Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or propsed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* September 1, 2023.

• **5739** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Prove that for any triangle $\triangle ABC$:

$$\frac{h_a^3}{h_b^3}+\frac{h_b^3}{h_c^3}+\frac{h_c^2}{h_a^3}\geqslant\frac{\sin^2 B}{\sin^2 A}+\frac{\sin^2 C}{\sin^2 B}+\frac{\sin^2 A}{\sin^2 C}$$

where h_a, h_b, h_c are the altitudes respectively issued from the vertices A, B, C.

• **5740** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

How many integers m are there for which the volume of parallelepiped determined by vectors $u=\langle 2023,\, m,\, 1-m\rangle$, $v=\langle m,\, 2-m,\, 4046\rangle$, $w=\langle 6069,\, 3-m,\, m\rangle$ is equal to $6(2023^2+2023)$?

• **5741** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Let $(a_k)_{k=1}^{\infty}$ be a non-decreasing sequence with $0<\ a_k\leqslant k$. Define $A_n:=\sum_{k=1}^{n}a_k$ and $S_n:=\sum_{k=1}^{n}a_k^{\alpha}$ for $\alpha>1$. Determine whether or not the following series converges:

$$\sum_{k=1}^{\infty}S_k\left(\frac{1}{A_k^{\alpha}}-\frac{1}{A_{k+1}^{\alpha}}\right).$$

• **5742** *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, and Romania, Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Suppose $(x_n)_{n\geqslant 1}$ is a sequence of postive terms with $\lim\limits_{n\to\infty} x_n/n! = \pi$. Define $E_n := \sum\limits_{k=0}^{n} 1/k!$ and let $0 < a \neq 1$. If $f : (0,\infty) \to (0,\infty)$ is a continuous function, then compute

$$L = \lim_{n\to\infty} x_{n+1} \int_{a^{E_n}}^{a^{E_{n+1}}} f(x)\,dx.$$

• **5743** *Proposed by Shivam Sharma, Delhi University, New Delhi, India.*

$$\int_1^{\infty} \left(\frac{\sum_{k=1}^{\infty} \{kx\} \frac{H_k}{k^8}}{x^9} \right) dx = \frac{\zeta(7)}{8} + \frac{3}{28}\zeta(3)\zeta(5) - \frac{\pi^8}{39200}$$

where $\{.\}$ denotes the Fractional Part, H_k denotes the Harmonic Number and $\zeta(s)$ denotes the Riemann Zeta Function.

• **5744** *Proposed by Toyesh Prakash Sharma (Student) Agra College, India.*

Show that

$$\left(\int_1^{\infty} \frac{\cos\left(\ln x^2\right)}{x^2\sqrt{\ln x}} \mathrm{dx} \right)^2 + \left(\int_1^{\infty} \frac{\sin\left(\ln x^2\right)}{x^2\sqrt{\ln x}} \mathrm{dx} \right)^2 = \frac{\pi}{\sqrt{5}}.$$

Solutions

To Formerly Published Problems

• **5715** *Proposed by Kenneth Korbin, New York, NY.*

Find the dimensions of a triangle with integer length sides in which the area divided by the perimeter is equal to the sine of 60 degrees.

Solution 1 by Brian D. Beasley, Presbyterian College, Clinton, SC.

By Heron's formula, the area A of a triangle satisfies

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, and c are the side lengths and $s = (a+b+c)/2$ is the semiperimeter. Then we seek the dimensions of a triangle with $A = \sqrt{3}s$, which yields

$$A^2 = 3s^2 = s(s-a)(s-b)(s-c),$$

or equivalently

$$12(a+b+c) = (-a+b+c)(a-b+c)(a+b-c).$$

Taking $a \leqslant b \leqslant c$ without loss of generality, we note that a computer search produces six solutions for (a, b, c):

$$(4, 14, 14); \quad (5, 7, 8); \quad (5, 16, 19); \quad (6, 6, 6); \quad (6, 10, 14); \quad (7, 8, 13).$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

For an equilateral triangle with sides of length s,

$$A = \frac{s^2\sqrt{3}}{4} \quad \text{and} \quad P = 3s,$$

so

$$\frac{A}{P} = \frac{s\sqrt{3}}{12}.$$

Thus, an equilateral triangle with sides of length $s = 6$ satisfies

$$\frac{A}{P} = \frac{\sqrt{3}}{2} = \sin 60^\circ.$$

Solution 3 by by G. C. Greubel, Newport News, VA.

For an integer side triangle with sides (a, b, c) consider the case when $(a, b, c) = (x, x + 2, x + 3)$. Since the area and perimeter are defined by

$$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}, \; s = \frac{a+b+c}{2} \quad \text{and} \quad P = a + b + c$$

then

$$\begin{aligned} A &= \frac{1}{4}\sqrt{(a+b-c)(a-b+c)(b+c-a)(a+b+c)} \\ &= \frac{1}{4}\sqrt{(x-1)(x+1)(x+5)(3x+5)} \end{aligned}$$

and $P = 3x + 5$. Now, since $A = \sin\left(\frac{\pi}{3}\right) P$, this leads to

$$\frac{A}{P} = \frac{1}{4}\sqrt{\frac{(x-1)(x+1)(x+5)}{3x+5}} = \frac{\sqrt{3}}{2}$$

which reduces to finding the integer roots of the equation $(x^2 - 1)(x + 5) = 12\,(3x + 5)$ or $(x - 5)(x^2 + 10x + 13) = 0$. Note that the roots are $x \in \{5, -5 + 2\sqrt{3}, -5 - 2\sqrt{3}\}$. The integer value is $x = 5$ and gives the integer sides as $(a, b, c) = (5, 7, 8)$.

Solution 4 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Without loss of generality, let $a + b$, $a + c$, $b + c$ be the length of sides of that triangle, where

$a, b, c \in \mathbb{Z}^+$ and $a \geqslant b \geqslant c$. Then, $2(a+b+c)$ and $\sqrt{abc(a+b+c)}$ are respectively the perimeter and the area of the triangle. By the given condition,

$$\frac{\sqrt{abc(a+b+c)}}{2(a+b+c)} = \sin 60\circ \quad \Longrightarrow \quad abc = 3a + 3b + 3c \quad .....(1)$$

By (1) and $a \geqslant b \geqslant c$, we have $\frac{1}{3} = \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} \leqslant \frac{3}{c^2}$ and it gives $c \leqslant 3$.

If $c = 3$. By (1), we get $(a-1)(b-1) = 4$ and $(a, b) = (3, 3)$.
If $c = 2$. By (2), we get $(2a-3)(2b-3) = 21$ and $(a, b) = (12, 2), (5, 3)$.
If $c = 1$. By (3), we get $(a-3)(b-3) = 12$ and $(a, b) = (15, 4), (9, 5), (7, 6)$.

By collecting all the lists (a, b, c), there are 6 possibilities of the dimension of triangle:

$$6, 6, 6; \quad 4, 14, 14; \quad 5, 7, 8; \quad 5, 16, 19; \quad 6, 10, 14; \quad 7, 8, 13.$$

Solution 5 by Michel Bataille, Rouen, France.

Let a, b, c denote the side lengths of the triangle. There are six solutions for a, b, c:

$$6, 6, 6\ ;\ 14, 14, 4\ ;\ 8, 7, 5\ ;\ 19, 16, 5\ ;\ 14, 10, 6\ ;\ 13, 8, 7.$$

From Heron's formula for the area of the triangle, the condition writes as $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{3}s$ where s is the semi-perimeter or, equivalently $(b+c-a)(c+a-b)(a+b-c) = 12(a+b+c)$. To solve this equation, we set $x = b+c-a, y = c+a-b, z = a+b-c$ (that is, $a = \frac{y+z}{2}, b = \frac{z+x}{2}, c = \frac{x+y}{2}$), reducing the problem to solving $xyz = 12(x+y+z)$ for positive integers x, y, z with the same parity.
Let x, y, z be a solution. Note that if these numbers are odd, then xyz is odd, in contradiction with the equation. Thus, x, y, z are even.
Without loss of generality, we suppose that $x \leqslant y \leqslant z$. Since $x^2 z \leqslant xyz \leqslant 12 \cdot 3z$, we see that $x \leqslant 6$ and $xy \leqslant 36$. Since $z(xy-12) = 12(x+y)$, we have $xy \geqslant 14$. This said, we consider the three possible cases $x = 6, 4, 2$.
If $x = 6$, then $y \geqslant 6$ (since $y \geqslant x$) and $y \leqslant 6$ (since $xy \leqslant 36$), hence $y = 6$ and the equation now gives $z = 6$. Thus $x = y = z = 6$ and $a = b = c = 6$.
If $x = 4$, then $y \geqslant 4$ and $y \leqslant 9$. If $y = 4$, the equation yields $z = 24$ and $a = b = 14, c = 4$. Similarly, $y = 6$ gives $z = 10$ and then $a = 8, b = 7, c = 5$. The case $y = 8$ does not provide an integer value of z.
If $x = 2$, in the same way we obtain $y = 8, z = 30$ (hence $a = 19, b = 16, c = 5$) or $y = 10, z = 18$ (and $a = 14, b = 10, c = 6$) or $y = 12, z = 14$ (and $a = 13, b = 8, c = 7$).
Conversely, if a, b, c is any of the six triples announced at the beginning, it is readily checked that the relation $(b+c-a)(c+a-b)(a+b-c) = 12(a+b+c)$ holds.

Solution 6 by Yoojin Choi and Trey Smith, Angelo State University, San Angelo, TX.

We start by demonstrating that for positive numbers x, y, z that satisfy the equation

$$xyz = 3(x + y + z),$$

it must be the case that at least one of them is less than or equal to 3. To see this, suppose all three are greater than 3. So $x = 3 + i$, $y = 3 + j$, and $z = 3 + k$ where $i, j, k > 0$. Then

$$\begin{aligned} & xyz \\ &= (3+i)(3+j)(3+k) \\ &= 27 + 9(i+j+k) + 3(ij+ik+jk) + ijk \\ &> 27 + 3(i+j+k) \\ &= 3(9+i+j+k) \\ &= 3((3+i)+(3+j)+(3+k)) \\ &= 3(x+y+z). \end{aligned}$$

In other words, when all x, y, z are greater than 3, $xyz > 3(x + y + z)$. So, evidently, if $xyz = 3(x + y + z)$ at least one of the x, y, z must be less than or equal to 3.

The goal is now to find all positive integer values that satisfy $xyz = 3(x + y + z)$. We start by assuming that z is less than or equal to 3. Since z must also be greater than or equal to 1, we have the following three equations:

$$y_1 = \frac{3(x+1)}{x-3}, \quad y_2 = \frac{3(x+2)}{2x-3}, \quad y_3 = \frac{x+3}{x-1}.$$

They are obtained by substituting 1, 2, and 3 respectively into the original equation, and solving for y. Notice that each of these may be viewed as a rational function. Working with y_2, we have that x must be an integer greater than $3/2$. Also, the graph of y_2 has a horizontal asymptote $y = 3/2$. Then for all $x > 12$, $3/2 < y_2 < 2$. So the only integer values for x that could possibly result in an integer value for y are 2 through 12. Substituting each of these in for x, only four do yield integers, and of those, only two are unique as *multisets* of three numbers. Those two solutions are $\{2, 2, 12\}$ and $\{2, 3, 5\}$.

Similarly, y_1 yields $\{1, 4, 15\}$, $\{1, 6, 7\}$, and $\{1, 5, 9\}$. The equation y_3 yields one previously undiscovered solution, $\{3, 3, 3\}$.

