, where $(1-b)_k$ is the Pochhammer symbole: $\frac{\Gamma(k+1-b)}{\Gamma(1-b)}$. Using these in the integral we get:

$$I = -\frac{1}{6^3}\frac{d^2}{dadb}\Big(\beta(\frac{1}{2}+a+b,0)-\beta(\frac{1}{6}+a+b,0)\Big)_{|a=b=0} =$$

$$= -\frac{1}{6^3}\frac{d^2}{dadb}\Big(\sum_{k=0}^{\infty}\frac{1}{k+\frac{1}{2}+a+b}-\sum_{k=0}^{\infty}\frac{1}{k+\frac{1}{6}+a+b}\Big)_{|a=b=0}$$

After performing the derivation, we get that:

$$I = -\frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}+2\sum_{k=0}^{\infty}\frac{1}{(6k+1)^3}$$

4. Determination of the $\sum_{k=0}^{\infty}\frac{1}{(k+\frac{1}{2})^3}$

$\zeta(3)$ can be written based on summation separated by even and odd numbers:

$$\zeta(3) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}+\sum_{k=1}^{\infty}\frac{1}{(2k)^3}$$

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3)$$

5. Determination of the $\sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{6})^3}$

Similar to the principle used in the previous point $\zeta(3)$ can be broken into six sums:

$$\zeta(3) = \sum_{n=1}^{6} \sum_{k=0}^{\infty} \frac{1}{(6k+n)^3}$$

6. $Li_3(-1)$ can also be broken into six sums, where $Li_3(-1)$ is the trilogarithm function in the point (-1). It equals to $-\frac{3}{4}\zeta(3)$. So

$$Li_3(-1) = -\frac{3}{4}\zeta(3) = \sum_{n=1}^{6} \sum_{k=0}^{\infty} \frac{(-1)^n}{(6k+n)^3}$$

Let's make the difference of $\zeta(3)$ and $Li_3(-1)$:

$$\zeta(3) - Li_3(-1) = \frac{7}{4}\zeta(3) = 2\sum_{k=0}^{\infty} \frac{1}{(6k+1)^3} + 2\underbrace{\sum_{k=0}^{\infty} \frac{1}{(6k+3)^3}}_{\frac{1}{27}\frac{7}{8}\zeta(3)} + 2\sum_{k=0}^{\infty} \frac{1}{(6k+5)^3}$$

7. After performing the possible cancellations and introduce the polygamma function $\psi_2(z) = -2!\sum_{k=0}^{\infty} \frac{1}{(k+z)^3}$ we get:

$$364\zeta(3) = -\psi_2(\frac{1}{6}) - \underline{\psi_2(\frac{5}{6})}$$

Applying the reflexion relation for polygamma functions:

$$\psi_2(1-\frac{1}{6}) - \psi_2(\frac{1}{6}) = \pi\frac{d^2(cot(\pi z))}{dz^2}\Big|_{z=\frac{1}{6}} = 2\pi^3\frac{\sin(\pi\frac{1}{6})}{(\cos)^3(\pi\frac{1}{6})} = 8\sqrt{3}\pi^3$$

$$\psi_2(\frac{5}{6}) = \underline{8\sqrt{3}\pi^3 + \psi_2(\frac{1}{6})}$$

Put it back to into the place of $\psi_2(\frac{5}{6})$ we get:

$$\psi_2(\frac{1}{6}) = -182\zeta(3) - 4\sqrt{3}\pi^3 = -2\sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{6})^3}$$

The value of the sum can be calculated:

$$\sum_{k=0}^{\infty} \frac{1}{(6k+1)^3} = \frac{91}{216}\zeta(3) + \frac{\pi^3}{36\sqrt{3}}$$

Finally

$$I = -\frac{2}{27}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} + 2\sum_{k=0}^{\infty} \frac{1}{(6k+1)^3} = -\frac{7}{108}\zeta(3) + \frac{91}{108}\zeta(3) + \frac{\pi^3}{18\sqrt{3}}$$

$$I = \frac{7}{9}\zeta(3) + \frac{\pi^3}{18\sqrt{3}}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Kaushik Mahanta, NIT Silchar, Assam, India; Moti Levy, Rehovot, Israel; and the problem proposer.

