Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please make sure every proposed problem or propsed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

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Solutions to the problems published in this issue should be submitted *before* February 1, 2024.

• **5745** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

For real x, solve the equation

$$2^{(2^x-1)^2} + 4^x = \sqrt{x} + 2^{x+1} + \log_2\left(1 + \sqrt{x}\right).$$

• **5746** *Proposed by Problem proposed by Albert Stadler, Herrliberg, Switzerland.*

Let $m, n \geqslant 1$ and let $p_{m,n}(x)$ be the polynomial defined by

$$p_{m,n}(x) = e^{-x^m}\frac{d^n}{dx^n}e^{x^m}.$$

Prove that for all $n \geqslant 0$:

$$\left((x^m + y^m)^{n/m}\right) p_{m,n}\left((x^m + y^m)^{1/m}\right) = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} x^k y^{n-k} p_{m,k}(x)\, p_{m,n-k}(y).$$

• **5747** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose $f : (2,3) \to (0,\infty)$ is a function with $f'(x) < 0$ and $f''(x) < 0$ for all x in $(2,3)$. Show that for a, b, c in $(1,2)$:

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant 2\cdot\sqrt[4]{\prod_{cyc} f(a+1)\cdot\sum_{cyc} f(a+1)}.$$

• **5748** *Proposed by Narendra Bhandari, Bajura, Nepal.*

Let B_k denote the kth Bernoulli number. For positive integers m and n prove that

$$\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n} k^m \sin(kx)\right]^2 dx = \frac{\pi}{2m+1}\sum_{k=0}^{2m}\binom{2m+1}{k}B_k n^{2m-n+1}.$$

• **5749** *Proposed by Prakash Pant, Mathematics Initiatives in Nepal(MIN), Bardiya, Nepal.*

Let x, y, z be positive real numbers with $x + y + z = 3$. Prove that:

$$\prod_{x,y,z}(x^{1/x}e^e) \leqslant e^{\sum_{x,y,z} e^x}$$

For what values of x, y and z does equality hold?

• **5750** *Proposed by Albert Natian, Problem Section Editor.*

Assuming all the radicands are non-negative, solve the system of equations for real x and y:

$$\left\{\begin{array}{c} \sqrt[3]{xy+3x-y+3} + \sqrt[|x|]{5xy-x+y+4} = \sqrt[|x|]{3xy-7x+3y-2} \\ \sqrt{3x+9y+15} + \sqrt[|yx|]{5y-2x+34} = \sqrt[|yx|]{2y-3x+29} \end{array}\right\}.$$

Solutions

To Formerly Published Problems

• **5721** *Proposed by Albert Stadler, Herrliberg, Switzerland.*

Triangle $\triangle$ABC has angles α, β, γ (expressed all in radians), inradius r and circumradius R. Prove that

$$\left(\frac{\sin\alpha}{\alpha}\right)^{\frac{3}{2}} + \left(\frac{\sin\beta}{\beta}\right)^{\frac{3}{2}} + \left(\frac{\sin\gamma}{\gamma}\right)^{\frac{3}{2}} \geqslant 2 + \frac{r}{2R}.$$

Solution 1 by Michel Bataille, Rouen, France.

First, we prove two auxiliary results:

$$2 + \frac{r}{2R} = \cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} \qquad (1)$$

$$\text{for } x \in (0, \frac{\pi}{2}), \quad \left(\frac{\sin x}{x}\right)^3 \geqslant \cos x. \qquad (2)$$

Proof of (1). From the well-known formula $\cos\alpha + \cos\beta + \cos\gamma = 1 + \dfrac{r}{R}$, we deduce

$$2 + \frac{r}{2R} = \frac{3 + \cos\alpha + \cos\beta + \cos\gamma}{2} = \frac{3 + 2\cos^2\frac{\alpha}{2} - 1 + 2\cos^2\frac{\beta}{2} - 1 + 2\cos^2\frac{\gamma}{2} - 1}{2}$$

and (1) follows.

Proof of (2). We have to prove that $f(x) \geqslant 0$ for $0 < x < \dfrac{\pi}{2}$ where $f(x) = \sin x(\cos x)^{-1/3} - x$. To this aim, we calculate $f'(x) = \dfrac{2}{3}(\cos x)^{2/3} + \dfrac{1}{3}(\cos x)^{-4/3} - 1$ and observe that from the weighted AM-GM, $f'(x) \geqslant (\cos x)^{4/9}(\cos x)^{-4/9} - 1 = 0$. It follows that f is nondecreasing on $[0, \dfrac{\pi}{2})$ and therefore $f(x) \geqslant f(0) = 0$.

Now, using (2), we obtain

$$\left(\frac{\sin\alpha}{\alpha}\right)^{\frac{3}{2}} = \left(\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\alpha}\right)^{\frac{3}{2}} = \left(\cos\frac{\alpha}{2}\right)^{3/2}\left(\frac{\sin\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)^{3/2} \geqslant \left(\cos\frac{\alpha}{2}\right)^{3/2}\left(\cos\frac{\alpha}{2}\right)^{1/2} = \cos^2\frac{\alpha}{2}.$$

Similar results hold for $\left(\dfrac{\sin\beta}{\beta}\right)^{\frac{3}{2}}$ and $\left(\dfrac{\sin\gamma}{\gamma}\right)^{\frac{3}{2}}$ and the desired inequality then directly follows from (1).

Solution 2 by Moti Levy, Rehovot, Israel.

The ratio of inradius to circumradius is

$$\frac{r}{R} = 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right). \tag{1}$$

We reformulate by plugging (1) into the original inequality

$$\left(\frac{\sin(\alpha)}{\alpha}\right)^{\frac{3}{2}} + \left(\frac{\sin(\beta)}{\beta}\right)^{\frac{3}{2}} + \left(\frac{\sin(\gamma)}{\gamma}\right)^{\frac{3}{2}} - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) - 2 \geqslant 0. \tag{2}$$

It is known that (see the reference)

$$\left(\frac{\sin(x)}{x}\right)^{\frac{3}{2}} \geqslant \cos^2\left(\frac{x}{2}\right), \qquad x \in (0, \pi). \tag{3}$$

Hence by (3) it suffices to prove that

$$\cos^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\beta}{2}\right) + \cos^2\left(\frac{\gamma}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) - 2 \geqslant 0. \tag{4}$$

Setting $\gamma = \pi - (\alpha + \beta)$ in (4) we get

$$\cos^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\beta}{2}\right) + \cos^2\left(\frac{\gamma}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) - 2$$
$$= \cos^2\left(\frac{\alpha}{2}\right) + \cos^2\left(\frac{\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right) - 2\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) - 2 = 0,$$

and this completes the proof. ■

We give here the proof of statement (3), after the referred article:

The following power series expansions are known:

$$\ln\left(\frac{\sin(x)}{x}\right) = -\sum_{k=1}^{\infty} \frac{2^{2k-1}\,|B_{2k}|}{k\,(2k)!} x^{2k}, \quad \text{for } 0 < x < \pi, \tag{5}$$

$$\ln\left(\cos(x)\right) = -\sum_{k=1}^{\infty} \frac{2^{2k-1}\left(2^{2k}-1\right)|B_{2k}|}{k\,(2k)!} x^{2k}, \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}, \tag{6}$$

where $B_i \ \ (i \in \mathbb{N})$ are the Bernoulli's numbers.

By (5) and (6) we have

$$f(x) := \frac{3}{2}\ln\left(\frac{\sin(x)}{x}\right) - 2\ln\left(\cos\left(\frac{x}{2}\right)\right) = \sum_{k=1}^{\infty} E_k x^{2k}, \quad \alpha > 1 \quad \text{and} \quad x \in (0,\pi), \tag{7}$$

where

$$E_k := \frac{\left(2^{2k-1}-2\right)|B_{2k}|}{2k\,(2k)!}.$$

Now, we have $E_1 = 0$ and $E_k > 0$ for $k > 2$. Thus from (7) we have $f(x) > 0$ for $x \in (0,\pi)$. ■

Reference:

Rašajski, M., Lutovac, T. & Malešević, B. "*About some exponential inequalities related to the sinc function*". J Inequal Appl 2018, 150 (2018). https://doi.org/10.1186/s13660-018-1740-9

Solution 3 by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

We know that $1 + \frac{r}{R} = \cos\alpha + \cos\beta + \cos\gamma$ (D.S.Mitrinovicć, J.E.Pečarić, V.Volenec, "Recent Advances in Geometric Inequalities", Kluwer Academic Publisher (1989) p.55). The inequality is

$$\left(\frac{\sin\alpha}{\alpha}\right)^{\frac{3}{2}} + \left(\frac{\sin\beta}{\beta}\right)^{\frac{3}{2}} + \left(\frac{\sin\gamma}{\gamma}\right)^{\frac{3}{2}} \geqslant \frac{3}{2} + \frac{1}{2}(\cos\alpha + \cos\beta + \cos\gamma) \tag{1}$$

$\left(\dfrac{\sin x}{x}\right)^{\frac{3}{2}}$ changes concavity once in $0 \leqslant x \leqslant \pi$ as well as $\cos x$. Of course we understand that $(\sin x)/x = 1$ for $x = 0$. It follows that the minimum of (1) for $\alpha+\beta+\gamma=\pi$ occurs when at least two of them are equal and the same happens to the maximum of $\cos\alpha + \cos\beta + \cos\gamma$ hence we must show that for any $0 \leqslant x \leqslant \pi/2$

$$2\left(\frac{\sin x}{x}\right)^{\frac{3}{2}} + \left(\frac{\sin(\pi-2x)}{\pi-2x}\right)^{\frac{3}{2}} - \cos x - \frac{1}{2}\cos(\pi-2x) \geqslant \frac{3}{2}$$

We show that

$$\text{(I) } 2\left(\frac{\sin x}{x}\right)^{\frac{3}{2}} - \cos x \geqslant 1, \qquad \text{(II) } \left(\frac{\sin(\pi-2x)}{\pi-2x}\right)^{\frac{3}{2}} - \frac{1}{2}\cos(\pi-2x) \geqslant \frac{1}{2}$$

(I). Upon squaring we get

$$4\frac{\sin^3 x}{x^3} \geqslant 4\cos^4\frac{x}{2} \iff \frac{\sin^3 y}{y^3} \geqslant \cos y, \quad 0 \leqslant y \leqslant \pi/4$$

$$\left(\sin^3 y - y^3\cos y\right)' \geqslant 0 \iff 3\sin^2 y\cos x + y^3\sin y \geqslant 3y^2\cos y$$

$$\frac{\sin^2 y}{y^2} + \frac{y\tan y}{3} \geqslant \frac{\left(y - \frac{y^3}{6}\right)^2}{y^2} + \frac{y^2}{3} \geqslant 1 \iff \frac{y^4}{36} \geqslant 0$$

because $\sin x \geqslant x - x^3/6$ and $\tan x \geqslant x$. It follows that $\sin^3 y - y^3\cos y$ increases starting by the value zero thus is nonnegative.