Assume we have a triangle with integer sides a, b, c that satisfies the desired condition. By Heron's formula, we have that

$$\frac{\sqrt{s(s-a)(s-b)(s-c)}}{2s} = \frac{\sqrt{3}}{2}$$

where $s = (a + b + c)/2$. Assuming that $x = (s - a)$, $y = (s - b)$, and $z = (s - c)$, we have that $x + y + z = s$, and

$$\frac{\sqrt{s(s-a)(s-b)(s-c)}}{2s} = \frac{\sqrt{3}}{4}$$

$$\iff \frac{\sqrt{sxyz}}{2s} = \frac{\sqrt{3}}{2}$$

$$\iff \frac{sxyz}{4s^2} = \frac{3}{4}$$

$$\iff \frac{xyz}{s} = 3$$

$$\iff xyz = 3(x+y+z).$$

The double implication is appropriate since everything is positive. Since we have the solutions for the last equation, we will simply use them to find all triangle solutions to obtain the following six triangles:

$$(4,14,14),\ (5,7,8),\ (5,16,19),\ (6,6,6),\ (6,10,14),\ (7,8,13).$$

Also solved by Albert Stadler, Herrliberg, Switzerland; the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA; and the problem proposer.

• **5716** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania.*

Prove:

$$\text{If } x, y \in \mathbb{R}, \text{ then } \left|\cos x \cos y \sin(x+y)\right| \leqslant \frac{3\sqrt{3}}{8}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We need to prove that $f(x,y) := \cos^2 x \cos^2 y \sin^2(x+y) \leqslant \frac{27}{64}$. f(x,y) is periodic with respect to x and y. Hence the extrema of f(x,y) are assumed at points where the partial derivatives with respect to x and y vanish. We find

$$\frac{\partial}{\partial x} f(x,y) = -2\sin x \cos x \cos^2 y \sin^2(x+y) + 2\cos^2 x \cos^2 y \sin(x+y)\cos(x+y) = 0$$

$$\frac{\partial}{\partial y} f(x,y) = -2\sin y \cos y \cos^2 x \sin^2(x+y) + 2\cos^2 x \cos^2 y \sin(x+y)\cos(x+y) = 0$$

$$0 = \frac{\partial}{\partial x} f(x,y) - \frac{\partial}{\partial y} f(x,y) = 2\cos x \cos y \sin(y-x)\sin^2(x+y).$$

So either x $\equiv \pi/2 \pmod{\pi}$, or y $\equiv \pi/2 \pmod{\pi}$, or x $\equiv$ -y $\pmod{\pi}$, or x $\equiv$ y $\pmod{\pi}$.

The first three alternatives lead to f(x,y)=0, while the last one leads to

$$0 = -2\sin x \cos^3 x \sin^2(2x) + 2\cos^4 x \sin(2x)\cos(2x) =$$

$$= 2\cos^3 x \left(-\sin x \left(4\sin^2 x \cos^2 x\right) + \cos x \,(2\sin x \cos x)\cos(2x)\right) =$$

$$= 4\cos^5 x \sin x \left(2\cos(2x) - 1\right).$$

So either x $\equiv \pi/2 \pmod{\pi}$, or x $\equiv 0 \pmod{\pi}$, or x $\equiv \pm\pi/6 \pmod{\pi}$. When combined withy $\equiv$ x $\pmod{\pi}$ we get indeed $f(x,y) \leqslant \cos^2\left(\frac{\pi}{6}\right)\cos^2\left(\frac{\pi}{6}\right)\sin^2\left(\frac{\pi}{3}\right) = \frac{27}{64}$.

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Using well-known trigonometric formulas we obtain

$$\begin{aligned} f(x,y) := \quad & \left|\cos x \cos y \sin(x+y)\right| = \frac{1}{2}\left|\cos(x+y) + \cos(x-y)\right| \sqrt{1-\cos^2(x+y)} \\ \leqslant \quad & \frac{1}{2}|z+1|\sqrt{1-z^2}, \end{aligned}$$

where $z = \left|\cos(x+y)\right|$. Hence,

$$f^2(x,y) \leqslant \frac{1}{4}(z+1)^2\left(1-z^2\right) = \frac{27}{64} - \left(z-\frac{1}{2}\right)^2\left(\frac{11}{16} + \frac{3}{4}z + \frac{1}{4}z^2\right).$$

Since $0 \leqslant z \leqslant 1$, we conclude that

$$f(x,y) \leqslant \sqrt{27/64} = 3\sqrt{3}/8.$$

Remark: The inequality is sharp. Equality occurs if $x = y = \pi/6$

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $f(x,y) = \cos x \cos y \sin(x+y)$. Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= -\sin x \cos y \sin(x+y) + \cos x \cos y \cos(x+y) \\ &= \cos y \left(\cos x \cos(x+y) - \sin x \sin(x+y)\right) = \cos y \cos(2x+y), \end{aligned}$$

which is equal to 0 when

$$y = \left(n + \frac{1}{2}\right)\pi \quad \text{or} \quad 2x + y = \left(n + \frac{1}{2}\right)\pi,$$

for some integer n. Similarly,

$$\frac{\partial f}{\partial y} = \cos x \cos(x + 2y),$$

which is equal to 0 when

$$x = \left(m + \frac{1}{2}\right)\pi \quad \text{or} \quad x + 2y = \left(m + \frac{1}{2}\right)\pi,$$

for some integer m. It follows that f has four categories of critical points:

1. $\left(\left(m + \frac{1}{2}\right)\pi, \left(n + \frac{1}{2}\right)\pi\right)$, for any integers m and n
2. $\left(\left(m + \frac{1}{2}\right)\pi, \left(n - 2m - \frac{1}{2}\right)\pi\right)$, for any integers m and n
3. $\left(\left(m - 2n - \frac{1}{2}\right)\pi, \left(n + \frac{1}{2}\right)\pi\right)$, for any integers m and n
4. $\left(\frac{1}{3}\left(2n - m + \frac{1}{2}\right)\pi, \frac{1}{3}\left(2m - n + \frac{1}{2}\right)\pi\right)$, for any integers m and n

When evaluated at any critical point from the first three categories, f is equal to 0. For the critical points in the fourth category, note

$$2m - n = 2n - m + 3(m - n) \quad \Rightarrow \quad 2m - n \equiv 2n - m \pmod 3.$$

This leads to three cases to consdier:

Case 1: $2n - m \equiv 0 \pmod 3$

Then

$$x = j\pi + \frac{\pi}{6}, \quad y = k\pi + \frac{\pi}{6}, \quad \text{and} \quad x + y = (j + k)\pi + \frac{\pi}{3}$$

for some integers j and k, and

$$f(x, y) = \pm\frac{3\sqrt{3}}{8}.$$

Case 2: $2n - m \equiv 1 \pmod 3$

Then

$$x = j\pi + \frac{\pi}{2}, \quad y = k\pi + \frac{\pi}{2}, \quad \text{and} \quad x + y = (j + k + 1)\pi$$

for some integers j and k, and

$$f(x, y) = 0.$$

Case 3: $2n - m \equiv 2 \pmod 3$

Then

$$x = j\pi + \frac{5\pi}{6}, \quad y = k\pi + \frac{5\pi}{6}, \quad \text{and} \quad x + y = (j+k)\pi + \frac{5\pi}{3}$$

for some integers j and k, and

$$f(x,y) = \pm\frac{3\sqrt{3}}{8}.$$

Thus, for all $x, y \in \mathbb{R}$,

$$-\frac{3\sqrt{3}}{8} \leqslant f(x,y) \leqslant \frac{3\sqrt{3}}{8},$$

or

$$|f(x,y)| \leqslant \frac{3\sqrt{3}}{8}.$$

Solution 4 by David Huckaby, Angelo State University, San Angelo, TX.

Let $f(x,y) = \cos x \cos y \sin(x+y)$. Note that $f(x+\pi, y) = \cos(x+\pi)\cos y \sin(x+\pi+y) = -\cos x \cos y[-\sin(x+y)] = f(x,y)$. Similarly, $f(x, y+\pi) = f(x,y)$. So we need only consider the square $\left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We first note that since $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$, $f(x,y) = 0$ for every point on the boundary of $\left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$.

To find extrema for f in the interior of $\left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$, we compute $\frac{\partial f}{\partial x} = -\sin x \cos y \sin(x+y) + \cos x \cos y \cos(x+y) = \cos y[\cos x \cos(x+y) - \sin x \sin(x+y)]$. From the symmetry of $f(x,y)$ in x and y, $\frac{\partial f}{\partial y} = \cos x[\cos y \cos(x+y) - \sin y \sin(x+y)]$.

Setting $\frac{\partial f}{\partial x} = 0$ gives $\cos y = 0$ or $\cos x \cos(x+y) - \sin x \sin(x+y) = 0$. Since $\cos y \neq 0$ in the interior of $\left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have $\cos x \cos(x+y) - \sin x \sin(x+y) = 0$. Now

$$\begin{aligned}
&\cos x \cos(x+y) - \sin x \sin(x+y) \\
&= \cos x[\cos x \cos y - \sin x \sin y] - \sin x[\sin x \cos y + \cos x \sin y] \\
&= \cos^2 x \cos y - \sin^2 x \cos y - 2\cos x \sin x \sin y \\
&= \cos 2x \cos y - \sin 2x \sin y \\
&= \cos(2x + y).
\end{aligned}$$

So $\frac{\partial f}{\partial x} = 0$ implies $\cos(2x + y) = 0$. By symmetry, $\frac{\partial f}{\partial y} = 0$ implies $\cos(x + 2y) = 0$. Now $\cos(2x+y) = 0$ when $2x+y = \frac{\pi}{2}+\pi n$ for any interger n. Solving for y gives $y = -2x+\frac{\pi}{2}+\pi n$. Similarly, $\cos(x+2y) = 0$ when $x+2y = \frac{\pi}{2}+\pi n$ for some integer n. Solving for y gives $y = -\frac{x}{2}+\frac{\pi}{4}+\frac{\pi n}{2}$. Setting these two values of y equal to each other yields $-2x + \frac{\pi}{2} + \pi n = -\frac{x}{2} + \frac{\pi}{4} + \frac{\pi n}{2}$, whence $x = \frac{\pi}{6} + \frac{\pi n}{3}$.

The only values of $x = \frac{\pi}{6} + \frac{\pi n}{3}$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ are $x = \pm\frac{\pi}{6}$. So any point (x, y) yielding an extremum of f in the interior of $\left[-\frac{\pi}{2}, -\frac{\pi}{2}\right] \times \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ must lie on $\left(\frac{\pi}{6}, y\right)$ or $\left(-\frac{\pi}{6}, y\right)$. By symmetry, any extremum must also lie on $\left(x, \frac{\pi}{6}\right)$ or $\left(x, -\frac{\pi}{6}\right)$. So there are only four possible points that could yield an extremum.

Note that if $x + y = 0$, then $\sin(x + y) = 0$ so that $f(x, y) = 0$. So we need only check two points: $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$ and $f\left(-\frac{\pi}{6}, -\frac{\pi}{6}\right) = -\frac{3\sqrt{3}}{8}$. (Note that rather than using direct calculation, the latter can be obtained from the former by noting that $f(-x, -y) = \cos(-x)\cos(-y)\sin(-(x + y)) = \cos x \cos y[-\sin(x + y)] = -f(x, y)$.)

So f attains a maximum value of $\frac{3\sqrt{3}}{8}$ and a minimum value of $-\frac{3\sqrt{3}}{8}$. Thus $|\cos x \cos y \sin(x + y)| = |f(x, y)| \leqslant \frac{3\sqrt{3}}{8}$.

Solution 5 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Let $f(x, y) = \cos x \cos y \sin(x+y)$, and consider $g(x) = f(x, x) = \cos^2 x \sin(2x)$, which has period π. Since $g'(x) = (2\cos(2x) - 1)(\cos(2x) + 1)$, by the first derivative test we see that g achieves its maximum value of $\frac{3\sqrt{3}}{8}$ at $x = \frac{\pi}{6} + n\pi$ and its minimum value of $-\frac{3\sqrt{3}}{8}$ at $x = -\frac{\pi}{6} + n\pi$, where n is an integer. Thus

$$f(\frac{\pi}{6} + n\pi, \frac{\pi}{6} + n\pi) = \frac{3\sqrt{3}}{8} \text{ and } f(-\frac{\pi}{6} + n\pi, -\frac{\pi}{6} + n\pi) = -\frac{3\sqrt{3}}{8}.$$

Since $f(x, y)$ attains the two values above, in searching for absolute extreme values of $f(x, y)$ we may assume $f(x, y) \neq 0$; that is, we assume $\cos x$, $\cos y$, and $\sin(x + y)$ are all nonzero.