• **5714** *Proposed by Peter Fulop, Gyomro, Hungary.*

Prove

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right).$$

Solution 1 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

We use the following well-known identity, see

`https://en.wikipedia.org/wiki/Hyperbolic_functions`. For any complex number z,

$$\sinh(z) = z\prod_{k=1}^{\infty}\left(1 + \frac{z^2}{\pi^2 k^2}\right).$$

Thus

$$\sinh(\pi) = \pi\prod_{k=1}^{\infty}\left(1 + \frac{1}{k^2}\right) = \pi\prod_{k=1}^{\infty}\left(\frac{k^2+1}{k^2}\right),$$

$$\frac{\pi}{\sinh(\pi)} = \prod_{k=1}^{\infty}\left(\frac{k^2}{k^2+1}\right),$$

and

$$\frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right) = \ln\sqrt{\prod_{k=1}^{\infty}\left(\frac{k^2}{k^2+1}\right)} = \sum_{k=1}^{\infty}\ln\left(\frac{k}{\sqrt{k^2+1}}\right).$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Using the product formula for the sine

$$\frac{\sin(\pi z)}{\pi z} = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)$$

and the relation $\sin(iz) = i\sinh(z)$ we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \ln \frac{k}{\sqrt{k^2+1}} &= \sum_{k=1}^{\infty} \ln \frac{1}{\sqrt{1+k^{-2}}} \\ &= -\frac{1}{2}\ln \prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right) \\ &= -\frac{1}{2}\ln\left(\frac{\sin(i\pi)}{i\pi}\right) \\ &= \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right).\end{aligned}$$

Solution 3 by Toyesh Prakash Sharma, Agra College, Agra, India.

As is well known,

$$\sinh x = x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{\pi^2 k^2}\right)$$

from which we have for $x = \pi$:

$$\sinh\pi = \pi\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right)$$

which implies

$$\frac{\pi}{\sinh\pi} = \prod_{k=1}^{\infty}\left(\frac{k^2}{1+k^2}\right) = \left(\prod_{k=1}^{\infty}\left(\frac{k}{\sqrt{1+k^2}}\right)\right)^2$$

$$\Rightarrow \frac{1}{2}\ln\frac{\pi}{\sinh\pi} = \ln\prod_{k=1}^{\infty}\left(\frac{k}{\sqrt{1+k^2}}\right) = \sum_{k=1}^{\infty}\ln\left(\frac{k}{\sqrt{1+k^2}}\right)$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

By the product representation of the hyperbolic sine, e.g.,

https://en.wikipedia.org/wiki/Hyperbolic_functions, we have

$$\frac{\sinh(\pi x)}{\pi x} = \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right).$$

Hence

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \ln\left(\frac{k^2}{k^2+1}\right) = -\frac{1}{2}\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k^2}\right)$$
$$= -\frac{1}{2}\ln\left(\frac{\sinh(\pi)}{\pi}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right).$$

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The identity to prove may be written as

$$\prod_{k=1}^{\infty} \frac{k^2}{k^2+1} = \frac{\pi}{\sinh(\pi)}.$$

By using Euler's product formula for $\sin z$,

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty}\left(1-\frac{z^2}{\sqrt{\pi^2 z^2}}\right)$$

it follows that

$$\prod_{k=1}^{\infty} \frac{k^2}{k^2+1} = \frac{\pi i}{\sin(\pi i)} = \frac{2\pi}{e^{\pi}-e^{\pi}} = \frac{\pi}{\sinh(\pi)}.$$

Solution 6 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We rewrite the left side of the given equation as

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \ln\left(\frac{k^2}{k^2+1}\right) = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\frac{k^2}{k^2+1}\right).$$

Then it suffices to show that

$$\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right) = \frac{\sinh(\pi)}{\pi}.$$

So we apply Euler's formula, which states that for any complex number z,

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right).$$

Substituting $z = i$ and noting that $\sin(\pi i) = i\sinh(\pi)$ yields the desired result.

Solution 7 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

First,

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\sum_{k=1}^{\infty} \ln\left(\frac{k^2}{k^2+1}\right) = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\frac{k^2}{k^2+1}\right).$$

Next,

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right).$$

Replacing x with $i\pi x$ and using that $\sin i\theta = i\sinh\theta$ yields

$$\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right) = \frac{\sin(i\pi x)}{i\pi x} = \frac{\sinh \pi x}{\pi x}.$$

With $x = 1$, this becomes

$$\prod_{k=1}^{\infty}\frac{k^2+1}{k^2} = \frac{\sinh\pi}{\pi}.$$

Thus,

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh\pi}\right).$$

Solution 8 by G. C. Greubel, News, VA.

The sum begins as

$$S = \sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \ln\left(\prod_{k=1}^{\infty}\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\ln\left(\prod_{k=1}^{\infty}\frac{k^2}{k^2+1}\right).$$

The product takes the form

$$\begin{aligned} P = \prod_{k=1}^{\infty}\frac{k^2}{k^2+1} &= \lim_{m\to\infty}\prod_{k=1}^{m}\frac{k^2}{k^2+1} \\ &= \lim_{m\to\infty}\frac{\Gamma^2(m+1)\,\Gamma(1+i)\,\Gamma(1-i)}{\Gamma(m+1+i)\,\Gamma(m+1-i)} \\ &= |\Gamma(1+i)|^2 \lim_{m\to\infty}\frac{\Gamma^2(m+1)}{|\Gamma(m+1+i)|^2} \\ &= |\Gamma(1+i)|^2 = \frac{\pi}{\sinh(\pi)}. \end{aligned}$$

This leads to

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right)$$

which is the desired result.