(II). Upon setting $\pi - 2x = y$, we obtain the same inequality as in (I) hence completing the proof.

Also solved by the problem proposer.

• **5722** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Let p_n be the n-th prime number. Prove the following inequality

$$p_{n+1} < 3p_{\lfloor \frac{n}{2} \rfloor + 1} \text{ for } n \geqslant 1$$

where $\lfloor . \rfloor$ denotes the integer part function.

Hint: Use the Rosser-Schoenfeld inequalities $p_n < n\log n + n\log\log n - \frac{1}{2}n$ for $n \geqslant 20$ and $p_n > n\log n$ for $n \geqslant 1$ along with a small table of primes.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

This is a refined version of Bertrand's postulate which states that $p_{n+1} < 2p_n$. The Mathematica command Table[Prime[n + 1] < 3 Prime[Floor[n/2] + 1], {n, 1, 100}] confirms the truth of the claim for all n with $1 \leqslant n \leqslant 100$. It remains to prove that

$$(n+1)\log(n+1) + (n+1)\log\log(n+1) - \frac{n+1}{2} <$$

$$< 3\left(\frac{n+1}{2}\right)\log\left(\frac{n+1}{2}\right) \leqslant 3\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)\log\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right).$$

The second inequality holds true, since $\frac{n+1}{2} \leqslant \left\lfloor\frac{n}{2}\right\rfloor + 1$ and x→x log x is monotonically increasing for x⩾1. The first inequality is equivalent to

$$3\log 2 - 1 < \log(n+1) - 2\log\log(n+1).$$

This inequality holds true, since $x \to \log(x) - 2\log\log(x)$ is monotonically increasing for $x \geqslant e^2$ and $3\log 2 - 1 < \log 64 - 2\log\log 64 = 6\log 2 - 2\log(6\log 2)$.

Also solved by the problem proposer.

• **5723** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.*

For real x, solve the equation $\left(\sqrt[5]{x+2} - \sqrt[5]{2x+1} - \sqrt[5]{4x+7}\right)^5 = 3x+8$.

Editor's comments: Unfortunately the proposer of this problem has made an error. The solution set foe the equation provided by the porposer is $\{1, 3, -9/5\}$. It's immediate to see that 1 is not a solution since for $x = 1$ the left-hand side of the equation becomes -11, whereas the right-hand side becomes $+11$. It's also immediate to see that 3 cannot be a solution to the equation either since for $x = 3$, the left-hand side of the equation is negative, whereas the right-hand side is positive. We all make mistakes. Some mistakes are fated to occur, it seems, despite our best efforts to forestall them. Other than the fated mistakes, we can take proactive steps to minimize the frequency of many other — especially mundane — mistakes. Resovle to catch your *misteaks*! Please make it an unviolable rule to check all your solutions BEFORE you submit your problems for publication in this journal's Problem Section. You can also ask a friend to go over your solution(s) and to check your answers for accuracy before you email your problems to me. *Thanks a million!*

Solution 1 by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

Instead of solving the given equation, let's solve a slightly different one:

$$\left(\sqrt[5]{x+2} - \sqrt[5]{2x+1} + \sqrt[5]{4x+7}\right)^5 = 3x+8.$$

The equation can be rewritten as

$$\sqrt[5]{2x+1}+\sqrt[5]{3x+8}-\sqrt[5]{x+2}=\sqrt[5]{4x+7}=\sqrt[5]{((2x+1)+(3x+8)-(x+2))}$$

which we rewrite as $A=2x+1$, $B=3x+8$, $C=x+2$

$$A^{1/5}+B^{1/5}-C^{1/5}=(A+B-C)^{1/5} \qquad (A^{1/5}+B^{1/5}-C^{1/5})^5=A+B-C$$

$$(a+b-c)^5-(a^5+b^5-c^5)=5(b-c)(a-c)(a+b)(a^2+ab-ac+c^2-bc+b^2)$$

$$b=c \iff B^{1/5}=C^{1/5} \iff 3x+8=x+2 \iff x=-3$$

$$a=c \iff A^{1/5}=C^{1/5} \iff 2x+1=x+2 \iff x=1$$

$$b=-a \iff 3x+8=-2x-1 \iff x=-9/5$$

$$a^2+ab-ac+c^2-bc+b^2=((a+b)^2+(b-c)^2+(a-c)^2)/2=0 \iff a=b=c$$

but the system $\{2x+1=3x+8, 3x+8=x+2\}$ has no solutions.

Solution 2 by Michel Bataille, Rouen, France.

Instead of solving the given equation, let's solve a slightly different one:

$$\left(\sqrt[5]{x+2}-\sqrt[5]{2x+1}+\sqrt[5]{4x+7}\right)^5=3x+8,$$

which we rewrite as

$$\sqrt[5]{4x+7}-\sqrt[5]{3x+8}=\sqrt[5]{2x+1}-\sqrt[5]{x+2}.$$

Let x be a solution and let $a=\sqrt[5]{4x+7}, b=\sqrt[5]{3x+8}, c=\sqrt[5]{2x+1}, d=\sqrt[5]{x+2}$. Using the easily checked identity $(u-v)^5=u^5-v^5-5uv(u-v)(u^2-uv+v^2)$ with $u=a, v=b$ and with $u=c, v=d$, we obtain

$$ab(a-b)(a^2-ab+b^2)=cd(c-d)(c^2-cd+d^2). \tag{1}$$

Since $a-b=c-d$ and $a^2-ab+b^2=(a-b)^2+ab, c^2-cd+d^2=(c-d)^2+cd=(a-b)^2+cd$, (1) provides

$$(a-b)(ab-cd)((a-b)^2+ab+cd)=0,$$

that is,

$$(a-b)(ab-cd)(a^2+b^2+c^2+d^2)=0.$$

(note that $2[(a-b)^2+ab+cd]=2a^2+2b^2+2cd-2ab=2a^2+2b^2+c^2+d^2-(c-d)^2+(a-b)^2-a^2-b^2=2a^2+2b^2+c^2+d^2-a^2-b^2$.)

Since there is no x such that $a=b=c=d=0$, we see that $a=b$ or $ab=cd$, that is, $4x+7=3x+8$ or $(4x+7)(3x+8)=(2x+1)(x+2)$. Thus, $x\in\{1,-3,-\frac{9}{5}\}$.

Conversely, it is readily checked that the numbers $1,-3,-\frac{9}{5}$ are indeed solutions.

We conclude that the solutions are these three numbers.

Solution 3 by Brian D. Beasley, Presbyterian College, Clinton, SC.

We first solve the equation as given. We then change one sign in the equation and find the solutions of the new version.

I. The given equation is equivalent to

$$\sqrt[5]{x+2} - \sqrt[5]{2x+1} - \sqrt[5]{4x+7} = \sqrt[5]{3x+8}.$$

For each real number x, let $f(x) = \sqrt[5]{x+2} - \sqrt[5]{2x+1} - \sqrt[5]{4x+7} - \sqrt[5]{3x+8}$. For $x \geqslant 1$, we have

$$0 < x+2 \leqslant \min\{2x+1, 4x+7, 3x+8\},$$

so $f(x) < 0$. For $x \leqslant -3$, we have

$$0 < -x-2 \leqslant \min\{-2x-1, -4x-7, -3x-8\},$$

so $f(x) > 0$. Hence any real zero of $f(x)$ must lie in the interval $(-3, 1)$. A graph of $f(x)$ shows that there is one real zero, $x \approx -1.69437$.

II. Next, for real x, we seek to solve instead the equation

$$\left(\sqrt[5]{x+2} - \sqrt[5]{2x+1} + \sqrt[5]{4x+7}\right)^5 = 3x+8.$$

This version is equivalent to

$$\sqrt[5]{x+2} + \sqrt[5]{4x+7} = \sqrt[5]{2x+1} + \sqrt[5]{3x+8}.$$

Let $a = \sqrt[5]{x+2}$, $b = \sqrt[5]{4x+7}$, $c = \sqrt[5]{2x+1}$, and $d = \sqrt[5]{3x+8}$. Then we must solve $a+b = c+d$, so we apply the fact that $a^5 + b^5 = c^5 + d^5$ to the equation $(a+b)^5 = (c+d)^5$ to obtain

$$a^4b + 2a^3b^2 + 2a^2b^3 + ab^4 = c^4d + 2c^3d^2 + 2c^2d^3 + cd^4.$$

This in turn yields

$$ab(a+b)(a^2+ab+b^2) = cd(c+d)(c^2+cd+d^2).$$

(i) If $a+b = c+d = 0$, then $b^5 = -a^5$ and $d^5 = -c^5$, so $x = -\dfrac{9}{5}$.

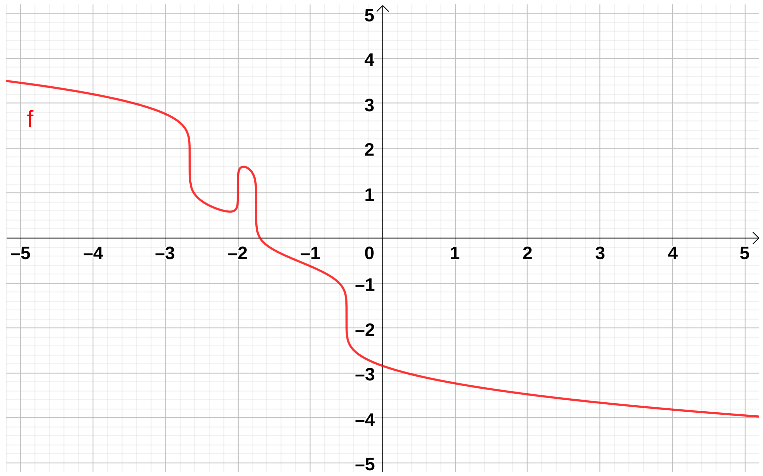
(ii) If $a+b = c+d \neq 0$, then $ab(a^2+ab+b^2) = cd(c^2+cd+d^2)$, so

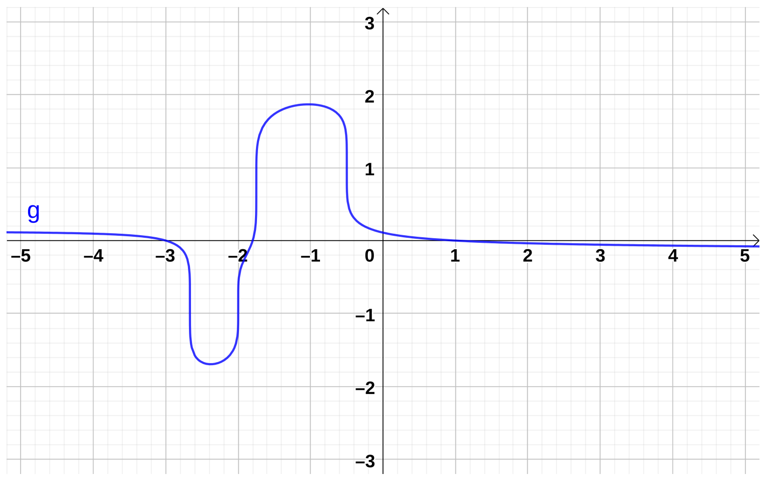
$$ab[(a+b)^2 - ab] = cd[(c+d)^2 - cd] = cd[(a+b)^2 - cd].$$

Thus we have

$$(ab-cd)[(a+b)^2 - (ab+cd)] = 0,$$

so $cd = ab$ or $cd = (a+b)^2 - ab$. If $cd = ab$, then $x = -3$ or $x = 1$. If $cd = (a+b)^2 - ab$, then substituting $d = a+b-c$ produces $a^2 + b^2 + c^2 + ab - bc - ca = 0$. Thus





$$c = \frac{a + b \pm \sqrt{-3a^2 - 2ab - 3b^2}}{2}.$$

But no two of a, b, and c can equal zero, so we use the facts that $a^2 + ab + b^2 > 0$ and $a^2 + ab + b^2 > ab$ to conclude

$$-3a^2 - 2ab - 3b^2 = (-3)(a^2 + ab + b^2) + ab < (-2)(a^2 + ab + b^2) < 0.$$

Hence the three real solutions of this equation are $x = -3$, $x = -\dfrac{9}{5}$, and $x = 1$.