Since the partial derivatives of $f(x, y) = \cos x \cos y \sin(x + y)$ are

$$f_x(x, y) = \cos y \left(\cos x \cos(x + y) - \sin x \sin(x + y)\right) \text{ and}$$
$$f_y(x, y) = \cos x \left(\cos y \cos(x + y) - \sin y \sin(x + y)\right),$$

then any critical points with $f(x,y) \neq 0$ must satisfy

$$\sin x \cos y \sin(x+y) = \cos x \cos y \cos(x+y) = \cos x \sin y \sin(x+y),$$

and $\tan x = \tan y$. Thus, $y = x + n\pi$, where n is an integer, and since $\cos^2(n\pi) = 1$, then

$$f(x,y) = \cos x \cos(x+n\pi)\sin(2x+n\pi) = \cos^2 x \cos^2(n\pi)\sin(2x) = \cos^2 x \sin(2x) = g(x).$$

From the analysis of $g(x)$ above, $f(x,y)$ must achieve its maximum at $\dfrac{3\sqrt{3}}{8}$ and its minimum at $-\dfrac{3\sqrt{3}}{8}$.

Solution 6 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Note that

$$\left|\cos x \cos y \sin(x+y)\right| \leqslant \frac{3\sqrt{3}}{8} \iff (\cos x \cos y \sin(x+y))^2 \leqslant \frac{27}{64}$$
$$\iff$$
$$(\sin(x+y) + \sin x + \sin y)^2 \leqslant \frac{27}{4},$$

which must be proved.
Let $f(x,y) = \sin(x+y) + \sin x + \sin y$, over $x, y \in \mathbb{R}$. It is enough to show that $f(x,y)^2 \leqslant \dfrac{27}{4}$.
Observe that $f(x,y) = f(2a\pi + x, 2b\pi + y), \forall a, b \in \mathbb{Z}$; so, WLOG, $x, y \in [0, 2\pi]$.

CASE 1: If $x, y \in [0, \pi]$.
We have

$$-1 \leqslant \sin(x+y) \leqslant f(x,y) \leqslant \sin(x+y) + 2\sin\left(\frac{x+y}{2}\right) \quad .....(1)$$

Consider the function $f_1(x) = \sin 2x + 2\sin x$, $\forall x \in [0, \pi]$. Then, $f_1'(x) = 2(2\cos x - 1)(\cos x + 1)$; f_1 is increasing when $x \in [0, \frac{\pi}{3}]$ and decreasing when $x \in [\frac{\pi}{3}, \pi]$. Therefore, $\sin(x+y) + 2\sin\left(\frac{x+y}{2}\right) \leqslant f(\frac{2\pi}{3}) = \frac{3\sqrt{3}}{2}$. By (1), we get $-1 \leqslant f(x,y) \leqslant \frac{3\sqrt{3}}{2}$ and thus, $f(x,y)^2 \leqslant \frac{27}{4}$.

CASE 2: If $x, y \in [\pi, 2\pi]$.
Let $x = \pi + x_1$ and $y = \pi + y_1$ where $x_1, y_1 \in [0, \pi]$. Then, $f(x,y) = \sin(x_1 + y_1) - \sin x_1 - \sin y_1$.
We have

$$\sin(x_1+y_1) - 2\sin\left(\frac{x_1+y_1}{2}\right) \leqslant f(x,y) \leqslant \sin(x_1+y_1) \leqslant 1 \quad .....(2)$$

Consider the function $f_2(x) = \sin 2x - 2\sin x$, $\forall x \in [0, \pi]$. Then, $f_2'(x) = 2(2\cos x + 1)(\cos x - 1)$; f_2 is decreasing for $x \in [0, \frac{2\pi}{3}]$ and increasing for $x \in [\frac{2\pi}{3}, \pi]$. Therefore, $\sin(x_1+y_1) - 2\sin\left(\frac{x_1+y_1}{2}\right) \geqslant f_2(\frac{2\pi}{3}) = \frac{-3\sqrt{3}}{2}$. By (2), we get $\frac{-3\sqrt{3}}{2} \leqslant f(x,y) \leqslant 1$ and thus, $f(x,y)^2 \leqslant$

$\frac{27}{4}$.

CASE 3: If one of x and y is in $[0, \pi]$, while another one is in $[\pi, 2\pi]$.
By symmetry, WLOG $x \in [0, \pi]$ and $y \in [\pi, 2\pi]$.
We have $-1 \leqslant \sin(x + y) \leqslant 1$, $0 \leqslant \sin x \leqslant 1$, and $-1 \leqslant \sin y \leqslant 0$. Summing up these 3 inequalities gives us $-2 \leqslant f(x, y) \leqslant 2$, so $f(x, y)^2 \leqslant 4 < \frac{27}{4}$.

All 3 cases above yield that $f(x, y)^2 \leqslant \frac{27}{4}$ and the result follows.

Solution 7 by Michael C. Faleski, Delta College, University Center, MI.

Let P be the product in question. We want to maximize the quantity $P = \cos(x)\cos(y)\sin(x + y)$
So, we take derivatives of the expression finding

$$\frac{\partial P}{\partial y} = -\cos(x)\sin(y)\sin(x + y) + \cos(x)\cos(y)\cos(x + y) = 0$$

$$\cos(x)\left(-\sin(y)\sin(x + y) + \cos(y)\cos(x + y)\right) = 0$$

$$\cos(x)\cos(x + 2y) = 0 \to x = \frac{(2p + 1)\pi}{2} \quad ; \quad x + 2y = \frac{(2n + 1)\pi}{2}$$

and

$$\frac{\partial P}{\partial x} = -\sin(x)\cos(y)\sin(x + y) + \cos(x)\cos(y)\cos(x + y) = 0$$

$$\cos(y)\left(-\sin(x)\sin(x + y) + \cos(x)\cos(x + y)\right) = 0$$

$$\cos(y)\cos(2x + y) = 0 \to y = \frac{(2q + 1)\pi}{2} \quad ; \quad 2x + y = \frac{(2m + 1)\pi}{2}$$

with $m, n, p, q \in \mathbb{Z}$.

We analyze the results by cases.

CASE 1: $\cos(x) = 0$ or $\cos(y) = 0$
Arbitrarily choosing the case of $\cos(x) = 0$ leads to

$$P = (1)\cos(y)\sin(y) = \frac{1}{2}\sin(2y)$$

The maximum value of $\sin(2y) = 1$ leading to $|P| = \frac{1}{2} < \frac{3\sqrt{3}}{8}$

For the other conditions, by taking the difference in the equations gives

$$y - x = (n - m)\pi = r\pi \to \quad y = x + r\pi \quad r \in \mathbb{Z}$$

Because of the periodicty involved with the problem, we can restrict $r = 0, 1$.
By adding the expressions, one finds

$$y + x = \frac{1}{3}(n + m)\pi + \frac{\pi}{3}$$

Combining our relations together allows for solutions to the angles of x and y as

$$y = \frac{\pi}{6} + \frac{\pi}{3}(2n - m) \qquad x = \frac{\pi}{6} + \frac{\pi}{3}(2m - n)$$

CASE 2: $n - m = r = 0$
This restriction makes $x = y = \frac{\pi}{6} + \frac{n\pi}{3}$. Hence,

n	$x = y$	$\|P\|$
0	$\frac{\pi}{6}$	$\| \cos(\frac{\pi}{6}) \cos(\frac{\pi}{6}) \sin(\frac{2\pi}{6})\| = \|(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
1	$\frac{3\pi}{6}$	$\| \cos(\frac{3\pi}{6}) \cos(\frac{3\pi}{6}) \sin(\frac{6\pi}{6})\| = \|(0)(0)(0)\| = 0$
2	$\frac{5\pi}{6}$	$\| \cos(\frac{5\pi}{6}) \cos(\frac{5\pi}{6}) \sin(\frac{10\pi}{6})\| = \|(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
3	$\frac{7\pi}{6}$	$\| \cos(\frac{7\pi}{6}) \cos(\frac{7\pi}{6}) \sin(\frac{14\pi}{6})\| = \|(\frac{-\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})(\frac{\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$
4	$\frac{9\pi}{6}$	$\| \cos(\frac{9\pi}{6}) \cos(\frac{9\pi}{6}) \sin(\frac{18\pi}{6})\| = \|(0)(0)(0)\| = 0$
5	$\frac{11\pi}{6}$	$\| \cos(\frac{11\pi}{6}) \cos(\frac{11\pi}{6}) \sin(\frac{22\pi}{6})\| = \|(\frac{\sqrt{3}}{2})(\frac{\sqrt{3}}{2})(\frac{-\sqrt{3}}{2})\| = \frac{3\sqrt{3}}{8}$

CASE 3: $n - m = r = 1$
Since $x + y = \frac{1}{3}(m + n)\pi + \frac{\pi}{3}$ and $m + n$ must be odd, we restrict $m + n = 1, 3, 5$ as $\sin(2\pi + x) = \sin(x)$.
Therefore, we have cases: $n = 1, m = 0$; $\quad n = 2, m = 1$; $\quad$ and $n = 3, m = 2$ to consider.

n	m	x	y	$\|P\|$
1	0	$\frac{-\pi}{6}$	$\frac{5\pi}{6}$	$\| \cos(\frac{-\pi}{6}) \cos(\frac{5\pi}{6}) \sin(\frac{4\pi}{6})\| = \frac{3\sqrt{3}}{8}$
2	1	$\frac{\pi}{6}$	$\frac{7\pi}{6}$	$\| \cos(\frac{\pi}{6}) \cos(\frac{7\pi}{6}) \sin(\frac{8\pi}{6})\| = \frac{3\sqrt{3}}{8}$
3	2	$\frac{3\pi}{6}$	$\frac{9\pi}{6}$	$\| \cos(\frac{3\pi}{6}) \cos(\frac{9\pi}{6}) \sin(\frac{12\pi}{6})\| = 0$

Consequently, there is no value of $|P| > \frac{3\sqrt{3}}{8}$. This means that
If $x, y \in \mathbb{R}$, then $\| \cos(x) \cos(y) \sin(x + y)\| \leqslant \frac{3\sqrt{3}}{8}$.

Solution 8 by Michel Bataille, Rouen, France.