This result can be extended to

$$\sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+x^2}}\right) = \frac{1}{2}\ln\left(\frac{\pi x}{\sinh(\pi x)}\right).$$

The proof follows, with use of some of the efforts in the main problem,

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \ln\left(\frac{k}{\sqrt{k^2+x^2}}\right) = \ln\left(\prod_{k=1}^{\infty} \frac{k}{\sqrt{k^2+x^2}}\right) \\ &= \frac{1}{2}\ln\left(\lim_{m\to\infty} \prod_{k=1}^{m} \frac{k^2}{k^2+x^2}\right) \\ &= \frac{1}{2}\ln\left(\lim_{m\to\infty} \frac{\Gamma^2(m+1)\,|\Gamma(1+i\,x)|^2}{|\Gamma(m+1+i\,x)|^2}\right) \\ &= \frac{1}{2}\ln(|\Gamma(1+i\,x)|^2) \\ &= \frac{1}{2}\ln\left(\frac{\pi x}{\sinh(\pi x)}\right). \end{aligned}$$

When $x = 1$ the desired result of the main problem is obtained.

Solution 9 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Weierstrass factorization of $\sinh(x)$ is as follows:

$$\sinh(x) = x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right) \quad ....(1)$$

By setting $x := \pi$ to (1), we obtain that

$$\frac{\pi}{\sinh(\pi)} = \prod_{k=1}^{\infty}\left(\frac{1}{1+\frac{1}{k^2}}\right) = \prod_{k=1}^{\infty}\left(\frac{k}{\sqrt{k^2+1}}\right)^2 \quad \Longrightarrow \quad \ln\left(\frac{\pi}{\sinh(\pi)}\right) = 2\sum_{k=1}^{\infty}\ln\left(\frac{k}{\sqrt{k^2+1}}\right).$$

Thus,

$$\sum_{k=1}^{\infty}\ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right).$$

Solution 10 by Michel Bataille, Rouen, France.

We use the following well-known result:

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = \frac{\sin(\pi z)}{\pi z} \quad (z \in \mathbb{C},\ z \neq 0).$$

With $z = i$, this gives

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right) = \frac{\sin(\pi i)}{\pi i} = \frac{\sinh(\pi)}{\pi}.$$

Thus

$$\begin{aligned}
\frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right) &= \lim_{n\to\infty}\left(\frac{1}{2}\ln\left(\frac{1}{\prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right)}\right)\right)\\
&= \lim_{n\to\infty}\ln\left(\prod_{j=1}^{n}\frac{1}{\sqrt{1+\frac{1}{k^2}}}\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{1}{\sqrt{1+\frac{1}{k^2}}}\right)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{k}{\sqrt{1+k^2}}\right)\\
&= \sum_{k=1}^{\infty}\ln\left(\frac{k}{\sqrt{k^2+1}}\right)
\end{aligned}$$

and the result follows.

Solution 11 by Perfetti Paolo, dipartimento di matematica, Università di “Tor Vergata", Roma, Italy.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\frac{k}{\sqrt{k^2+1}}\right) = \lim_{n\to\infty}\frac{1}{2}\ln\prod_{k=1}^{n}\frac{k^2}{k^2+1}$$

Now set $z = 2\pi$:

$$\frac{2}{z}\sinh\frac{z}{2} = \prod_{k=1}^{\infty}\left(1+\frac{z^2}{4k^2\pi^2}\right) \implies \frac{\sinh\pi}{\pi} = \prod_{k=1}^{\infty}\frac{k^2+1}{k^2}$$

hence

$$\lim_{n\to\infty}\frac{1}{2}\ln\prod_{k=1}^{n}\frac{k^2}{k^2+1} = \frac{1}{2}\ln\left(\frac{\pi}{\sinh\pi}\right)$$

Solution 12 Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\text{LHS} = \frac{1}{2}\sum_{k=1}^{\infty}\ln\frac{k^2}{1+k^2} = -\frac{1}{2}\ln\frac{k^2+1}{k^2} = -\frac{1}{2}\ln\left(1+\frac{1}{10^2}\right) = -\frac{1}{2}\ln\prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right).$$

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right),$$

then, for $x = \pi i$:

$$\frac{\sin \pi i}{\pi i} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k^2}\right) = \frac{e^{\pi}-e^{-\pi}}{2\pi} = \frac{\sinh \pi}{\pi}.$$

We thus conclude

$$\text{LHS} = -\frac{1}{2}\ln\left(\frac{\sinh \pi}{\pi}\right) = \frac{1}{2}\ln\left(\frac{\pi}{\sinh \pi}\right).$$

Also solved by Moti Levy, Rehovot, Israel; the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike

the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica ($\longleftarrow$ You may choose to not include a title.)

Statement of the problem:

Compute $\displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