Addendum. The first graph shows the one real zero of $f(x)$, corresponding to the solution of the first equation. The second graph shows that the function

$$g(x) = \sqrt[5]{x+2} + \sqrt[5]{4x+7} - \sqrt[5]{2x+1} - \sqrt[5]{3x+8}$$

has the three real zeros $x = -3$, $x = -\dfrac{9}{5}$, and $x = 1$, corresponding to the solutions of the second equation. (Both graphs were generated using GeoGebra.)

Proposed solutions were also provided by Albert Stadler, Herrliberg, Switzerland; and the problem proposer.

• **5724** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the integral:

$$\int_0^\infty \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}dx.$$

Solution 1 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\begin{aligned}
I &= \int_0^1 \frac{\sqrt{x}\ln^2 x}{1+x^2+x^4}\mathrm{d}x + \int_1^\infty \frac{\sqrt{x}\ln^2 x}{1+x^2+x^4}\mathrm{d}x \\
&= \int_0^1 \frac{\sqrt{x}\ln^2 x}{1+x^2+x^4}\mathrm{d}x + \int_1^0 \frac{\sqrt{\frac{1}{x}}\ln^2 x}{1+\frac{1}{x^2}+\frac{1}{x^4}}\mathrm{d}x\left(-\frac{1}{x^2}\right) \\
&= 2\int_0^1 \frac{\sqrt{x}\ln^2 x}{1+x^2+x^4}\mathrm{d}x \\
&= 2\sum_{n=0}^\infty \int_0^1 \left(1-x^2\right)x^{6n}\sqrt{x}\ln^2 x\mathrm{d}x \\
\because &\quad \int_0^1 x^{6n+\frac{1}{2}}\ln^2 x\mathrm{d}x = \frac{16}{27(4n+1)^3}, \int_0^1 x^{6n+\frac{5}{2}}\ln^2 x\mathrm{d}x = \frac{16}{(12n+7)^3} \\
\therefore &\quad I = 2\sum_{n=0}^\infty \left[\frac{16}{27(4n+1)^3} - \frac{16}{(12n+7)^3}\right] \\
\because &\quad \sum_{n=0}^\infty \frac{1}{(4n+1)^3} = \frac{1}{64}\left[28\xi(3)+\pi^3\right], \quad \sum_{n=6}^\infty \frac{1}{(12n+7)^3} = \frac{\psi^{(2)}\left(\frac{7}{12}\right)}{3456} \\
\therefore &\quad I = \frac{1}{54}\left[28\xi(3)+\pi^3\right] - \frac{1}{108}\psi^{(2)}\left(\frac{7}{12}\right) \\
&= \frac{1}{108}\left[56\xi(3)+2\pi^3-\psi^{(2)}\left(\frac{7}{12}\right)\right]
\end{aligned}$$

Solution 2 by Péter Fülöp, Gyömrő, Hungary.

We split the interval of the integral into two parts:

$$I = \int_0^1 \frac{\sqrt{x}\ln^2(x)}{x^4+x^2+1}dx + \int_1^\infty \frac{\sqrt{x}\ln^2(x)}{x^4+x^2+1}dx$$

After performing the $x = \dfrac{1}{t}$ substitution in the second integral we get:

$$I = \int_0^1 \frac{\sqrt{x}\ln^2(x)}{x^4 + x^2 + 1}dx + \int_0^1 \frac{t\sqrt{t}\ln^2(t)}{t^4 + t^2 + 1}dt$$

We can combine them:

$$I = \int_0^1 \frac{(x^{\frac{1}{2}} + x^{\frac{3}{2}})\ln^2(x)}{x^4 + x^2 + 1}dx$$

Introducing x^a(where $a = \frac{1}{2}$) instead of $\sqrt{x}$ and realizing that $\frac{d^2(x^a)}{da^2} = x^a \ln^2(x)$ we get:

$$I = \frac{d^2}{da^2}\int_0^1 \frac{x^a + x^{a+1}}{x^4 + x^2 + 1}dx$$

Using the fact that $x^4 + x^2 + 1 = \frac{x^6 - 1}{x^2 - 1}$ and performing the $u = x^6$ substitution:

$$I = \frac{1}{6}\frac{d^2}{da^2}\int_0^1 \frac{-u^{\frac{a-2}{6}} - u^{\frac{a-3}{6}} + u^{\frac{a-4}{6}} + u^{\frac{a-5}{6}}}{1 - u}du$$

The integral can be expressed as the function of β function:

$$I = \frac{1}{6}\frac{d^2}{da^2}\Big[-\beta(\frac{a+4}{6}, 0) - \beta(\frac{a+3}{6}, 0) + \beta(\frac{a+2}{6}, 0) + +\beta(\frac{a+1}{6}, 0)\Big]$$

Known that the incomplete beta function $\beta(z; a, b)$ can be written as the following sum: $\sum_{k=0}^{\infty} \frac{(1-b)_k}{k!(k+a)} z^{a+k}$, where $(x)_k$ is a Pochhammer symbol and $z = 1$ in our case.

$$I = \frac{1}{6}\frac{d^2}{da^2}\Big[-\sum_{k=0}^{\infty}\frac{1}{k + \frac{a+4}{6}} - \sum_{k=0}^{\infty}\frac{1}{k + \frac{a+3}{6}} + \sum_{k=0}^{\infty}\frac{1}{k + \frac{a+2}{6}} + \sum_{k=0}^{\infty}\frac{1}{k + \frac{a+1}{6}}\Big]$$

After performing the derivations we get:

$$I = \frac{1}{108}\Big[-\sum_{k=0}^{\infty}\frac{1}{(k + \frac{a+4}{6})^3} - \sum_{k=0}^{\infty}\frac{1}{(k + \frac{a+3}{6})^3} + \sum_{k=0}^{\infty}\frac{1}{(k + \frac{a+2}{6})^3} + \sum_{k=0}^{\infty}\frac{1}{(k + \frac{a+1}{6})^3}\Big]$$

The integral can be expressed as the function of *polygamma* function:

$$\psi^{(m)}(z) = (-1)^m m! \sum_{k=0}^{\infty}\frac{1}{(k + z)^{m+1}}.$$

In our case $m = 2$.

We express I, which is a function of a, as follows:

$$I(a) = \frac{1}{216}\Big[\psi^{(2)}(\frac{a+4}{6}) + \psi^{(2)}(\frac{a+3}{6}) - \psi^{(2)}(\frac{a+2}{6}) - \psi^{(2)}(\frac{a+1}{6})\Big]$$

We get the value of the original integral at $a = \frac{1}{2}$:

$$I(a = \frac{1}{2}) = \frac{1}{216}\Big[\psi^{(2)}(\frac{9}{12}) + \psi^{(2)}(\frac{7}{12}) - \psi^{(2)}(\frac{5}{12}) - \psi^{(2)}(\frac{3}{12})\Big]$$

We can realize that $\frac{9}{12} = 1 - \frac{3}{12}$ and $\frac{7}{12} = 1 - \frac{5}{12}$ so:

$$I(\frac{1}{2}) = \frac{1}{216}\Big[\psi^{(2)}(1-\frac{3}{12}) - \psi^{(2)}(\frac{3}{12}) + \psi^{(2)}(1-\frac{5}{12}) - \psi^{(2)}(\frac{5}{12})\Big]$$

Applying the reflection relation of the m-th order polygamma function $(-1)^m\psi^{(m)}(1-z) - \psi^{(m)}(z) = \pi\frac{d^m(\cot(\pi z))}{dz^m}$. In case of $m = 2$ we get:

$$\psi^{(2)}(1-z) - \psi^{(2)}(z) = 2\pi^2\cot(\pi z)\csc^2(\pi z)$$

At $z = \frac{3}{12}$ and $z = \frac{5}{12}$ we have the result:

$$I = \frac{4\pi^3}{216} + \frac{2\pi^3}{216}\frac{1}{(1+\frac{\sqrt{3}}{2})^2}$$

$$I = \frac{\pi^3}{54}(15 - 8\sqrt{3}) \approx 0,656440327.....$$

Solution 3 by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

We first address the integrals

$$I_1 = \int_0^\infty \frac{\sqrt{x}}{x^4+x^2+1}dx, \qquad I_2 = \int_0^\infty \frac{\sqrt{x}\ln x}{x^4+x^2+1}dx$$

Case I_1 : Residue's theory applied to the contour integral over the path made by the four curves: Let's pass to the complex function $f_1(z) = \sqrt{z}/(z^4+z^2+1)$

$$\gamma_1(t) = t + i\varepsilon,\ 0 \leqslant t \leqslant R, \qquad \gamma_2(t) = \sqrt{R^2+\varepsilon^2}e^{it},\ \varphi_0 \leqslant t \leqslant 2\pi - \varphi_0,\ \varphi_0 = \arctan(\varepsilon/R)$$

$$\gamma_3(t) = (-t - i\varepsilon)e^{2i\pi},\ -R \leqslant t \leqslant 0, \qquad \gamma_4(t) = \varepsilon e^{-it},\ -3\pi/2 \leqslant t \leqslant -\pi/2$$

It is easy to show that

$$\lim_{R\to\infty}\int_{\gamma_2} f_1(z)dz = 0, \qquad \lim_{\varepsilon\to 0}\int_{\gamma_4} f_1(z)dz = 0$$

$$\int_{\gamma_1} f_1(z)dz = \int_0^R \frac{\sqrt{t}\,dt}{t^4+t^2+1}$$

$$\int_{\gamma_1} f_3(z)dz = \int_{-R}^0 \frac{\sqrt{-te^{2i\pi}}\,(-dt)e^{2i\pi}}{t^4+t^2+1} \underbrace{=}_{t=-\tau} -\int_0^\infty \frac{e^{i\pi}\sqrt{\tau}\,d\tau}{t^4+t^2+1} = \int_0^R \frac{\sqrt{\tau}\,d\tau}{\tau^4+\tau^2+1}$$