We have

$$\begin{aligned} \cos x \cos y \sin(x + y) &= \sin x \cos x \cos^2 y + \sin y \cos y \cos^2 x \\ &= \frac{1}{4}((1 + \cos 2y) \sin 2x + (1 + \cos 2x) \sin 2y) \\ &= \frac{1}{4}(\sin 2x + \sin 2y + \sin(2x + 2y)), \end{aligned}$$

hence the problem boils down to proving that $|f(x,y)| \leqslant \dfrac{3\sqrt{3}}{2}$ for all $x, y \in \mathbb{R}$ where

$$f(x,y) = \sin x + \sin y + \sin(x+y).$$

Note that due to periodicity it suffices to prove the inequality for $(x,y) \in [-\pi,\pi] \times [-\pi,\pi]$.
Now, if $(u,v) \in \mathbb{R}^2$ and $f(u,v)$ is a local extremum of f, we must have $\dfrac{\partial f}{\partial x}(u,v) = \dfrac{\partial f}{\partial y}(u,v) = 0$, that is, $\cos u + \cos(u+v) = \cos v + \cos(u+v) = 0$ or equivalently: ($u = v \pmod{2\pi}$ and $\cos 2u + \cos u = 0$) or ($u = -v \pmod{2\pi}$ and $\cos u = -1$). Thus, the candidates for an extremum in $[-\pi,\pi] \times [-\pi,\pi]$ are $(\frac{\pi}{3}, \frac{\pi}{3})$, $(-\frac{\pi}{3}, -\frac{\pi}{3})$, (π,π), $(-\pi,-\pi)$, $(-\pi,\pi)$, $(\pi,-\pi)$. Being continuous on the compact set $[-\pi,\pi] \times [-\pi,\pi]$, the function f attains its (absolute) maximum and minimum on this set (and on $\mathbb{R}^2$) at one of these pairs. However, we have $f(\pi,\pi) = f(-\pi,-\pi) = f(-\pi,\pi) = f(\pi,-\pi) = 0$ while $f(\frac{\pi}{4}, \frac{\pi}{4}) > 0$ and $f(-\frac{\pi}{4}, -\frac{\pi}{4}) < 0$, hence no extremum is attained at (π,π), $(-\pi,-\pi)$, $(-\pi,\pi)$ or $(\pi,-\pi)$. It follows that the maximum and the minimum of f are $f(\frac{\pi}{3}, \frac{\pi}{3}) = \dfrac{3\sqrt{3}}{2}$ and $f(-\frac{\pi}{3}, -\frac{\pi}{3}) = -\dfrac{3\sqrt{3}}{2}$. Thus we have

$$-\frac{3\sqrt{3}}{2} \leqslant f(x,y) \leqslant \frac{3\sqrt{3}}{2}$$

for all $(x,y) \in [-\pi,\pi] \times [-\pi,\pi]$ (and all $(x, y \in \mathbb{R}^2)$. The result follows.

Solution 9 by Moti Levy, Rehovot, Israel.

Since

$$\left|\cos(x)\right| = \left|\sin\left(\frac{\pi}{2} - x\right)\right| = \left|\sin\left(\left(\frac{\pi}{2} - x\right) \bmod \pi\right)\right| = \sin\left(\left(\frac{\pi}{2} - x\right) \bmod \pi\right),$$
$$\left|\sin(x)\right| = \sin(x \bmod \pi),$$

the original inequality can be rewritten as follows:

$$\sin\left(\left(\frac{\pi}{2} - x\right) \bmod \pi\right) \sin\left(\left(\frac{\pi}{2} - y\right) \bmod \pi\right) \sin\left((x+y) \bmod \pi\right) \leqslant \frac{3\sqrt{3}}{8}.$$

By AM-GM inequality,

$$\sin\left(\left(\frac{\pi}{2} - x\right) \bmod \pi\right) \sin\left(\left(\frac{\pi}{2} - y\right) \bmod \pi\right) \sin\left((x+y) \bmod \pi\right)$$
$$\leqslant \left(\frac{\sin\left(\left(\frac{\pi}{2} - x\right) \bmod \pi\right) + \sin\left(\left(\frac{\pi}{2} - y\right) \bmod \pi\right) + \sin\left((x+y) \bmod \pi\right)}{3}\right)^3.$$

By Jensen's inequality

$$\left(\frac{\sin\left(\left(\frac{\pi}{2}-x\right)\bmod\pi\right)+\sin\left(\left(\frac{\pi}{2}-y\right)\bmod\pi\right)+\sin\left((x+y)\bmod\pi\right)}{3}\right)^3$$

$$\leqslant\left(\sin\left(\frac{\left(\frac{\pi}{2}-x\right)\bmod\pi+\left(\frac{\pi}{2}-y\right)\bmod\pi+\sin\left((x+y)\bmod\pi\right)}{3}\right)\right)^3$$

$$=\sin^3\left(\frac{\left(\left(\frac{\pi}{2}-x\right)+\left(\frac{\pi}{2}-y\right)+(x+y)\right)\bmod\pi}{3}\right)$$

$$=\sin^3\left(\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{8}.$$

Solution 10 by Perfetti Paolo, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

It is equivalent

$$F(x,y)\doteq(\cos x)^2(\cos y)^2(\sin(x+y))^2\leqslant\frac{27}{64}$$

$F(x,y)$ is π–periodic both in x and y.

We search the maximum of $F(x,y)$ which exists because $F(x,y)$ is continuous and periodic hence it suffices to search the maximum in $[0,\pi,]\times[0,\pi]$ which is compact.

Let's observe that $F(0,y)\equiv F(x,0)=0$ and $F(\pi,y)=(\sin(2y))^2/4$, $F(x,\pi)=(\sin(2x))^2/4$ thus on the boundary of the square $[0,\pi]\times[0,\pi]$ the functions does not exceed the value $1/4$.

$$F_x=\Big(-2\sin(2x)(\sin(x+y))^2+(\cos x)^2\sin 2(x+y)\Big)(\cos y)^2=0$$

$$F_y=\Big(-2\sin(2x)(\sin(x+y))^2+(\cos y)^2\sin 2(x+y)\Big)(\cos x)^2=0$$

$$F_x=\Big(-2\sin x\sin(x+y)+2\cos x\cos(x+y)\Big)\cos x(\cos y)^2\sin(x+y)=0$$

$$F_y=\Big(-2\sin y\sin(x+y)+2\cos y\cos(x+y)\Big)\cos y(\cos x)^2\sin(x+y)=0$$

$(x,y)=(\pi/2,y)$, $y\in\mathbb{R}$ and $(x,y)=(x,\pi/2)$, $x\in\mathbb{R}$ all are critical points. Moreover $\{(x,y)\in[0,\pi]\times[0,\pi]\colon x+y=k\pi, k=0,1,2\}$ also are critical points. Since $F(x,y)$ annihilates on each of the above points, no one of them can be point of maximum. Actually they are all point of minimum.

Based on that we can write

$$F_x = -\sin x \sin(x+y) + \cos x \cos(x+y) = 0 \implies \operatorname{cotg}(x+y) = \tan x \qquad (1)$$

$$F_y = -\sin y \sin(x+y) + \cos y \cos(x+y) \implies \operatorname{cotg}(x+y) = \tan y \qquad (2)$$

hence $\tan x = \tan y$, $y = x$. It follows

$$\tan x = \frac{1}{\tan(2x)} \iff \tan x = \frac{1-(\tan x)^2}{2\tan x} \implies \tan x = \frac{1}{\sqrt{3}} \implies x = \frac{\pi}{6} + k\pi$$

Clearly by periodicity of $F(x,y)$ it suffices to consider $x = \pi/6$ and then $y = \pi/6$.

$$F(\pi/6, \pi/6) = 27/64 > 1/4$$

and then $(\pi/6, \pi/6)$ is the point of the searched maximum.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; and the problem proposer.

• **5717** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Let h_1 and h_2 be nonnegative integers. Prove the following limit:

$$\lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_2}{k}}\right)^{\frac{1}{n}} = e^{h_1-h_2}.$$

Solution 1 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

It is enough to prove that for a a nonnegative integer

$$\lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+a+1}{k}}{\prod_{k=1}^{n}\binom{n+a}{k}}\right)^{\frac{1}{n}} = e.$$

Notice that for each $k = 1, 2, \ldots, n$, $\dfrac{\binom{n+a+1}{k}}{\binom{n+a}{k}} = \dfrac{n+a+1}{n+a+1-k}$, and therefore

$$\begin{aligned}
\frac{\prod_{k=1}^{n}\binom{n+a+1}{k}}{\prod_{k=1}^{n}\binom{n+a}{k}} &= \frac{(n+a+1)^n a!}{(n+a)!} \\
\lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+a+1}{k}}{\prod_{k=1}^{n}\binom{n+a}{k}}\right)^{\frac{1}{n}} &= \lim_{n\to\infty}\frac{(n+a+1)\sqrt[n]{a!}}{\sqrt[n]{(n+a)^{n+a}e^{-n-a}\sqrt{2\pi(n+a)}}} \\
&= e.
\end{aligned}$$

Solution 2 by Moti Levy, Rehovot, Israel.

Let $L := \lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_2}{k}}\right)^{\frac{1}{n}}$. We may assume that $h_1 \geqslant h_2$, without loss of generality.

$$\begin{aligned}
\frac{\prod_{k=1}^{n}\binom{n+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_2}{k}} &= \prod_{k=1}^{n}\frac{\binom{n+h_1}{k}}{\binom{n+h_2}{k}} = \prod_{k=1}^{n}\frac{(n+h_2-k)!\,(n+h_1)!}{(n+h_1-k)!\,(n+h_2)!} \\
&= \prod_{k=0}^{n-1}\frac{(k+1+h_1)\,(k+2+h_1)\cdots(n+h_1)}{(k+1+h_2)\,(k+2+h_2)\cdots(n+h_2)} \\
&= \prod_{k=0}^{n-1}\frac{(n+h_2+1)\cdots(n+h_1)}{(k+h_2+1)\cdots(k+h_1)} \\
&= \prod_{k=1}^{n}\frac{\left(1+\frac{h_2+1}{n}\right)\cdots\left(1+\frac{h_1}{n}\right)}{\left(\frac{k+h_2}{n}\right)\cdots\left(\frac{k+h_1-1}{n}\right)}
\end{aligned}$$

$$L = \lim_{n\to\infty}\left(\left(1+\frac{h_2+1}{n}\right)\cdots\left(1+\frac{h_1}{n}\right)\right)\lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{1}{\left(\frac{k+h_2}{n}\right)\cdots\left(\frac{k+h_1-1}{n}\right)}\right)^{\frac{1}{n}},$$

but

$$\lim_{n\to\infty}\left(\left(1+\frac{h_2+1}{n}\right)\cdots\left(1+\frac{h_1}{n}\right)\right) = \lim_{n\to\infty}\left(1+\frac{h_2+1}{n}\right)\cdots\lim_{n\to\infty}\left(1+\frac{h_1}{n}\right) = 1,$$

hence

$$L = \lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{1}{\left(\frac{k+h_2}{n}\right)\cdots\left(\frac{k+h_1-1}{n}\right)}\right)^{\frac{1}{n}}.$$

$$
\begin{aligned}
\ln(L) &= \lim_{n\to\infty}\left(-\frac{1}{n}\sum_{k=1}^{n}\sum_{m=h_2}^{h_1-1}\ln\left(\frac{k+m}{n}\right)\right)\\
&= \lim_{n\to\infty}\left(\sum_{m=h_2}^{h_1-1}\left(-\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k+m}{n}\right)\right)\right)\\
&= \sum_{m=h_2}^{h_1-1}\lim_{n\to\infty}\left(-\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k+m}{n}\right)\right).
\end{aligned}
$$

Now we have

$$
\lim_{n\to\infty}\left(-\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k+m}{n}\right)\right) = -\int_0^1 \ln(x)\,dx = 1.
$$