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\gamma_1} f_1(z)dz + \int_{\gamma_3} f_1(z)dz = 2\int_0^\infty \frac{\sqrt{x}\,dx}{x^4+x^2+1} = 2\pi i\sum_{k=0}^{3}\text{Res}\frac{\sqrt{z_k}}{z_k^4+z_k^2+1}$$

The poles of $f_1(z)$ correspond to the zeroes of z^4+z^2+1 and are of first order

$$z_0 = e^{i\pi/3},\ z_1 = e^{i4\pi/3},\ z_2 = e^{i2\pi/3},\ z_3 = e^{i5\pi/3}$$

$$\begin{aligned}
2I_1 &= 2\pi i\sum_{k=0}^{3}\frac{\sqrt{z_k}}{4z_k^3+2z_k} = \\
&= 2\pi i\left[\frac{\frac{\sqrt{3}}{2}+\frac{i}{2}}{-4+2(\frac{1}{2}+\frac{i\sqrt{3}}{2})} + \frac{\frac{-1}{2}+\frac{i\sqrt{3}}{2}}{4+2(\frac{-1}{2}+\frac{-i\sqrt{3}}{2})} + \frac{\frac{1}{2}+\frac{i\sqrt{3}}{2}}{4+2(\frac{-1}{2}+\frac{i\sqrt{3}}{2})} + \frac{-\sqrt{3}2+\frac{i}{2}}{-4+2(\frac{1}{2}+\frac{-i\sqrt{3}}{2})}\right] = \\
&= \frac{2\pi i}{2\sqrt{3}}\left[\frac{i+\sqrt{3}}{i-\sqrt{3}} + \frac{-1+i\sqrt{3}}{\sqrt{3}-i} + \frac{1+i\sqrt{3}}{i+\sqrt{3}} + \frac{-i+\sqrt{3}}{i+\sqrt{3}}\right] = \\
&= \frac{2\pi i}{2\sqrt{3}}\left[\frac{-\sqrt{3}-1+i(-1+\sqrt{3})}{\sqrt{3}-i} + \frac{\sqrt{3}+1+i(\sqrt{3}-1)}{\sqrt{3}+i}\right] = \\
&= \frac{2\pi i}{2\sqrt{3}}2i\text{Im}\left[\frac{\sqrt{3}+1+i(\sqrt{3}-1)}{\sqrt{3}+i}\right] = \pi - \frac{\pi}{\sqrt{3}}
\end{aligned}$$

whence $I_1 = \frac{\pi}{2} - \frac{\pi\sqrt{3}}{6}$

Case I_2 : Let $f_2(z) = (\sqrt{z}\text{Ln}(z))/(z^4+z^2+1)$. The contour integral over the same path as above yields

$$\begin{aligned}
&\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\gamma_1} f_1(z)dz + \int_{\gamma_3} f_1(z)dz = \int_0^\infty \frac{\sqrt{x}\ln x dx}{x^4+x^2+1} + \int_0^\infty \frac{\sqrt{x}(\ln x + 2i\pi)dx}{x^4+x^2+1} = \\
&= 2\pi i\sum_{k=0}^{3}\text{Res}\frac{\sqrt{z_k}\text{Ln}(z_k)}{z_k^4+z_k^2+1} = \\
&= \frac{2\pi^2 i\cdot i}{6\sqrt{3}}\left[\frac{i+\sqrt{3}}{i-\sqrt{3}} + 4\frac{-1+i\sqrt{3}}{\sqrt{3}-i} + 2\frac{1+i\sqrt{3}}{i+\sqrt{3}} + 5\frac{-i+\sqrt{3}}{i+\sqrt{3}}\right] = \\
&= \frac{\pi^2\sqrt{3}}{9}(3i(\sqrt{3}-1) - 2 + \sqrt{3})
\end{aligned}$$

Then

$$2I_2 = -2\pi i\left(\frac{\pi}{2} - \frac{\pi\sqrt{3}}{6}\right) + \frac{\pi^2\sqrt{3}}{9}(3i(\sqrt{3}-1) - 2 + \sqrt{3}) \implies I_2 = \frac{\pi^2}{6} - \frac{\pi^2\sqrt{3}}{9}$$

Finally we can evaluate the integral in the statement (which we call I.) The contour integral over the same path yields

$$I + \int_0^\infty \frac{\sqrt{x}(\ln x + 2\pi i)^2 dx}{x^4 + x^2 + 1} = 2\pi i \sum \text{Res}\frac{\sqrt{z_k}\text{Ln}^2(z_k)}{z_k^4 + z_k^2 + 1}$$

thus

$$2I + 4\pi i I_2 - 4\pi^2 I_1 = 2\pi i \sum \text{Res}\frac{\sqrt{z_k}\text{Ln}^2(z_k)}{z_k^4 + z_k^2 + 1},$$

$$\begin{aligned}
&2\pi i \sum \text{Res}\frac{\sqrt{z_k}\text{Ln}^2(z_k)}{z_k^4 + z_k^2 + 1} = \\
&= \frac{2\pi i}{2\sqrt{3}}\frac{-\pi^2}{9}\left[\frac{i+\sqrt{3}}{i-\sqrt{3}} + \frac{-1+i\sqrt{3}}{\sqrt{3}-i}16 + \frac{1+i\sqrt{3}}{i+\sqrt{3}}4 + \frac{-i+\sqrt{3}}{i+\sqrt{3}}25\right] = \\
&= \frac{\pi^3\sqrt{3}}{27}\left(10 - 13\sqrt{3} + i(6\sqrt{3} - 12)\right)
\end{aligned}$$

and

$$\begin{aligned}
I &= -2\pi i\left(\frac{\pi^2}{6} - \frac{\pi^2\sqrt{3}}{9}\right) + 2\pi^2\left(\frac{\pi}{2} - \frac{\pi\sqrt{3}}{6}\right) + \frac{\pi^3\sqrt{3}}{54}\left(10 - 13\sqrt{3} + i(6\sqrt{3} - 12)\right) = \\
&= \frac{5\pi^3}{18} - \frac{4\pi^3\sqrt{3}}{27}.
\end{aligned}$$

Solution 4 by Moti Levy, Rehovot, Israel.

Let

$$I := \int_0^\infty \frac{\sqrt{x}\ln^2(x)}{x^4 + x^2 + 1}dx,$$

then

$$I = \int_0^\infty \frac{\sqrt{x}\ln^2(x)\left(1 - x^2\right)}{1 - x^6}dx = \int_0^\infty \frac{x^{\frac{1}{2}}\ln^2(x)}{1 - x^6}dx - \int_0^\infty \frac{x^{\frac{5}{2}}\ln^2(x)}{1 - x^6}dx$$

The first integral is

$$\begin{aligned}
\int_0^\infty \frac{x^{\frac{1}{2}}\ln^2(x)}{1 - x^6}dx &= \int_0^1 \frac{x^{\frac{1}{2}}\ln^2(x)}{1 - x^6}dx + \int_1^\infty \frac{x^{\frac{1}{2}}\ln^2(x)}{1 - x^6}dx \\
&= \int_0^1 \frac{x^{\frac{1}{2}}\ln^2(x)}{1 - x^6}dx - \int_0^1 \frac{x^{\frac{7}{2}}\ln^2(x)}{1 - x^6}dx.
\end{aligned}$$

The second integral is

$$\int_0^\infty \frac{x^{\frac{5}{2}}\ln^2(x)}{1-x^6}dx = \int_0^1 \frac{x^{\frac{5}{2}}\ln^2(x)}{1-x^6}dx + \int_1^\infty \frac{x^{\frac{5}{2}}\ln^2(x)}{1-x^6}dx$$
$$= \int_0^1 \frac{x^{\frac{5}{2}}\ln^2(x)}{1-x^6}dx - \int_0^1 \frac{x^{\frac{3}{2}}\ln^2(x)}{1-x^6}dx.$$

Let

$$J(n) := \int_0^1 \frac{x^{\frac{n}{2}}\ln^2(x)}{1-x^6}dx,$$

then

$$I = J(1) + J(3) - J(5) - J(7).$$

Evaluation of $J(n)$:

$$J(n) = \int_0^1 \left(x^{\frac{n}{2}}\ln^2(x)\sum_{k=0}^{\infty} x^{6k}\right) dx$$
$$= \sum_{k=0}^{\infty}\int_0^1 x^{6k+\frac{n}{2}}\ln^2(x)\,dx$$

Now

$$\int_0^1 x^r \ln^2(x)\,dx = \frac{2}{(1+r)^3}, \quad r > -1.$$

Hence

$$J(n) = 2\sum_{k=0}^{\infty}\frac{1}{\left(1+6k+\frac{1}{2}\right)^3} = -\frac{1}{216}\psi^{(2)}\left(\frac{n+2}{12}\right),$$

where $\psi^{(2)}(x)$ is the Polygamma function of order 2.

$$I = -\frac{1}{216}\left(\psi^{(2)}\left(\frac{1}{4}\right) - \psi^{(2)}\left(\frac{3}{4}\right) + \psi^{(2)}\left(\frac{5}{12}\right) - \psi^{(2)}\left(\frac{7}{12}\right)\right).$$

To evaluate I we use the reflection relation

$$\psi^{(2)}(1-z) - \psi^{(2)}(z) = \pi\frac{d^2}{dz^2}\cot(\pi z) = 2\pi^3(\cot\pi z)\left(\cot^2\pi z + 1\right).$$

$$I = -\frac{2\pi^3}{216}\left(\left(\cot\left(\frac{3\pi}{4}\right)\right)\left(\cot^2\left(\frac{3\pi}{4}\right)+1\right) + \left(\cot\left(\frac{7\pi}{12}\right)\right)\left(\cot^2\left(\frac{7\pi}{12}\right)+1\right)\right)$$
$$= \left(\frac{5}{18} - \frac{4}{27}\sqrt{3}\right)\pi^3 \cong 0.656\,64.$$

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

In the calculations below, we use the following well-known definitions and formulas: (1) $\int_0^1 x^a \ln^2 x\, dx = 2/(a+1)^3$ for $a \geqslant 0$; (2) the polygamma function, $\psi^{(2)}(x) = -2\sum_{n=0}^{\infty} 1/(x+n)^3$; (3) $\psi^{(2)}(1-x) - \psi^{(2)}(x) = \pi d^2(\cot \pi x)/dx^2$.

First we write

$$I = \int_0^{\infty} \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\, dx = \overbrace{\int_0^1 \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\, dx}^{A} + \overbrace{\int_1^{\infty} \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\, dx}^{B}.$$

In integral A, the substitution $x^2 = y$ yields

$$\begin{aligned}
\frac{1}{8}\int_0^1 \frac{y^{-1/4}\ln^2 y}{y^2+y+1}\, dy &= -\frac{1}{8}\int_0^1 \frac{(y-1)y^{-1/4}\ln^2 y}{1-y^3}\, dy \\
&= -\frac{1}{8}\left(\int_0^1 \frac{y^{3/4}\ln^2 y}{1-y^3}\, dy - \int_0^1 \frac{y^{-1/4}\ln^2 y}{1-y^3}\, dy\right) \\
&= -\frac{1}{8}\left(\int_0^1 \sum_{k=0}^{\infty} y^{3k+3/4}\ln^2 y\, dy - \int_0^1 \sum_{k=0}^{\infty} y^{3k-1/4}\ln^2 y\, dy\right) \\
&\overset{\text{dom. conv.}}{=} -\frac{1}{8}\sum_{k=0}^{\infty}\left(\int_0^1 y^{3k+3/4}\ln^2 y\, dy - \int_0^1 y^{3k-1/4}\ln^2 y\, dy\right) \\
&\overset{(1)}{=} -\frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{2}{(3k+7/4)^3} - \frac{2}{(3k+3/4)^3}\right) \\
&= -\frac{1}{108}\sum_{k=0}^{\infty}\frac{1}{(k+7/12)^3} + \frac{1}{108}\sum_{k=0}^{\infty}\frac{1}{(k+1/4)^3} \\
&\overset{(2)}{=} \frac{1}{216}\left(\psi^{(2)}\left(\frac{7}{12}\right) - \psi^{(2)}\left(\frac{1}{4}\right)\right)
\end{aligned}$$

In the same way, the substitutions $u = 1/x$ followed by $u^2 = y$ in integral B yield

$$\int_0^1 \frac{u^{3/2}\ln^2 u}{1+u^2+u^4}\, du = \frac{1}{8}\int_0^1 \frac{(1-y)y^{1/4}\ln^2 y}{1-y^3}\, dy = -\frac{1}{216}\left(\psi^{(2)}\left(\frac{5}{12}\right) - \psi^{(2)}\left(\frac{3}{4}\right)\right).$$

Therefore,

$$\begin{aligned}
I = A + B &= \frac{1}{216}\left(\psi^{(2)}\left(\frac{7}{12}\right) - \psi^{(2)}\left(\frac{1}{4}\right)\right) - \frac{1}{216}\left(\psi^{(2)}\left(\frac{5}{12}\right) - \psi^{(2)}\left(\frac{3}{4}\right)\right) \\
&= \frac{1}{216}\left(\psi^{(2)}\left(\frac{7}{12}\right) - \psi^{(2)}\left(\frac{5}{12}\right)\right) + \frac{1}{216}\left(\psi^{(2)}\left(\frac{3}{4}\right) - \psi^{(2)}\left(\frac{1}{4}\right)\right).
\end{aligned}$$

Now we use formula (3) to get $\psi^{(2)}(1-x)-\psi^{(2)}(x) = 2\pi^3\cot\pi x(1+\cot^2\pi x)$. Substituting and using elementary trigonometric identities, we find that $\cot(5\pi/12) = 2-\sqrt{3}$, $\cot(\pi/4) = 1$, $\psi^{(2)}\left(\frac{7}{12}\right)-\psi^{(2)}\left(\frac{5}{12}\right) = 8\pi^3(7-4\sqrt{3})$, $\psi^{(2)}\left(\frac{3}{4}\right)-\psi^{(2)}\left(\frac{1}{4}\right) = 4\pi^3$, giving us

$$I = A+B = \frac{8\pi^3(7-4\sqrt{3})+4\pi^3}{216} = \frac{\pi^3}{54}(15 - 8\sqrt{3}) \approx 0.6566.$$

Solution 6 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

We proceed as follows:

$$\begin{aligned}
\int_0^\infty \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\,dx &= \int_0^1 \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\,dx + \int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\,dx \\
&= \int_0^1 \frac{(x^{1/2}+x^{3/2})\ln^2 x}{x^4+x^2+1}\,dx \\
&= \int_0^1 \frac{(x^{1/2}+x^{3/2}-x^{5/2}-x^{7/2})\ln^2 x}{1-x^6}\,dx \\
&= \sum_{k=0}^\infty \int_0^1 \left(x^{6k+1/2}+x^{6k+3/2}-x^{6k-5/2}-x^{6k-7/2}\right)\ln^2 x\,dx \\
&= 2\sum_{k=0}^\infty \left[\frac{1}{(6k+\frac{3}{2})^3}+\frac{1}{(6k+\frac{5}{2})^3}-\frac{1}{(6k+\frac{7}{2})^3}-\frac{1}{(6k+\frac{9}{2})^3}\right] \\
&= \frac{1}{108}\sum_{k=0}^\infty \left[\frac{1}{(k+\frac{1}{4})^3}+\frac{1}{(k+\frac{5}{12})^3}-\frac{1}{(k+\frac{7}{12})^3}-\frac{1}{(k+\frac{3}{4})^3}\right] \\
&= -\frac{1}{216}\left[\psi_2\left(\frac{1}{4}\right)-\psi_2\left(\frac{3}{4}\right)+\psi_2\left(\frac{5}{12}\right)-\psi_2\left(\frac{7}{12}\right)\right],
\end{aligned}$$

where $\psi_2(x)$ is a polygamma function. It is known that

$$\psi_2(1-z)-\psi_2(z) = \pi\frac{d^2}{dz^2}\cot(\pi z) = 2\pi^3\csc^2(\pi z)\cot(\pi z),$$

so

$$\psi_2\left(\frac{1}{4}\right)-\psi_2\left(\frac{3}{4}\right) = 2\pi^3\csc^2\frac{3\pi}{4}\cot\frac{3\pi}{4} = -4\pi^3$$

and

$$\psi_2\left(\frac{5}{12}\right)-\psi_2\left(\frac{7}{12}\right) = 2\pi^3\csc^2\frac{7\pi}{12}\cot\frac{7\pi}{12} = 8\pi^3(4\sqrt{3}-7).$$

Finally,

$$\int_0^\infty \frac{\sqrt{x}\ln^2 x}{x^4+x^2+1}\,dx = \frac{(15-8\sqrt{3})\pi^3}{54}.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; and the problem proposer.

• **5725** *Proposed by Narendra Bhandari, Bajura District, Nepal.*

Prove

$$\sum_{n=1}^{\infty} \frac{n\cdot 4^n}{(2n-1)^2(4n+5)}\left[\frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}}\right] = \frac{1559\sqrt{2}-1216}{58800}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

By Stirling's asymptotic formula,

$$\frac{n\cdot 4^n}{(2n-1)^2(4n+5)}\frac{\begin{pmatrix}2n\\n\end{pmatrix}}{\begin{pmatrix}4n+4\\2n+2\end{pmatrix}} = O\left(\frac{4^n}{n^2}\frac{\begin{pmatrix}2n\\n\end{pmatrix}}{\begin{pmatrix}4n\\2n\end{pmatrix}}\right) = O\left(\frac{4^n}{n^2}\frac{(2n)!^3}{n!^2(4n)!}\right) =$$

$$= O\left(\frac{4^n}{n^2}\frac{(2n)^{6n+\frac{3}{2}}e^{-6n}}{n^{2n+1}e^{-2n}(4n)^{4n+\frac{1}{2}}e^{-4n}}\right) = O\left(\frac{1}{n^2}\right),\ \ n\to\infty.$$

So the given sum converges absolutely. Euler's evaluation of the beta function gives

$$\frac{1}{(4n+5)\begin{pmatrix}4n+4\\2n+2\end{pmatrix}} = \frac{(2n+2)!\,(2n+2)!}{(4n+5)!} = \int_0^1 x^{2n+2}(1-x)^{2n+2}dx.$$

Furthermore

$$\frac{n\begin{pmatrix}2n\\n\end{pmatrix}}{(2n-1)^2} = \frac{2\,(2n-2)!}{(2n-1)\,(n-1)!^2} = \frac{2}{2n-1}\begin{pmatrix}2n-2\\n-1\end{pmatrix}.$$

The generating function of the central binomial coefficients is given by

$$\frac{1}{\sqrt{1-y}} = \sum_{n=0}^{\infty}\frac{1}{4^n}\begin{pmatrix}2n\\n\end{pmatrix}y^n,\ \ |y|<1.$$

We replace y by y^2 and integrate to obain

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)\,4^n}\begin{pmatrix}2n\\n\end{pmatrix}y^{2n} = \frac{1}{y}\int_0^y\frac{1}{\sqrt{1-t^2}}dt = \frac{\arcsin(y)}{y},\ \ 0\leqslant y<1.$$

By Stirling's asymptotic formula, $\frac{1}{(2n+1)4^n}\binom{2n}{n} = O\left(\frac{1}{n^{3/2}}\right)$, as n→∞. Hence, by Abel's theorem, above equation holds true even for y=1. We find

$$\sum_{n=1}^{\infty}\frac{n\cdot 4^n}{(2n-1)^2(4n+5)}\frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}}y^{2n} = \sum_{n=1}^{\infty}\frac{n\cdot 4^n}{(2n-1)^2}\binom{2n}{n}y^{2n}\int_0^1 x^{2n+2}(1-x)^{2n+2}dx =$$

$$= 2\sum_{n=1}^{\infty}\frac{4^n}{2n-1}\binom{2n-2}{n-1}y^{2n}\int_0^1 x^{2n+2}(1-x)^{2n+2}dx =$$

$$= 8y^2\sum_{n=0}^{\infty}\frac{1}{(2n+1)4^n}\binom{2n}{n}\int_0^1 x^4(1-x)^4\left(4xy(1-x)\right)^{2n}dx =$$

$$= 8y^2\int_0^1 x^4(1-x)^4\frac{\arcsin\left(4xy(1-x)\right)}{4xy(1-x)}dx =$$

$$= 2y\int_0^1 x^3(1-x)^3\arcsin\left(4xy(1-x)\right)\,dx.$$

We let y tend to 1 and find

$$\sum_{n=1}^{\infty}\frac{n\cdot 4^n}{(2n-1)^2(4n+5)}\frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} = 2\int_0^1 x^3(1-x)^3\arcsin\left(4x(1-x)\right)\,dx.$$