It follows that

$$
\ln(L) = \sum_{m=h_2}^{h_1-1}(1) = h_1 - h_2,
$$

and that

$$
L = e^{h_1-h_2}.
$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We may assume that $h_2 \geqslant h_1 \geqslant 0$ by considering, if necessary, the reciprocal of the limit. We have, by Stirling's asymptotic formula for the factorials,

$$
\frac{\prod_{k=1}^{n}\begin{pmatrix} n+h_1 \\ k \end{pmatrix}}{\prod_{k=1}^{n}\begin{pmatrix} n+h_2 \\ k \end{pmatrix}} = \prod_{k=1}^{n}\frac{(n+h_1)!\,(n+h_2-k)!}{(n+h_1-k)!\,(n+h_2)!} =
$$

$$
= \prod_{k=1}^{n}\frac{(n+h_1)^{n+h_1+\frac{1}{2}}e^{-n-h_1+O\left(\frac{1}{n}\right)}(n+h_2-k)^{n+h_2-k+\frac{1}{2}}e^{-n-h_2+k+O\left(\frac{1}{n+1-k}\right)}}{(n+h_1-k)^{n+h_1-k+\frac{1}{2}}e^{-n-h_1+k+O\left(\frac{1}{n+1-k}\right)}(n+h_2)^{n+h_2+\frac{1}{2}}e^{-n-h_2+O\left(\frac{1}{n}\right)}} =
$$

$$
= e^{O(\log n)}\prod_{k=1}^{n}\frac{(n+h_1)^{n+h_1+\frac{1}{2}}(n+h_2-k)^{n+h_2-k+\frac{1}{2}}}{(n+h_1-k)^{n+h_1-k+\frac{1}{2}}(n+h_2)^{n+h_2+\frac{1}{2}}} =
$$

$$
= e^{O(\log n)}\left(\frac{(n+h_1)^{n+h_1+\frac{1}{2}}}{(n+h_2)^{n+h_2+\frac{1}{2}}}\right)^{n}\prod_{k=0}^{n-1}\frac{(h_2+k)^{h_2+k+\frac{1}{2}}}{(h_1+k)^{h_1+k+\frac{1}{2}}} =
$$

$$
= e^{O(\log n)}\left(\frac{(n+h_1)^{n}(n+h_1)^{h_1+\frac{1}{2}}}{(n+h_2)^{n}(n+h_2)^{h_2+\frac{1}{2}}}\right)^{n}\prod_{k=0}^{h_2-h_1-1}\frac{1}{(h_1+k)^{h_1+k+\frac{1}{2}}}\prod_{k=0}^{h_2-h_1-1}(h_1+n+k)^{h_1+n+k+\frac{1}{2}}.
$$

We note that

$$\lim_{n\to\infty} \frac{(n+h_1)^n}{(n+h_2)^n} = \lim_{n\to\infty} \frac{\left(1+\frac{h_1}{n}\right)^n}{\left(1+\frac{h_2}{n}\right)^n} = e^{h_1-h_2},$$

$$\lim_{n\to\infty} \prod_{k=0}^{h_2-h_1-1} \left(\frac{1}{(h_1+k)^{h_1+k+\frac{1}{2}}}\right)^{\frac{1}{n}} = 1,$$

and

$$\lim_{n\to\infty} \prod_{k=0}^{h_2-h_1-1} (h_1+n+k)^{\frac{1}{n}\left(h_1+k+\frac{1}{2}\right)} = 1.$$

Hence

$$\lim_{n\to\infty} \left(\frac{\prod_{k=1}^{n} \begin{pmatrix} n+h_1 \\ k \end{pmatrix}}{\prod_{k=1}^{n} \begin{pmatrix} n+h_2 \\ k \end{pmatrix}} \right)^{\frac{1}{n}} = e^{h_1-h_2} \lim_{n\to\infty} \frac{(n+h_1)^{h_1+\frac{1}{2}}}{(n+h_2)^{h_2+\frac{1}{2}}} \prod_{k=0}^{h_2-h_1-1} (h_1+n+k)^{\frac{1}{n}\left(h_1+n+k+\frac{1}{2}\right)} =$$

$$= e^{h_1-h_2} \lim_{n\to\infty} \frac{1}{n^{h_2-h_1}} \prod_{k=0}^{h_2-h_1-1} (h_1+n+k) \prod_{k=0}^{h_2-h_1-1} (h_1+n+k)^{\frac{1}{n}\left(h_1+k+\frac{1}{2}\right)} =$$

$$= e^{h_1-h_2} \prod_{k=0}^{h_2-h_1-1} \left(1+\frac{h_1+k}{n}\right) = e^{h_1-h_2}.$$

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

If $h_1 = h_2$, then

$$\lim_{n\to\infty} \left(\frac{\prod_{k=1}^{n} \binom{n+h_1}{k}}{\prod_{k=1}^{n} \binom{n+h_2}{k}} \right)^{1/n} = 1 = e^0 = e^{h_1-h_2}.$$

Suppose $h_1 > h_2$. Because

$$\binom{n+h_1}{0} = \binom{n+h_2}{0} = 1,$$

each product inside the desired limit can start from $k = 0$. Now

$$\frac{\binom{n+h_1}{k}}{\binom{n+h_2}{k}} = \prod_{j=0}^{h_1-h_2-1} \frac{n+h_1-j}{n+h_1-k-j},$$

so

$$\frac{\prod_{k=0}^{n} \binom{n+h_1}{k}}{\prod_{k=0}^{n} \binom{n+h_2}{k}} = \prod_{k=0}^{n} \frac{\binom{n+h_1}{k}}{\binom{n+h_2}{k}} = \prod_{j=0}^{h_1-h_2-1} \frac{(h_1-1-j)! \cdot (n+h_1-j)^{n+1}}{(n+h_1-j)!}.$$

Using Stirling's approximation, for large n,

$$\frac{(n+h_1-j)^{n+1}}{(n+h_1-j)!} \sim \frac{(n+h_1-j)^{n+1}}{(n+h_1-j)^{n+h_1-j+1/2}e^{-(n+h_1-j)}} = (n+h_1-j)^{1/2+j-h_1}e^{n+h_1-j},$$

so

$$\frac{\prod_{k=0}^{n}\binom{n+h_1}{k}}{\prod_{k=0}^{n}\binom{n+h_2}{k}} \sim \prod_{j=0}^{h_1-h_2-1}(h_1-1-j)!(n+h_1-j)^{1/2+j-h_1}e^{n+h_1-j}.$$

For each $j = 0, 1, 2, \ldots, h_1 - h_2 - 1$,

$$\begin{aligned}\lim_{n\to\infty}\left((h_1-1-j)!\right)^{1/n} &= 1,\\ \lim_{n\to\infty}\left((n+h_1-j)^{1/2+j-h_1}\right)^{1/n} &= 1, \text{ and}\\ \lim_{n\to\infty}\left(e^{n+h_1-j}\right)^{1/n} &= e.\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_2}{k}}\right)^{1/n} = \prod_{j=0}^{h_1-h_2-1} e = e^{h_1-h_2}.$$

Finally, suppose $h_1 < h_2$. Then, by the above analysis,

$$\lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+h_2}{k}}{\prod_{k=1}^{n}\binom{n+h_1}{k}}\right)^{1/n} = e^{h_2-h_1},$$

or

$$\lim_{n\to\infty}\left(\frac{\prod_{k=1}^{n}\binom{n+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_2}{k}}\right)^{1/n} = e^{-(h_2-h_1)} = e^{h_1-h_2}.$$

Solution 5 by G. C. Greubel, Newport News, VA.

In this solution the use x and y will be used for h_1 and h_2, respectively. By using

$$P_n(x) = \prod_{k=1}^{n}\binom{n+x}{k} = \frac{\Gamma^n(n+x+1)\,G(x+1)}{G(n+2)\,G(n+x+1)},$$

where $\Gamma(x)$ is the Gamma function and $G(x)$ is the Barnes G-function, then

$$\left(\frac{P_n(x)}{P_n(y)}\right)^{1/n} = \frac{\Gamma(n+x+1)}{\Gamma(n+y+1)}\left(\frac{G(x+1)}{G(y+1)}\right)^{1/n}\left(\frac{G(n+y+1)}{G(n+x+1)}\right)^{1/n}.$$

The Stirling approximation for the Gamma function and the Stirling-like approximation for the Barnes G-function are

$$\begin{aligned}\Gamma(x+1) &\approx \sqrt{2\pi x}\,x^x\,e^{-x}\left(1+O\left(\frac{1}{x}\right)\right)\\ G(x+1) &\approx (2\pi)^{x/2}\,x^{x^2/2-1/(12)}\,e^{-3x^2/4+\zeta'(-1)}\left(1+O\left(\frac{1}{x}\right)\right)\end{aligned}$$

and lead to

$$\left(\frac{\Gamma(n+x+1)}{\Gamma(n+y+1)}\right)^{1/n} \approx n^{x-y} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)$$

$$\begin{aligned}\left(\frac{G(n+y+1)}{G(n+x+1)}\right)^{1/n} &\approx (2\pi)^{\frac{y-x}{2n}} \left(n\, e^{-3/4}\right)^{(y-x)\left(1+\frac{x+y}{2n}\right)} \\ &\quad \cdot \left(\frac{n+y}{n+x}\right)^{n/2} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\ &\approx n^{y-x}\, e^{x-y} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right),\end{aligned}$$

for $n \to \infty$, and

$$\left(\frac{P_n(x)}{P_n(y)}\right)^{1/n} \approx e^{x-y} \left(\frac{G(x+1)}{G(y+1)}\right)^{1/n} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

Now taking the desired limit leads to the expected result, namely,

$$\lim_{n\to\infty} \left(\frac{\prod\limits_{k=1}^{n} \binom{n+x}{k}}{\prod\limits_{k=1}^{n} \binom{n+y}{k}}\right)^{\frac{1}{n}} = e^{x-y}.$$

Solution 6 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Let H be the initial limit. We define the function $f(x) = \log\left(\frac{h_1+x}{h_2+x}\right)^x$ for all $x \in \mathbb{Z}^+$. Besides that, we rewrite the following expression.

$$\frac{\prod_{k=1}^{n}\binom{n+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_2}{k}} = \prod_{k=1}^{n}\prod_{i=0}^{k-1} \frac{h_1+n-i}{h_2+n-i} = \prod_{k=1}^{n}\left(\frac{h_1+k}{h_2+k}\right)^k.$$

Observe that

$$H = \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{h_1+k}{h_2+k}\right)^{\frac{k}{n}},$$

and

$$\log H = \lim_{n\to\infty} \frac{f(1)+f(2)+....+f(n)}{n}.$$

Also,

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \log\left(1+\frac{h_1-h_2}{h_2+n}\right)^n = \log\left(\lim_{n\to\infty}\left(1+\frac{h_1-h_2}{h_2+n}\right)^n\right) = \log e^{h_1-h_2} = h_1-h_2.$$

By Cauchy first limit theorem,

$$\log H = \lim_{n\to\infty} \frac{f(1)+f(2)+....+f(n)}{n} = \lim_{n\to\infty} f(n) = h_1 - h_2 \quad \Longrightarrow \quad H = e^{h_1-h_2}.$$

The proof completes.

Solution 7 by Michel Bataille, Rouen, France.

Let m be a fixed nonnegative integer and let $p_n^{[m]} = \prod_{k=1}^{n} \binom{n+m}{k}$. Using the identity $\binom{n+1+m}{k} = \frac{n+1+m}{k}\binom{n+m}{k-1}$, we easily obtain

$$p_{n+1}^{[m]} = \frac{(n+1+m)^{n+1}}{(n+1)!} \cdot p_n^{[m]}.$$

It follows that

$$\frac{p_{n+1}^{[m]}}{p_n^{[m]}} = \frac{(n+m+1)^n}{n!} \cdot \frac{n+1+m}{n+1} \sim \frac{n^n\left(1+\frac{m+1}{n}\right)^n}{\sqrt{2\pi n}\, n^n e^{-n}} \sim \frac{e^{m+1}}{\sqrt{2\pi}} \cdot \frac{e^n}{\sqrt{n}}$$

as $n \to \infty$.

Now, let $u_n = e^{-\frac{n^2}{2}} n^{\frac{n}{2}} p_n^{[m]}$. We readily obtain that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{e^{m+1}}{\sqrt{2\pi}}$. Thus, we also have $\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{e^{m+1}}{\sqrt{2\pi}}$, which means that

$$\left(p_n^{[m]}\right)^{\frac{1}{n}} \sim \frac{e^{m+1}}{\sqrt{2\pi}} \cdot \frac{e^{\frac{n}{2}}}{\sqrt{n}}.$$

As a result, the ratio $q_n = \frac{\left(p_n^{[h_1]}\right)^{\frac{1}{n}}}{\left(p_n^{[h_2]}\right)^{\frac{1}{n}}}$ satisfies

$$q_n \sim \frac{(e^{h_1+1}/\sqrt{2\pi}) \cdot (e^{\frac{n}{2}}/\sqrt{n})}{(e^{h_2+1}/\sqrt{2\pi}) \cdot (e^{\frac{n}{2}}/\sqrt{n})},$$

as $n \to \infty$, hence

$$\lim_{n\to\infty} q_n = e^{(h_1+1)-(h_2+1)} = e^{h_1-h_2},$$

as desired.