We perform the change of variables x = $\sin^2$ t, dx = 2 sin t cos t dt and get

$$2\int_0^1 x^3(1-x)^3\arcsin\left(4x(1-x)\right)\,dx = \frac{1}{32}\int_0^{\frac{\pi}{2}}(2\sin t\cos t)^7\arcsin\left((2\sin t\cos t)^2\right)\,dt =$$

$$= \frac{1}{32}\int_0^{\frac{\pi}{2}}\sin^7(2t)\arcsin\left(\sin^2(2t)\right)\,dt = \frac{1}{32}\int_0^{\frac{\pi}{2}}\left(\sin(t)\right)^7\arcsin\left(\left(\sin(t)\right)^2\right)\,dt =$$

$$= \frac{1}{64}\int_0^1\frac{x^3}{\sqrt{1-x}}\arcsin(x)\,dx,$$

by applying the same substitution (in reverse order).
We easily establish by repeated integration by parts that

$$\int\frac{x^3}{\sqrt{1-x}}dx = -\frac{2}{35}\sqrt{1-x}\left(5x^3+6x^2+8x+16\right)+C.$$

Hence integration by parts gives

$$\frac{1}{64}\int_0^1 \frac{x^3}{\sqrt{1-x}}\arcsin(x)\,dx = -\frac{2}{35}\sqrt{1-x}\left(5x^3+6x^2+8x+16\right)\frac{1}{64}\arcsin(x)\Big|_{x=0}^{x=1}+$$

$$+\frac{1}{64}\int_0^1 \frac{2}{35}\sqrt{1-x}\left(5x^3+6x^2+8x+16\right)\frac{1}{\sqrt{1-x^2}}dx =$$

$$= \frac{1}{1120}\int_0^1 \left(5x^3+6x^2+8x+16\right)\frac{1}{\sqrt{1+x}}dx = \frac{1559\sqrt{2}-1216}{58800},$$

where we have used that

$$\int \frac{1}{\sqrt{1+x}}dx = 2\sqrt{1+x}+C,$$

$$\int \frac{x}{\sqrt{1+x}}dx = \frac{2}{3}\sqrt{1+x}\ (x-2)+C,$$

$$\int \frac{x^2}{\sqrt{1+x}}dx = \frac{2}{15}\sqrt{1+x}\ \left(3x^2-4x+8\right)+C,$$

$$\int \frac{x^3}{\sqrt{1+x}}dx = \frac{2}{35}\sqrt{1+x}\ \left(5x^3-6x^2+8x-16\right)+C.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

In [1, Theorem 5], the problem proposer proves

$$\sum_{n=1}^{\infty}\frac{n\cdot 4^n}{(2n-1)^2(4n+m)}\cdot\frac{\binom{2n}{n}}{\binom{4n+m-1}{2n+\frac{m-1}{2}}} = \sqrt{2}\sum_{k=0}^{\frac{m+1}{2}}\frac{(-1)^k}{(2k+1)2^{m-k}}\binom{\frac{m+1}{2}}{k}\int_0^{1/2}\frac{t^k}{\sqrt{1-t}}\,dt.$$

With $m=5$, this becomes

$$\sum_{n=1}^{\infty}\frac{n\cdot 4^n}{(2n-1)^2(4n+5)}\cdot\frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}}$$
$$= \frac{\sqrt{2}}{32}\int_0^{1/2}\frac{1}{\sqrt{1-t}}\,dt-\frac{\sqrt{2}}{16}\int_0^{1/2}\frac{t}{\sqrt{1-t}}\,dt+\frac{3\sqrt{2}}{40}\int_0^{1/2}\frac{t^2}{\sqrt{1-t}}\,dt-\frac{\sqrt{2}}{28}\int_0^{1/2}\frac{t^3}{\sqrt{1-t}}\,dt$$

Now,

$$\int_0^{1/2}\frac{1}{\sqrt{1-t}}\,dt = -2\sqrt{1-t}\Big|_0^{1/2} = 2-\sqrt{2},$$

and with the substitution $u = 1 - t$,

$$\begin{aligned}
\int_0^{1/2} \frac{t}{\sqrt{1-t}}\,dt &= \int_{1/2}^{1} (u^{-1/2} - u^{1/2})\,du = 2\sqrt{u} - \frac{2}{3}u^{3/2}\Big|_{1/2}^{1} = \frac{8-5\sqrt{2}}{6};\\
\int_0^{1/2} \frac{t^2}{\sqrt{1-t}}\,dt &= \int_{1/2}^{1} (u^{-1/2} - 2u^{1/2} + u^{3/2})\,du\\
&= 2\sqrt{u} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2}\Big|_{1/2}^{1} = \frac{64-43\sqrt{2}}{60};\\
\int_0^{1/2} \frac{t^3}{\sqrt{1-t}}\,dt &= \int_{1/2}^{1} (u^{-1/2} - 3u^{1/2} + 3u^{3/2} - u^{5/2})\,du\\
&= 2\sqrt{u} - 2u^{3/2} + \frac{6}{5}u^{5/2} - \frac{2}{7}u^{7/2}\Big|_{1/2}^{1} = \frac{256-177\sqrt{2}}{280}.
\end{aligned}$$

Bringing all of these results together yields

$$\sum_{n=1}^{\infty} \frac{n \cdot 4^n}{(2n-1)^2(4n+5)} \cdot \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} = \frac{1559\sqrt{2} - 1216}{58800}.$$

Reference:

1. Narendra Bhandari, "Infinite Series Associated with the Ratio and Product of Central Binomial Coefficients," *Journal of Integer Sequences*, volume 25 (2022), Article 22.6.5, 18 pages.

Solution 3 by Moti Levy, Rehovot, Israel.

Let

$$S := \sum_{n=1}^{\infty} \frac{n \cdot 4^n}{(2n-1)^2\,(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}}.$$

We present here two solutions; one is based on a paper by Narendra Bhandari [1] and the second is based on hypergeometric functions, see [2].

Solution I: Narendra Bhandari paper.

The generating function of the sequence $\left\{\binom{2n}{n}\right\}_{n\geqslant 0}$ is

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}} \tag{8}$$

It follows from (8) that

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-2}}{4^n}=\frac{1}{x^2\sqrt{1-x^2}}-\frac{1}{x^2}. \tag{9}$$

Integration of both sides of (9) gives

$$\int_0^x\left(\frac{1}{t^2\sqrt{1-t^2}}-\frac{1}{t^2}\right)dt=\frac{1-\sqrt{1-x^2}}{x},\quad 0\leqslant x<1.$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-2}}{(2n-1)\,4^n}=\frac{1-\sqrt{1-x^2}}{x^2}. \tag{10}$$

Integration of both sides of (10) gives

$$\int_0^x\left(\frac{1-\sqrt{1-t^2}}{t^2}\right)dt=\frac{-1+\sqrt{1-x^2}+x\arcsin(x)}{x},\quad 0\leqslant x<1.$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-1}}{(2n-1)^2\,4^n}=\frac{-1+\sqrt{1-x^2}+x\arcsin(x)}{x} \tag{11}$$

We multiply by x and then differentiae both sides of (12)

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n}}{(2n-1)^2\,4^n}=-1+\sqrt{1-x^2}+x\arcsin(x). \tag{12}$$

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,4^n}\binom{2n}{n}x^{2n-1}=\frac{1}{2}\frac{d\left(-1+\sqrt{1-x^2}+x\arcsin(x)\right)}{dx}=\frac{1}{2}\arcsin(x). \tag{13}$$

It follows that

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,4^n}\binom{2n}{n}x^{4n+5}=\frac{x^7}{2}\arcsin\left(x^2\right). \tag{14}$$

Change $x=\sin(y)$ in (14)

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,4^n}\binom{2n}{n}\sin(y)^{4n+5}=\frac{\sin^7(y)}{2}\arcsin\left(\sin^2(y)\right).$$

After integration of both sides,

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2\,4^n}\binom{2n}{n}\int_0^{\frac{\pi}{2}}\sin(y)^{4n+5}\,dy=\int_0^{\frac{\pi}{2}}\frac{\sin^7(y)}{2}\arcsin\left(\sin^2(y)\right)dy \tag{15}$$

We have the following integrals:

$$\int_0^{\frac{\pi}{2}}\sin(y)^{4n+5}\,dy=\frac{2^{4n+4}}{(4n+5)\binom{4n+4}{2n+2}}. \tag{16}$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{\sin^7(y)}{2} \arcsin\left(\sin^2(y)\right) dy \\
&= \int_0^{\frac{\pi}{2}} \frac{\left(1-\cos^2(y)\right)^3}{2} \sin(y) \arcsin\left(1-\cos^2(y)\right) dy \\
&= \frac{1}{2}\int_0^1 \left(\arcsin\left(1-t^2\right)\right)\left(1-t^2\right)^3 dt \\
&= \frac{1}{2}\sum_{k=0}^{3}(-1)^k \binom{3}{k} \int_0^1 t^{2k} \arcsin\left(1-t^2\right) dt. \qquad (17)
\end{aligned}$$

$$\int_0^1 \arcsin\left(1-t^2\right) dt = 2\left(\sqrt{2}-1\right), \qquad (18)$$

$$\int_0^1 t^2 \arcsin\left(1-t^2\right) dt = \frac{2}{9}\left(4\sqrt{2}-5\right), \qquad (19)$$

$$\int_0^1 t^4 \arcsin\left(1-t^2\right) dt = \frac{2}{75}\left(32\sqrt{2}-43\right), \qquad (20)$$

$$\int_0^1 t^6 \arcsin\left(1-t^2\right) dt = \frac{2}{245}\left(128\sqrt{2}-177\right). \qquad (21)$$

Substitution of (18), (19), (20) and (21) into (17) gives

$$\int_0^{\frac{\pi}{2}} \frac{\sin^7(y)}{2} \arcsin\left(\sin^2(y)\right) dy = \frac{3118\sqrt{2}-2432}{3675}. \qquad (22)$$

Plugging (16) and (22) into (15) produces the desired result:

$$\sum_{n=1}^{\infty} \frac{n}{(2n-1)^2 4^n} \binom{2n}{n} \frac{2^{4n+5}}{(4n+5)\binom{4n+4}{2n+2}} = \frac{3118\sqrt{2}-2432}{3675},$$

or

$$\sum_{n=1}^{\infty} \frac{n4^n}{(2n-1)^2(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} = \frac{1559\sqrt{2}-1216}{58800}.$$

Solution II: Hypergeometric functions.

Let

$$S(z) := \sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} z^n, \qquad (23)$$

then clearly $S = S(4)$.

To get rid of n in the numerator, we rewrite (23)

$$\frac{S(z)}{z} = \sum_{n=1}^{\infty} \frac{n}{(2n-1)^2(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} z^{n-1} \qquad (24)$$

and integrate both sides of (24) to get

$$T(z) := \int_0^z \left(\frac{S(t)}{t}\right) dt = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} z^n \tag{25}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n-1)^2(4n+5)} \frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}} z^n - \frac{1}{30}.$$

To evaluate binomial sum in (25), the first step is to express the binomial series as a hypergeometric function; the second step is application of some classical hypergeometric theorems and identities.