Solution 8 by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.

Let's employ Cesąro–Stolz theorem and study

$$\lim_{n\to\infty} \frac{\prod_{k=1}^{n+1}\binom{n+1+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_1}{k}} \frac{\prod_{k=1}^{n}\binom{n+h_2}{k}}{\prod_{k=1}^{n+1}\binom{n+1+h_2}{k}} = \lim_{n\to\infty} \underbrace{\frac{\prod_{k=1}^{n}\binom{n+1+h_1}{k}}{\prod_{k=1}^{n}\binom{n+h_1}{k}}}_{\mathbf{A}} \underbrace{\frac{\prod_{k=1}^{n}\binom{n+h_2}{k}}{\prod_{k=1}^{n}\binom{n+1+h_2}{k}}}_{\mathbf{B}} \underbrace{\frac{\binom{n+1+h_1}{n+1}}{\binom{n+1+h_2}{n+1}}}_{\mathbf{C}}$$

$$\frac{\binom{n+1+h_1}{k}}{\binom{n+h_1}{k}} = \frac{(n+1+h_1)!}{k!(n+1+h_1-k)!}\frac{k!(n+h_1-k)!}{(n+h_1)!} = \frac{n+1+h_1}{n+1+h_1-k}$$

thus we get

$$\lim_{n\to\infty} \underbrace{\prod_{k=1}^{n}\frac{n+1+h_1}{n+1+h_1-k}}_{\mathbf{A}} \underbrace{\prod_{k=1}^{n}\frac{n+1+h_2-k}{n+1+h_2}}_{\mathbf{B}} \underbrace{\frac{\binom{n+1+h_1}{n+1}}{\binom{n+1+h_2}{n+1}}}_{\mathbf{C}} =$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n}\frac{n+1+h_1}{n+1+h_2}\prod_{k=0}^{n-1}\frac{k+1+h_2}{k+1+h_1}\frac{\binom{n+1+h_1}{n+1}}{\binom{n+1+h_2}{n+1}}$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n}\frac{n+1+h_1}{n+1+h_2}\lim_{n\to\infty}\prod_{k=0}^{n-1}\frac{k+1+h_2}{k+1+h_1}\frac{\binom{n+1+h_1}{n+1}}{\binom{n+1+h_2}{n+1}}$$

$$\lim_{n\to\infty} \prod_{k=1}^{n}\frac{n+1+h_1}{n+1+h_2} = \lim_{n\to\infty}\frac{\left(1+\frac{h_1}{n+1}\right)^n}{\left(1+\frac{h_2}{n+1}\right)^n} = \frac{e^{h_1}}{e^{h_2}} = e^{h_1-h_2}$$

$$\prod_{k=0}^{n-1}\frac{k+1+h_2}{k+1+h_1}\frac{\binom{n+1+h_1}{n+1}}{\binom{n+1+h_2}{n+1}} = \frac{(n+h_2)!}{h_2!}\frac{h_1!}{(n+h_1)!}\frac{(n+1+h_1)!}{(n+1)!\,h_1!}\frac{(n+1)!\,h_2!}{(n+1+h_2)!}$$

$$= \frac{n+1+h_1}{n+1+h_2} \to 1$$

hence concluding the proof.

Also solved by the problem proposer.

• **5718** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

Find all real solutions of the system of equations

$$\left\{\begin{array}{rcl} x^4+25y^2+12 &=& 8y^3+28z+4x\sqrt{x-1} \\ y^4+25z^2+12 &=& 8z^3+28x+4y\sqrt{y-1} \\ z^4+25x^2+12 &=& 8x^3+28y+4z\sqrt{z-1} \end{array}\right\}.$$

Solution 1 by Michel Bataille, Rouen, France.

Let (x, y, z) be a solution. Then, $x, y, z \geqslant 1$ and the nonnegative real numbers $u = \sqrt{x-1}, v = \sqrt{y-1}, w = \sqrt{z-1}$ satisfy $x = 1 + u^2, y = 1 + v^2, w = 1 + z^2$. The system and a simple calculation lead to

$$p(u) = 8v^6 - v^4 - 26v^2 + 28w^2, \quad p(v) = 8w^6 - w^4 - 26w^2 + 28u^2, \quad p(w) = 8u^6 - u^4 - 26u^2 + 28v^2,$$

where $p(t) = t^8 + 4t^6 + 6t^4 - 4t^3 + 4t^2 - 4t + 2$.
We observe that $p(t) = (t-1)^4(1 + (t+1)^4) + 8t^6 - t^4 + 2t^2$ and deduce that

$$\left\{ \begin{array}{rcl} (u-1)^4(1+(u+1)^4) & = & 8(v^6-u^6)-(v^4-u^4)-26(v^2-u^2)+28(w^2-u^2) \\ (v-1)^4(1+(v+1)^4) & = & 8(w^6-v^6)-(w^4-v^4)-26(w^2-v^2)+28(u^2-v^2) \\ (w-1)^4(1+(w+1)^4) & = & 8(u^6-w^6)-(u^4-w^4)-26(u^2-w^2)+28(v^2-w^2) \end{array} \right\}.$$

By addition, we obtain

$$(u-1)^4(1+(u+1)^4) + (v-1)^4(1+(v+1)^4) + (w-1)^4(1+(w+1)^4) = 0$$

so that $u = v = w = 1$. It follows that $x = y = z = 2$.
Conversely, it is readily checked that $(2, 2, 2)$ is a solution for (x, y, z).
In conclusion, $(2, 2, 2)$ is the unique solution.

Solution 2 by Péter Fülöp, Gyömrő, Hungary.

Let's add the three equations and set them to zero:

$$x^4+y^4+z^4+25(x^2+y^2+z^2)-8(x^3+y^3+z^3)-28(x+y+z)-4x\sqrt{x-1}-4y\sqrt{y-1}-4z\sqrt{z-1} = 0$$

Rearranged the equation we get:

$$\left\{ \begin{array}{c} x^4 - 8x^3 + 25x^2 - 28x + 12 - 4x\sqrt{x-1}+ \\ y^4 - 8y^3 + 25y^2 - 28y + 12 - 4y\sqrt{y-1}+ \\ z^4 - 8z^3 + 25z^2 - 28z + 12 - 4z\sqrt{z-1} \end{array} \right\} = 0$$

We can separate it to three different equations are also equals to zero separately:

$$\begin{array}{c} x^4 - 8x^3 + 25x^2 - 28x + 12 - 4x\sqrt{x-1} = 0 \\ y^4 - 8y^3 + 25y^2 - 28y + 12 - 4y\sqrt{y-1} = 0 \\ z^4 - 8z^3 + 25z^2 - 28z + 12 - 4z\sqrt{z-1} = 0 \end{array}$$

Let's apply the following substitutions:

$$\begin{array}{rl} t = & \sqrt{x-1} \\ u = & \sqrt{y-1} \\ v = & \sqrt{z-1} \end{array}$$

Regarding the first equation we get:

$$(t^2+1)^4 - 8(t^2+1)^3 + 25(t^2+1)^2 - 28(t^2+1) + 12 - 4t(t^2+1) = 0$$

After performing the exponentiations:

$$t^8 - 4t^6 + 7t^4 - 4t^3 + 2t^2 - 4t + 2 = (t-1)^4[(t+1)^4+1] = 0$$

Performed the same operations with u and v we get the following equations:

$$\begin{gathered}(t-1)^4[(t+1)^4+1] = 0\\ (u-1)^4[(u+1)^4+1] = 0\\ (v-1)^4[(v+1)^4+1] = 0\end{gathered}$$

In the all three cases the solution equals to one. So

$$t = u = v = 1 \text{ and the solutions of the system equations: } x = y = z = 2$$

Solution 3 by Moti Levy, Rehovot, Israel.

Adding the three equations,

$$\begin{aligned}&x^4 + 25y^2 + 12 + 8y^3 + y^4 + 25z^2 + 12 + z^4 + 25x^2 + 12 \\ &= 8y^3 + 28z + 4x\sqrt{x-1} + 8z^3 + 28x + 4y\sqrt{y-1} + 8z^3 + 28y + 4z\sqrt{z-1}\end{aligned} \tag{1}$$

Let

$$f(x) := x^4 - 8x^3 + 25x^2 - 28x + 12 - 4x\sqrt{x-1}. \tag{2}$$

Equation (1) is a necessary condition for a solution and is equivalent to:

$$f(x) + f(y) + f(z) = 0. \tag{3}$$

Substitute $x = u^2 + 1$ in (2):

$$\begin{aligned}f(x) = f\left(u^2+1\right) &= \left(u^2+1\right)^4 - 8\left(u^2+1\right)^3 + 25\left(u^2+1\right)^2 - 28\left(u^2+1\right) + 4 - 4\left(u^2+1\right)u + 8\\ &= \left(4u + 6u^2 + 4u^3 + u^4 + 2\right)(u-1)^4.\end{aligned}$$

It follows that $f\left(u^2+1\right) > 0$ for $u > 0$ and $f\left(u^2+1\right) = 0$ for $u = 1$, hence

$$f(x) > 0 \text{ for } x > 1, \quad \text{and } f(2) = 0.$$

We conclude that only $x = y = z = 2$ meets the necessary condition (2) hence it is the sole solution of the system.

Solution 4 by G. C. Greubel, Newport News, VA.

For the triplet (x, y, z) the equations remain the same with the variable changes $(x, y, z) \to (y, z, x)$

and $(x, y, z) \to (z, x, y)$. The second consideration is the square root terms. If the solutions are to real, or integer, then the variables in the triplet will be of the form $n^2 + 1$. In terms of integer solutions then the variables will be values in the sequence $A002522$ of the OEIS. With these aspects in mind the equations reduce to solving

$$x^4 + 25\,x^2 + 12 = 8\,x^3 + 28\,x + 4\,x\,\sqrt{x-1}$$

and making the change $x \to t^2 + 1$ leads to $t^8 - 4t^6 + 7t^4 - 4t^3 + 2t^2 - 4t + 2 = 0$ or $(t-1)^4\,(t^4 + 4t^3 + 6t^2 + 4t + 2) = 0$. The real valued solution of this equation is $t = 1$ which gives $x = 2$ and $(x, y, z) = (2, 2, 2)$.

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

Clearly, x,y,z⩾1, since the square root of a negative number is not real. We have

$$0 = \sum_{cycl}\left(x^4 + 25y^2 + 12 - 8y^3 - 28z - 4x\sqrt{x-1}\right) =$$

$$= \sum_{cycl}\left(x^4 + 25x^2 + 12 - 8x^3 - 28x - 4x\sqrt{x-1}\right) =$$

$$= \sum_{cycl}\left((x-2)^4 + \left(x - 2\sqrt{x-1}\right)^2\right)$$

which implies that x=y=z=2. The soluition (x,y,z)=(2,2,2) is indeed a real solution of the given system of equations and it is the only one.

Also solved by and the problem proposer.

• **5719** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.*

Prove the compound inequality

$$\sqrt{2} < \int_0^1 x^x dx + \left(\int_0^1 x^{-x} dx\right)^{-1} < 2.$$

Solution 1 by Moti Levy, Rehovot, Israel.