The following lemma explains how to express a binomial sum as a hypergeometric function:

Lemma : Let $(\alpha_k)_{k\geqslant 0}$ be a sequence which satisfies the following conditions:

$$\alpha_0 = 1,$$

$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{1}{k+1} \frac{(k+a)(k+b)(k+c)(k+d)}{(k+e)(k+f)(k+g)} z.$$

Then

$$\sum_{k=0}^{\infty} \alpha_k = {}_4F_3\left(\left.\begin{array}{cccc} a, & b, & c, & d \\ e, & f, & g & \end{array}\right| z\right), \tag{26}$$

where ${}_4F_3\left(\left.\begin{array}{cccc} a, & b, & c, & d \\ e, & f, & g & \end{array}\right| z\right)$ is a generalized hypergeometric function.■

Evaluation of $T(z) = \sum_{n=1}^{\infty} \alpha_k$:

Let

$$\alpha_k := 30 \frac{1}{(2k-1)^2(4k+5)} \frac{\binom{2k}{k}}{\binom{4k+4}{2k+2}} z^k, \qquad \alpha_0 = 1$$

then

$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{1}{k+1} \frac{\left(k+\frac{3}{2}\right)(k+2)\left(k-\frac{1}{2}\right)\left(k-\frac{1}{2}\right)}{\left(k+\frac{1}{2}\right)\left(k+\frac{7}{4}\right)\left(k+\frac{9}{4}\right)} \frac{z}{4}$$

By the lemma,

$$T(z) = \frac{1}{30} {}_4F_3\left(\left.\begin{array}{cccc} \frac{3}{2}, & -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{1}{2}, & \frac{7}{4}, & \frac{9}{4} & \end{array}\right| \frac{z}{4}\right) - \frac{1}{30}. \tag{27}$$

By differentiation of both sides and multiplication by z, we obtain,

$$S(z) = \frac{1}{60}\left({}_4F_3\left(\left.\begin{array}{cccc} \frac{3}{2}, & -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{1}{2}, & \frac{7}{4}, & \frac{9}{4} & \end{array}\right| \frac{z}{4}\right) - {}_3F_2\left(\left.\begin{array}{ccc} \frac{3}{2}, & -\frac{1}{2}, & 2 \\ \frac{9}{4}, & \frac{7}{4} & \end{array}\right| \frac{z}{4}\right)\right).$$

$$S = S(4) = \frac{1}{60}\left({}_4F_3\left(\left.\begin{matrix} \frac{3}{2}, & -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{1}{2}, & \frac{7}{4}, & \frac{9}{4} & \end{matrix}\right| 1\right) - {}_3F_2\left(\left.\begin{matrix} \frac{3}{2}, & -\frac{1}{2}, & 2 \\ \frac{9}{4}, & \frac{7}{4} & \end{matrix}\right| 1\right)\right). \tag{28}$$

Now, to reduce the order of the hypergeometric function, we use the identity of hypergeometric functions with integral parameter difference, see [3]:

$$\begin{aligned} &{}_4F_3\left(\left.\begin{matrix} \frac{3}{2}, & -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{1}{2}, & \frac{7}{4}, & \frac{9}{4} & \end{matrix}\right| 1\right) \\ &= {}_3F_2\left(\left.\begin{matrix} -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{9}{4}, & \frac{7}{4} & \end{matrix}\right| 1\right) + \frac{16}{63}\,{}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & 3 \\ \frac{11}{4}, & \frac{13}{4} & \end{matrix}\right| 1\right). \end{aligned} \tag{29}$$

By plugging (29) in (28), the sum includes three ${}_3F_2(1)$ terms.

$$S = \frac{1}{60}\,{}_3F_2\left(\left.\begin{matrix} -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{9}{4}, & \frac{7}{4} & \end{matrix}\right| 1\right) + \frac{4}{945}\,{}_3F_2\left(\left.\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & 3 \\ \frac{11}{4}, & \frac{13}{4} & \end{matrix}\right| 1\right) - \frac{1}{60}\,{}_3F_2\left(\left.\begin{matrix} -\frac{1}{2}, & \frac{3}{2}, & 2 \\ \frac{7}{4}, & \frac{9}{4} & \end{matrix}\right| 1\right). \tag{30}$$

To evaluate the ${}_3F_2(1)$ terms, we use two special cases of the generalization of Whipple's theorem presented in [4], equation (4).

Theorem (Generalized Whipple):

If $a+b=1$ and $e+f=2c$ then

$${}_3F_2\left(\{a,b,c\},\{e,f\},1\right) = \frac{\Gamma(e)\Gamma(f)}{2^{2a}(c-1)\Gamma(e-a)\Gamma(f-a)}\left(\frac{\Gamma\left(\frac{e-a+1}{2}\right)\Gamma\left(\frac{f-a}{2}\right)}{\Gamma\left(\frac{e+a-1}{2}\right)\Gamma\left(\frac{f+a}{2}\right)} + \frac{\Gamma\left(\frac{e-a}{2}\right)\Gamma\left(\frac{f-a+1}{2}\right)}{\Gamma\left(\frac{e+a}{2}\right)\Gamma\left(\frac{f+a-1}{2}\right)}\right). \tag{31}$$

If $a+b=-1$ and $e+f=2c$ then

$${}_3F_2\left(\{a,b,c\},\{e,f\},1\right) = \frac{\Gamma(e)\Gamma(f)}{2^{2a+2}\Gamma(e-a)\Gamma(f-a)}\left(f\frac{\Gamma\left(\frac{e-a+1}{2}\right)\Gamma\left(\frac{f-a}{2}\right)}{\Gamma\left(\frac{e+a+1}{2}\right)\Gamma\left(\frac{f+a+2}{2}\right)} + e\frac{\Gamma\left(\frac{e-a}{2}\right)\Gamma\left(\frac{f-a+1}{2}\right)}{\Gamma\left(\frac{e+a+2}{2}\right)\Gamma\left(\frac{f+a+1}{2}\right)}\right). \blacksquare \tag{32}$$

Applying (31), we get

$${}_3F_2\left(\left.\begin{matrix} \frac{3}{2}, & -\frac{1}{2}, & 2 \\ \frac{7}{4}, & \frac{9}{4} & \end{matrix}\right| 1\right) = \frac{11\sqrt{2}-4}{28}. \tag{33}$$

Again, applying (31), we get

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & 3 \\ \frac{11}{4}, & \frac{13}{4} & \end{matrix}\middle| 1\right) = \frac{411\sqrt{2}-264}{280}. \tag{34} $$

Finally, applying (32), we get

$$ {}_3F_2\left(\begin{matrix} -\frac{1}{2}, & -\frac{1}{2}, & 2 \\ \frac{9}{4}, & \frac{7}{4} & \end{matrix}\middle| 1\right) = \frac{1184\sqrt{2}-841}{735} \tag{35} $$

After substitution of (34), (35) and into (30) we obtain the desired result,

$$ S = \frac{1}{60}\frac{1184\sqrt{2}-841}{735} + \frac{4}{945}\frac{411\sqrt{2}-264}{280} - \frac{1}{60}\frac{11\sqrt{2}-4}{28} = \frac{1559\sqrt{2}-1216}{58800}. $$

References:

[1] Narendra Bhandari, "*Infinite Series Associated with the ratio and Product of Central Binomial Coefficient*", Journal of Integer Sequences, Vol. 25 (2022).

[2] R. L. Graham, D. E. Knuth, O. Patashnik, "*Concrete Mathematics*", Addison-Wesley.

[3] W. Karlsson, "*Hypergeometric Functions with Integral Parameter Differences*", Journal of Mathematical Physics Vol. 12, No. 2, February 1971.

[4] J. L. Lavoie, F. Grondin, A. K. Rathie, "Generalizations of Whipple's theorem on the sum of a ${}_3F_2$", Journal of Computational and Applied Mathematics, 72 (1996) 293-300.

Solution 4 by Paolo Perfetti, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

$$ \int_0^{\pi/2} (\sin x)^{2n+1} dx = \frac{4^n}{(2n+1)\binom{2n}{n}}, \qquad \sum_{n=1}^{\infty} \binom{2n}{n} \frac{nx^n}{(2n-1)^2} = \sqrt{x}\arcsin(2\sqrt{x}) $$

$$ \int_0^{\pi/2} (\sin x)^{4n+5} dx = \frac{4^{2n+2}}{(4n+5)\binom{4n+4}{2n+2}} $$

$$\sum_{n=1}^{\infty}\frac{n4^n}{(2n-1)^2(4n+5)}\left[\frac{\binom{2n}{n}}{\binom{4n+4}{2n+2}}\right]=\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2 4^n 16}\binom{2n}{n}\int_0^{\pi/2}(\sin x)^{4n+5}dx=$$
$$=\frac{1}{16}\int_0^{\pi/2}(\sin x)^5\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2}\binom{2n}{n}\left(\frac{\sin^4 x}{4}\right)^n dx=$$
$$=\frac{1}{16}\int_0^{\pi/2}(\sin x)^5\frac{\sin^2 x}{2}\arcsin(\sin^2 x)dx\underbrace{=}_{\sin^2 x=t}$$
$$=\frac{1}{32}\int_0^1\frac{t^3}{\sqrt{1-t}}\arcsin t dt\doteq K$$

$$\int\frac{t^3\,dt}{\sqrt{1-t}}=-2\sqrt{1-t}+2(1-t)^{3/2}-\frac{6}{5}(1-t)^{5/2}+\frac{2}{7}(1-t)^{7/2}\doteq G(t)$$

$$K=\frac{1}{32}G(t)\arcsin t\Big|_0^1-\frac{1}{32}\int_0^1\frac{G(t)}{\sqrt{1-t^2}}=$$
$$=\frac{-1}{32}\int_0^1\left[\frac{-2}{\sqrt{1+t}}+\frac{2(1-t)}{\sqrt{1+t}}-\frac{6}{5}\frac{(1-t)^2}{\sqrt{1+t}}+\frac{2}{7}\frac{(1-t)^3}{\sqrt{1+t}}\right]dt=$$
$$=\frac{-1}{32}\int_0^1\left[\frac{-32}{35}-\frac{16t}{35}-\frac{12t^2}{35}-\frac{2t^3}{7}\right]\frac{dt}{\sqrt{1+t}}\underbrace{=}_{1+t=y}$$
$$=\frac{-1}{32}\frac{-2}{35}\int_1^2\left(\frac{9}{\sqrt{y}}+11\sqrt{y}-9y^{3/2}+5y^{5/2}\right)dy=\frac{1599\sqrt{2}-1216}{58800}.$$

Solution 5 by Péter Fülöp, Gyömrő, Hungary.