Since $x^x < 1$ and $x^{-x} > 1$ for $x \in (0, 1)$, we have

$$\int_0^1 x^x dx < 1, \quad \int_0^1 x^{-x} dx > 1.$$

It follows that

$$\int_0^1 x^x dx + \frac{1}{\int_0^1 x^{-x} dx} < 2.$$

$$x^x > 1 + x\ln(x), \quad x \in (0, 1),$$

hence

$$\int_0^1 x^x dx > \int_0^1 \left(1 + x\ln(x)\right) dx = \frac{3}{4}.$$

$$x^{-x} < e^{\frac{1}{e}}, \quad x \in (0,1),$$

hence

$$\int_0^1 x^{-x} dx < e^{\frac{1}{e}}$$

It follows that

$$\int_0^1 x^x dx + \frac{1}{\int_0^1 x^{-x} dx} > \frac{3}{4} + e^{-\frac{1}{e}}.$$

Now, $\frac{3}{4} + e^{-\frac{1}{e}} > \sqrt{2}$ since $\frac{3}{4} + e^{-\frac{1}{e}} \cong 1.4422$ and $\sqrt{2} \cong 1.4142$.

Solution 2 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Let

$$A = \int_0^1 x^x \mathrm{d}x = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} \cdots \qquad B = \int_0^1 x^{-x} \mathrm{d}x = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} \cdots$$

Clearly $A < 1$,$B > 1$. So $A + \frac{1}{B} < 2$. Also, $A + \frac{1}{B} \geqslant 2\sqrt{\frac{A}{B}}$ and $B < 2A$ which implies $\frac{A}{B} > \frac{1}{2}$. Therefore $A + \frac{1}{B} > 2\sqrt{\frac{1}{2}} = \sqrt{2}$. We conclude $\sqrt{2} < \int_0^1 x^x \mathrm{d}x + \left(\int_0^1 x^{-x} \mathrm{d}x\right)^{-1} < 2$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\int_0^1 x^x dx = \int_0^1 e^{x\log x} dx = \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 (x\log x)^k dx = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^k}$$

and similarly

$$\int_0^1 x^{-x} dx = \sum_{k=1}^{\infty} \frac{1}{k^k}.$$

We estimate the infinite sums as follows:

$$\sum_{k=1}^{2K} \frac{(-1)^{k-1}}{k^k} < \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^k} < \sum_{k=1}^{2K+1} \frac{(-1)^{k-1}}{k^k}$$

and

$$\sum_{k=1}^{2K} \frac{1}{k^k} < \sum_{k=1}^{\infty} \frac{1}{k^k} < \sum_{k=1}^{2K} \frac{1}{k^k} + \sum_{k=2K+1}^{\infty} \frac{1}{(2K+1)^k} = \sum_{k=1}^{2K} \frac{1}{k^k} + \frac{1}{2K(2K+1)^{2K}}.$$

K=2 then gives

$$\int_0^1 x^x dx + \left(\int_0^1 x^{-x} dx\right)^{-1} > \sum_{k=1}^{4} \frac{(-1)^{k-1}}{k^k} + \frac{1}{\sum_{k=1}^4 \frac{1}{k^k} + \frac{1}{4(5)^4}} > 1.55 > \sqrt{2}$$

and

$$\int_0^1 x^x dx + \left(\int_0^1 x^{-x} dx\right)^{-1} < \sum_{k=1}^{5} \frac{(-1)^{k-1}}{k^k} + \frac{1}{\sum_{k=1}^{4} \frac{1}{k^k}} < 1.56 < 2.$$

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let $f(x) = x^x$. With

$$\lim_{x\to 0^+} f(x) = 1, \quad f(1) = 1, \quad \text{and} \quad f'(x) = x^x(1 + \ln x),$$

it follows that

$$e^{-1/e} \leqslant f(x) \leqslant 1$$

for $x \in [0, 1]$. Thus,

$$e^{-1/e} < \int_0^1 x^x\, dx < 1, \quad 1 < \int_0^1 x^{-x}\, dx < e^{1/e}$$

and

$$2e^{-1/e} < \int_0^1 x^x\, dx + \left(\int_0^1 x^{-x}\, dx\right)^{-1} < 2.$$

This satisfies the desired inequality on the right, but not on the left as $2e^{-1/e} < \sqrt{2}$. To obtain tighter bounds, note

$$x^x = e^{x \ln x} = \sum_{k=0}^{\infty} \frac{(x \ln x)^k}{k!},$$

so

$$\int_0^1 x^x\, dx = \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^1 x^k \ln^k x\, dx = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (-1)^k \frac{k!}{(k+1)^{k+1}} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^k}.$$

By the approximation property for alternating series,

$$\left| \int_0^1 x^x\, dx - \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k^k} \right| < \frac{1}{(n+1)^{n+1}}.$$

In particular, with $n = 3$,

$$\left| \int_0^1 x^x\, dx - \frac{85}{108} \right| < \frac{1}{256}, \quad \text{or} \quad \frac{5413}{6912} < \int_0^1 x^x\, dx < \frac{5467}{6912}.$$

Next,

$$\int_0^1 x^{-x}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 x^k \ln^k x\, dx = \sum_{k=1}^{\infty} \frac{1}{k^k},$$

so

$$\sum_{k=1}^{n} \frac{1}{k^k} < \int_0^1 x^{-x}\, dx = \sum_{k=1}^{n} \frac{1}{k^k} + \sum_{k=n+1}^{\infty} \frac{1}{k^k} < \sum_{k=1}^{n} \frac{1}{k^k} + \int_n^{\infty} \frac{1}{x^x}\, dx.$$

Continuing,

$$\int_n^\infty \frac{1}{x^x}\,dx < \int_n^\infty \frac{1}{x^n}\,dx = \frac{1}{(n-1)n^{n-1}},$$

and

$$\sum_{k=1}^n \frac{1}{k^k} < \int_0^1 x^{-x}\,dx < \sum_{k=1}^n \frac{1}{k^k} + \frac{1}{(n-1)n^{n-1}};$$

in particular, with $n = 3$,

$$\frac{139}{108} < \int_0^1 x^{-x}\,dx < \frac{139}{108} + \frac{1}{18} = \frac{145}{108}.$$

Thus,

$$\frac{5413}{6912} + \frac{108}{145} < \int_0^1 x^x\,dx + \left(\int_0^1 x^{-x}\,dx\right)^{-1} < \frac{5467}{6912} + \frac{108}{139},$$

or

$$\frac{1531381}{1002240} < \int_0^1 x^x\,dx + \left(\int_0^1 x^{-x}\,dx\right)^{-1} < \frac{1506409}{960768}.$$

As

$$\frac{1531381}{1002240} > \sqrt{2} \quad \text{and} \quad \frac{1506409}{960768} < 2,$$

tighter bounds than those requested have been obtained.

Solution 5 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

Let $y = x^x$. Then $\ln y = x\ln x$ and $y' = x^x(1+\ln x)$, so that x^x is decreasing over the interval $\left(0, \frac{1}{e}\right)$, increasing over $\left(\frac{1}{e}, 1\right)$, and $x^x \geqslant \left(\frac{1}{e}\right)^{1/e}$ for all $x \in (0,1)$. For convenience, we set $m = \left(\frac{1}{e}\right)^{1/e} \approx 0.6922$.

A standard logarithmic application of L'Hospital's Rule reveals that

$$\lim_{x\to 0^+} x^x = 1,$$

so that

$$m \leqslant x^x \leqslant 1 \text{ for all } x \in (0,1].$$

Since $m < 1$, then

$$\int_0^1 x^x\,dx < \int_0^1 1\,dx = 1.$$

In addition,

$$y'' = x^x\left(\frac{1}{x} + (1+\ln x)^2\right) > 0,$$

so that $y = x^x$ is concave up over the interval $(0, 1)$. Since $y' = 1$ at $x = 1$, then $y = x^x$ lies above the tangent line $y = x$. Thus,

$$\int_0^1 x^x \geqslant \int_0^m m\,dx + \int_m^1 x\,dx = m + \frac{(1-m)^2}{2}.$$

Since $m \leqslant x^x \leqslant 1$, then $1 \leqslant x^{-x} \leqslant \dfrac{1}{m}$ for all $x \in (0, 1)$, so that

$$1 \leqslant \int_0^1 x^{-x}\,dx \leqslant \int_0^1 \frac{1}{m}\,dx = \frac{1}{m}$$

and

$$m \leqslant \left(\int_0^1 x^{-x}\,dx\right)^{-1} \leqslant 1.$$

Therefore,

$$\int_0^1 x^x\,dx + \left(\int_0^1 x^{-x}\,dx\right)^{-1} \geqslant 2m + \frac{(1-m)^2}{2} = \frac{(1+m)^2}{2} \approx 1.4318 > \sqrt{2},$$

and

$$\sqrt{2} < \int_0^1 x^x\,dx + \left(\int_0^1 x^{-x}\,dx\right)^{-1} < 2.$$

Solution 6 by G. C. Greubel, Newport News, VA.

It is fairly easy to show that

$$I = \int_0^1 x^x\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}.$$

From the series form of the integral

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n} = 1 - \frac{1}{4} + \frac{1}{27} - \cdots < 1$$

which leads to

$$I + \frac{1}{I} < 1 + \frac{1}{1} = 2.$$

Since

$$I = \int_0^1 x^x\,dx \geqslant \int_0^1 \min(x^x)\,dx = \int_0^1 e^{-1/e}\,dx = e^{-1/e}$$

then

$$I + \frac{1}{I} \geqslant e^{1/e} + e^{-1/e} = 2\cos\left(\frac{1}{e}\right) > \sqrt{2}$$

and gives

$$\sqrt{2} < 2\cos\left(\frac{1}{e}\right) \leqslant \int_0^1 x^x\,dx + \left(\int_0^1 x^x\,dx\right)^{-1} < 2.$$

Also solved by Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Rome, Italy; and the problem problem proposer.

• **5720** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Let $(a_n)_{n\geqslant 1}$ be sequence of real numbers such that $\forall n \geqslant 1:\ a_n \leqslant n$ and

$$\forall n \geqslant 2:\ \sum_{k=1}^{n} \cos\left(\frac{\pi a_k}{n}\right) = 0.$$

Find

$$\Omega = \lim_{n\to\infty}\left(a_n \cdot \sum_{k=0}^{2n}(-1)^k \frac{\binom{4n}{2n}}{\binom{2n}{k}}\right).$$

Solution 1 by G. C. Greubel, Newport News, VA.

For the condition that a_k satisfy the series

$$\sum_{k=1}^{n} \cos\left(\frac{\pi\, a_k}{n}\right) = 0$$

it is readily found that $a_k \in \left\{\pm(2k-1), \pm\frac{2k-1}{2}\right\}$. Since $a_k \leqslant k$ then it is determined that $a_k = \pm\left(k - \frac{1}{2}\right)$.

Using the series

$$\sum_{k=0}^{2n}(-1)^k \binom{2n}{k}^{-1} = \frac{2n+1}{n+1}$$

then

$$\Omega = \lim_{n\to\infty}\left(a_n\,\frac{2n+1}{n+1}\binom{4n}{2n}\right).$$

Since

$$\binom{2n}{n} \approx \frac{2^{4n}}{\sqrt{2\pi\, n}}\left(1 + O\left(\frac{1}{n}\right)\right)$$

then

$$\Omega = \lim_{n\to\infty} \left(a_n \frac{2n+1}{n+1} \binom{4n}{2n} \right)$$
$$= \sqrt{\frac{2}{\pi}} \lim_{n\to\infty} \left(\frac{4^{2n}\, a_n}{\sqrt{n}} \frac{1+\frac{1}{2n}}{1+\frac{1}{n}} \left(1 + O\left(\frac{1}{n}\right)\right)\right)$$
$$= \pm \frac{1}{\sqrt{2\,\pi}} \lim_{n\to\infty} \left(4^{2n} \sqrt{n} \left(1 + O\left(\frac{1}{n}\right)\right)\right)$$

This gives $\Omega \to \infty$ as $n \to \infty$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

Put $a_1 = a$. We calculate the first few values of a_k and find $a_2 = 2 - a$, $a_3 = 2 + a$, $a_4 = 4 - a$, $a_5 = 4+a$, $a_6 = 6-a$, $a_7 = 6+a$. A pattern emerges and we guess that $a_{2k} = 2k-a$, $a_{2k+1} = 2k+a$. Indeed,

$$\sum_{k=1}^{2n} \cos\left(\frac{\pi a_k}{2n}\right) = \frac{1}{2}\sum_{k=0}^{n-1}\left(e^{\frac{\pi i}{2n}a_{2k+1}} + e^{-\frac{\pi i}{2n}a_{2k+1}}\right) + \frac{1}{2}\sum_{k=1}^{n}\left(e^{\frac{\pi i}{2n}a_{2k}} + e^{-\frac{\pi i}{2n}a_{2k}}\right) =$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}\left(e^{\frac{\pi i}{2n}(2k+a)} + e^{-\frac{\pi i}{2n}(2k+a)}\right) + \frac{1}{2}\sum_{k=1}^{n}\left(e^{\frac{\pi i}{2n}(2k-a)} + e^{-\frac{\pi i}{2n}(2k-a)}\right) =$$

$$= \frac{1}{2}e^{\frac{\pi i a}{2n}}\left(\sum_{k=0}^{n-1} e^{\frac{\pi i}{2n}(2k)} + \sum_{k=1}^{n} e^{-\frac{\pi i}{2n}(2k)}\right) + \frac{1}{2}e^{-\frac{\pi i a}{2n}}\left(\sum_{k=0}^{n-1} e^{-\frac{\pi i}{2n}(2k)} + \sum_{k=1}^{n} e^{\frac{\pi i}{2n}(2k)}\right) = 0$$

and

$$\sum_{k=1}^{2n+1} \cos\left(\frac{\pi a_k}{2n+1}\right) = \frac{1}{2}\sum_{k=0}^{n}\left(e^{\frac{\pi i}{2n+1}a_{2k+1}} + e^{-\frac{\pi i}{2n+1}a_{2k+1}}\right) + \frac{1}{2}\sum_{k=1}^{n}\left(e^{\frac{\pi i}{2n+1}a_{2k}} + e^{-\frac{\pi i}{2n+1}a_{2k}}\right) =$$

$$= \frac{1}{2}\sum_{k=0}^{n}\left(e^{\frac{\pi i}{2n+1}(2k+a)} + e^{-\frac{\pi i}{2n+1}(2k+a)}\right) + \frac{1}{2}\sum_{k=1}^{n}\left(e^{\frac{\pi i}{2n+1}(2k-a)} + e^{-\frac{\pi i}{2n+1}(2k-a)}\right) =$$

$$= \frac{1}{2}e^{\frac{\pi i a}{2n+1}}\left(\sum_{k=0}^{n} e^{\frac{\pi i}{2n+1}(2k)} + \sum_{k=1}^{n} e^{-\frac{\pi i}{2n+1}(2k)}\right) + \frac{1}{2}e^{-\frac{\pi i a}{2n+1}}\left(\sum_{k=0}^{n-1} e^{-\frac{\pi i}{2n+1}(2k)} + \sum_{k=1}^{n} e^{\frac{\pi i}{2n+1}(2k)}\right) = 0.$$

We claim that

$$\sum_{k=0}^{2n} (-1)^k \frac{\binom{4n}{2k}}{\binom{2n}{k}} = -\frac{1}{2n-1},$$

for

$$\sum_{k=0}^{2n}(-1)^k\frac{\begin{pmatrix}4n\\2k\end{pmatrix}}{\begin{pmatrix}2n\\k\end{pmatrix}}=(2n+1)\sum_{k=0}^{2n}(-1)^k\begin{pmatrix}4n\\2k\end{pmatrix}\frac{k!\,(2n-k)!}{(2n+1)!}=$$

$$=(2n+1)\sum_{k=0}^{2n}(-1)^k\begin{pmatrix}4n\\2k\end{pmatrix}\int_0^1 t^k(1-t)^{2n-k}dt$$

$$=(2n+1)\int_0^1(1-t)^{2n}\sum_{k=0}^{4n}\frac{i^k+i^{-k}}{2}\begin{pmatrix}4n\\k\end{pmatrix}\left(\frac{t}{1-t}\right)^{k/2}dt=$$

$$=\frac{1}{2}(2n+1)\int_0^1(1-t)^{2n}\left(\left(1+i\sqrt{\frac{t}{1-t}}\right)^{4n}+\left(1-i\sqrt{\frac{t}{1-t}}\right)^{4n}\right)dt=$$

$$=\frac{1}{2}(2n+1)\int_0^1\left(\left(\sqrt{1-t}+i\sqrt{t}\right)^{4n}+\left(\sqrt{1-t}-i\sqrt{t}\right)^{4n}\right)dt\overbrace{=}^{t=\sin^2x}$$

$$=(2n+1)\int_0^{\frac{\pi}{2}}\left(e^{4inx}+e^{-4inx}\right)\sin x\cos x\,dt=$$

$$=\frac{1}{4i}(2n+1)\int_0^{\frac{\pi}{2}}\left(e^{4inx}+e^{-4inx}\right)\left(e^{2ix}-e^{-2ix}\right)dt=$$

$$=\frac{1}{4i}(2n+1)\left(\frac{i}{2n+1}-\frac{i}{2n-1}-\frac{i}{2n-1}+\frac{i}{2n+1}\right)=-\frac{1}{2n-1}.$$

So

$$=\lim_{n\to\infty}\left(a_n\cdot\sum_{k=0}^{2n}(-1)^k\frac{\begin{pmatrix}4n\\2k\end{pmatrix}}{\begin{pmatrix}2n\\k\end{pmatrix}}\right)=\lim_{n\to\infty}n\left(-\frac{1}{2n-1}\right)=-\frac{1}{2}.$$

Solution 3 by Moti Levy, Rehovot, Israel.

<u>Remarks:</u>

1) I took the liberty to correct what seems to me to be a typo error in the problem statement and replaced $\binom{4n}{2n}$ by $\binom{4n}{2k}$.

2) To make the problem well-defined, I add the condition that the sequence $(a_n)_{n\geqslant1}$ is strictly increasing.

Let us define the sequence $a_k = \dfrac{2k-1}{2}$. Then for $n \geqslant 2$ we have

$$\begin{aligned}\sum_{k=1}^{n} \cos\left(\frac{2\pi a_k}{n}\right) &= \sum_{k=1}^{n} \cos\left(\frac{\pi(2k-1)}{n}\right) \\ &= \frac{1}{2}\sum_{k=1}^{n} e^{i\frac{\pi(2k-1)}{n}} + \frac{1}{2}\sum_{k=1}^{n} e^{-i\frac{\pi(2k-1)}{n}} \\ &= \frac{1}{2}\frac{e^{i\frac{\pi}{n}} - e^{i\frac{\pi}{n}}e^{2i\pi}}{1-e^{2i\frac{\pi}{n}}} + \frac{1}{2}\frac{e^{-i\frac{\pi}{n}} - e^{-i\frac{\pi}{n}}e^{-2i\pi}}{1-e^{-2i\frac{\pi}{n}}} = 0.\end{aligned}$$

Now we evaluate the combinatorial sum

$$S := \sum_{k=0}^{2n} (-1)^k \frac{\binom{4n}{2k}}{\binom{2n}{k}} = \sum_{k=0}^{2n} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}-k\right)}.$$

$$\begin{aligned}S &= \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}-k\right)} - \sum_{k=2n+1}^{\infty} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}-k\right)} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}-k\right)} + \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+2n+\frac{3}{2}\right)\Gamma\left(-\frac{1}{2}-k\right)}\end{aligned}$$

Let

$$S1 := \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}-k\right)}$$

and

$$S2 := \sum_{k=0}^{\infty} (-1)^k \frac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(-k-\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}+k\right)}$$

Then the required sum S is $S = S1 + S2$. To evaluate the combinatorial sums $S1$ and $S2$, the first step is to express the binomial series as a hypergeometric function.
The second step is application of some classical hypergeometric theorems and identities.

The following lemma explains how to express the binomial sum as a hypergeometric function:

Lemma : Let $(\alpha_k)_{k\geqslant 0}$ be a sequence which satisfies the following conditions:

$$\begin{aligned}\alpha_0 &= 1, \\ \frac{\alpha_{k+1}}{\alpha_k} &= \frac{1}{k+1}\frac{(k+a)(k+b)}{(k+c)}z.\end{aligned}$$

Then

$$\sum_{k=0}^{\infty} \alpha_k = {}_2F_1\left[\begin{array}{cc|c} a & b & \\ & c & z \end{array}\right], \tag{4}$$

where ${}_2F_1\left[\begin{array}{cc|c} a & b & \\ & c & z \end{array}\right]$ is a hypergeometric function. ■

Evaluation of $S1$:

Let $\beta_k := (-1)^k \dfrac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n+\frac{1}{2}-k\right)}, \beta_0 = 1$ then

$$\begin{aligned} \frac{\beta_{k+1}}{\beta_k} &= -\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2n-k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(2n-k-\frac{1}{2}\right)} = -\frac{2}{2k+1}\left(-k+2n-\frac{1}{2}\right) \\ &= \frac{1}{k+1}\frac{(k+1)\left(k-2n+\frac{1}{2}\right)}{k+\frac{1}{2}} \end{aligned}$$

By the lemma,

$$S1 = \sum_{k=0}^{\infty} \beta_k = {}_2F_1\left[\begin{array}{cc|c} 1 & \frac{1}{2}-2n & \\ & \frac{1}{2} & 1 \end{array}\right]$$

By Gauss hypergeometric theorem

$$S1 = {}_2F_1\left[\begin{array}{cc|c} 1 & \frac{1}{2}-2n & \\ & \frac{1}{2} & 1 \end{array}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-1+2n-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}+2n\right)} = -\frac{1}{2(2n-1)}. ■$$

Evaluation of $S2$:

Let $\gamma_k := -(4n+1)(-1)^k \dfrac{\sqrt{\pi}\Gamma\left(2n+\frac{1}{2}\right)}{\Gamma\left(-k-\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}+k\right)}, \gamma_0 = 1$ then

$$\begin{aligned} \frac{\gamma_{k+1}}{\gamma_k} &= -\frac{\Gamma\left(-k-\frac{1}{2}\right)\Gamma\left(2n+\frac{3}{2}+k\right)}{\Gamma\left(-k-\frac{3}{2}\right)\Gamma\left(2n+\frac{5}{2}+k\right)} = \frac{2k| + 3}{2k+4n+3} \\ &= \frac{1}{k+1}\frac{(k+1)\left(k+\frac{3}{2}\right)}{k+2n+\frac{3}{2}} \end{aligned}$$

By the lemma,

$$S2 = -\frac{1}{4n+1}\sum_{k=0}^{\infty} \gamma_k = -\frac{1}{4n+1} {}_2F_1\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \\ & 2n+\frac{3}{2} & 1 \end{array}\right]$$

By Gauss hypergeometric theorem

$${}_2F_1\left[\begin{array}{cc|c} 1 & \frac{3}{2} & \\ & 2n+\frac{3}{2} & 1 \end{array}\right] = \frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma(2n-1)}{\Gamma\left(2n+\frac{1}{2}\right)\Gamma(2n)} = \frac{2n+\frac{1}{2}}{2n-1}$$

$$S2 = -\frac{1}{4n+1}\frac{2n+\frac{1}{2}}{2n-1} = -\frac{1}{2(2n-1)}.\blacksquare$$

We conclude that

$$S := \sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}} = -\frac{1}{2n-1}.$$

Finally,

$$a_n\sum_{k=0}^{2n}(-1)^k\frac{\binom{4n}{2k}}{\binom{2n}{k}} = \left(\frac{2n-1}{2}\right)\left(-\frac{1}{2n-1}\right) = -\frac{1}{2},$$

and trivially

$$\Omega = -\frac{1}{2}.$$

Also solved by the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

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"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