Let's re-express the part $\frac{1}{\binom{4n+4}{2n+2}}$ with the help of Γ function:

$$\frac{\Gamma(2n+3)\Gamma(2n+3)}{\Gamma(4n+5)}=\frac{(4n+5)\Gamma^2(2n+3)}{\Gamma(4n+6)}$$

Put it back into the sum we get:

$$S=\sum_{n=1}^{\infty}\frac{n4^n}{(2n-1)^2}\underbrace{\frac{\Gamma^2(2n+3)}{\Gamma(4n+6)}}_{\int_0^1[x(1-x)]^{2n+2}dx}\binom{2n}{n}$$

Using the relation between Γ and β functions and starting the sum from zero:

$$S = \sum_{n=0}^{\infty} \frac{(n+1)4^{n+1}}{(2n+1)^2} \binom{2n+2}{n+1} \int_0^1 [x(1-x)]^{2n+4} dx$$

Performing the possible simplifications then exchange the order of the summation and integration we get:

$$S = 8 \int_0^1 \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)} \binom{2n}{n} [x(1-x)]^{2n+4} dx$$

After further transformation:

$$S = 2 \int_0^1 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{[4x(1-x)]^{2n+1}}{4^n(2n+1)} [x(1-x)]^3 dx$$

Let's realize that $\arcsin[4x(1-x)] = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{[4x(1-x)]^{2n+1}}{4^n(2n+1)}$ we get the following integral:

$$S = 2 \int_0^1 [x(1-x)]^3 \arcsin[4x(1-x)] dx$$

Applying the $x = t + \frac{1}{2}$ and then the $t = \frac{1}{2}\sqrt{1-y}$ substitutions we have:

$$S = \frac{1}{32} \int_{-\frac{1}{2}}^{\frac{1}{2}} [1-4t^2]^3 \arcsin[1-4t^2] dt = \frac{1}{64} \int_0^1 \underbrace{\frac{y^3}{\sqrt{1-y}}}_{u'} \underbrace{\arcsin(y)}_{v} dy$$

Let's integrate it by parts:

$$u = \int \frac{y^3}{\sqrt{1-y}} dy =, \text{ and } v' = \frac{1}{\sqrt{1-y^2}}$$

Calculation of u(y); Performed the $z = 1 - y$ substitution:

$$u = \int \frac{1}{\sqrt{z}}(1 - 3z + 3z^2 - z^3) dz = \frac{\sqrt{z}}{35}(70 - 70z + 42z^2 - 10z^3\Big|_{z=1-y}$$

$$u = -2\frac{\sqrt{1-y}}{35}(16 + 8y + 6y^2 + 5y^3)$$

We can see that $uv\Big|_0^1 = 0$. The other part of the integral is $-\int_0^1 uv' dy$ we get:

$$S = \frac{1}{1120}\int_0^1 \frac{16 + 8y + 6y^2 + 5y^3}{\sqrt{1+y}}dy$$

Substitution $r = 1 + y$:

$$S = \frac{1}{1120}\int_1^2 \frac{16 + 8(1+r) + 6(1+r)^2 + 5(1+r)^3}{\sqrt{r}}dr$$

After performing the exponentiations and the integration we get the to the result:

$$S = \frac{\sqrt{r}}{560}\frac{(945 + 385r - 189r^2 + 75r^3)}{105}\Big|_1^2$$

$$S = \frac{1559\sqrt{2}}{58800} - \frac{76}{3675} = \frac{1559\sqrt{2} - 1216}{58800}.$$

Solution 6 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\binom{n}{k}^{-1} = (n+1)\int_0^1 x^{n-k}(1-x)^k \mathrm{d}x$$

$$\frac{1}{(4n+5)\binom{4n+4}{2n+2}} = \int_0^1 x^{2n+2}(1-x)^{2n+2}\mathrm{d}x$$

$$\because \quad \sum_{n=1}^{\infty}\binom{2n}{n}\frac{n}{(2n-1)^2}x^n = \sqrt{x}\arcsin(2\sqrt{x})$$

$$\therefore \quad \sum_{n=1}^{\infty}\frac{n4^n\binom{2n}{n}}{(2n-1)^2}\int_0^1 x^{2n+2}(1-x)^{2n+2}\mathrm{d}x$$

$$= \int_0^1\left[\sum_{n=1}^{\infty}\frac{n4^n\binom{2n}{n}}{(2n-1)^2}(x-x^2)^{2n+2}\right]\mathrm{d}x$$

$$= \int_0^1\left(x-x^2\right)^2\left[\sum_{n=1}^{\infty}\frac{n\binom{2n}{n}}{(2n-1)^2}\cdot\left(2x-2x^2\right)^{2n}\right]\mathrm{d}x$$

$$= \int_0^1\left(x-x^2\right)^2\sqrt{(2x-2x^2)^2}\arcsin\left(2\left(2x-2x^2\right)\right)\mathrm{d}x$$

$$= \int_0^1\left(x-x^2\right)^2\left(2x-2x^2\right)\arcsin\left(4x-4x^2\right)\mathrm{d}x$$

$$= \int_0^1 2\left(x-x^2\right)^3\arcsin\left(4x-4x^2\right)\mathrm{d}x$$

$$= 2\left(I_1 + I_2\right)$$

$$I_1 = \int_0^{1/2} \left(x - x^2\right)^3 \arcsin\left(4x - 4x^2\right) \mathrm{d}x$$

$$I_2 = \int_{1/2}^{1} \left(x - x^2\right)^3 \arcsin\left(4x - 4x^2\right) \mathrm{d}x$$

let $y = 1 - x, x = 1 - y$, then

$$I_2 = \int_{1/2}^{0} (1-y)^3 y^3 \arcsin\left(4y - 4y^2\right)(-1)\mathrm{d}y$$

$$= \int_0^{1/2} y^3(1-y)^3 \arcsin\left(4y - 4y^2\right) \mathrm{d}y = I_1$$

$$\therefore \quad I = 4I_1$$

now evaluate I_1

let $y = 4\left(x - x^2\right), \mathrm{d}y = (4 - 8x)\mathrm{d}x, \dfrac{y}{4} = x - x^2$

$$I_1 = \int_0^1 \frac{y^3}{64} \arcsin y \frac{\mathrm{d}y}{4(1-2x)}$$

$$\because \quad 2x = 1 - \sqrt{1-y},$$

$$\therefore \quad I_1 = \int_0^1 \frac{y^3}{64} \arcsin y \frac{\mathrm{d}y}{4\sqrt{1-y}}$$

$$\therefore \quad I = \frac{1}{64} \int_0^1 \frac{y^3 \arcsin y}{\sqrt{1-y}} \mathrm{d}y$$

$$= \frac{1}{64} \left\{ \frac{1}{3675\sqrt{1-y}} \cdot 2\left[2\sqrt{1-y^2}\left(75y^3 + 36y^2 + 232y + 1216\right)\right.\right.$$

$$\left.\left. +105\left(5y^4 + y^3 + 2y^2 + 8y - 16\right)\right] \arcsin y \right\}$$

$$= \frac{1599\sqrt{2} - 1216}{58800}.$$

Also solved by the problem proposer.

• **5726** *Proposed by Toyesh Prakash Sharma (Student) Agra College, Agra, India.*

Calculate

$$I := \int_0^1 \int_0^1 \int_0^1 \frac{\sqrt{x^3 y^2 z}\, dxdydz}{x\left(\sqrt{y} + \sqrt{z}\right) + y\left(\sqrt{z} + \sqrt{x}\right) + z\left(\sqrt{x} + \sqrt{y}\right)}.$$

Solution 1 by Ankush Kumar Parcha, New Delhi, India.

Given

$$I := \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}dxdydz}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}$$
$$= \int_0^1\int_0^1\int_0^1 \frac{xy\sqrt{xz}dxdydz}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}.$$

Then, by symmetry,

$$I+I+I = \int_0^1\int_0^1\int_0^1 \frac{\left(\mathrm{xy}\sqrt{\mathrm{xz}}+\mathrm{yz}\sqrt{\mathrm{yx}}+\mathrm{zx}\sqrt{\mathrm{zy}}\right)dxdydz}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}$$
$$= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{xyz}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\right)dxdydz}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}.$$

Once gain, by symmetry, we can obtain

$$3I+3I = \int_0^1\int_0^1\int_0^1 \frac{\sqrt{xyz}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}+x\sqrt{z}+y\sqrt{x}+z\sqrt{y}\right)dxdydz}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}.$$

Since

$$(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}+x\sqrt{z}+y\sqrt{x}+z\sqrt{y} = x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right),$$

then

$$6I = \int_0^1\int_0^1\int_0^1 \sqrt{xyz}dxdydz = \left(\int_0^1 \sqrt{x}\mathrm{dx}\right)\left(\int_0^1 \sqrt{y}dy\right)\left(\int_0^1 \sqrt{z}dz\right) = \frac{8}{27}.$$

So

$$I = \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}dxdydz}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)} = \frac{4}{81}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let

$$I = \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}.$$

Recognize that the value of the integral remains unchanged for each of the other five permutations of the variables x, y, and z under the radical in the numerator of the integrand. That is,

$$\begin{aligned}
I &= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3yz^2}\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})} \\
&= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^2y^3z}\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})} \\
&= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^2yz^3}\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})} \\
&= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{xy^3z^2}\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})} \\
&= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{xy^2z^3}\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}.
\end{aligned}$$

Summing these six variations of I yields

$$\begin{aligned}
6I &= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}+\sqrt{x^3yz^2}+\sqrt{x^2y^3z}+\sqrt{x^2yz^3}+\sqrt{xy^3z^2}+\sqrt{xy^2z^3}}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}\,dx\,dy\,dz \\
&= \int_0^1 \sqrt{x}\,dx \int_0^1 \sqrt{y}\,dy \int_0^1 \sqrt{z}\,dz \\
&= \left(\frac{2}{3}\right)^3.
\end{aligned}$$

Thus,

$$I = \frac{1}{6}\left(\frac{2}{3}\right)^3 = \frac{4}{81}.$$

Solution 3 by Michael Faleski, Delta College, University Center, MI.

We note that from the symmetry of the variables involved that one can permute the variables to obtain the same result. That is, the numerator can have the form $\sqrt{x^3z^2y}$, $\sqrt{y^3x^2z}$, $\sqrt{y^3z^2x}$, $\sqrt{z^3x^2y}$, or $\sqrt{z^3y^2x}$ and not change the evaluation of the integral.

By writing each of these separate identical integrals and adding them together yields

$$6I = \int_0^1\int_0^1\int_0^1 \frac{\left(\sqrt{x^3y^2z}+\sqrt{x^3z^2y}+\sqrt{y^3x^2z}+\sqrt{y^3z^2x}+\sqrt{z^3x^2y}+\sqrt{z^3y^2x}\right)\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{x}+\sqrt{z})+z(\sqrt{x}+\sqrt{y})}$$

For the numerator, we factor $\sqrt{xyz}$ out of each term and then reduce any $\sqrt{p^2}=p$ terms to yield

$$6I = \int_0^1\int_0^1\int_0^1 \frac{\sqrt{xyz}\left(x(\sqrt{y}+\sqrt{z})+y(\sqrt{x}+\sqrt{z})+z(\sqrt{x}+\sqrt{y})\right)\,dx\,dy\,dz}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{x}+\sqrt{z})+z(\sqrt{x}+\sqrt{y})}$$

which leaves the integral to be computed as

$$I = \frac{1}{6}\int_0^1\int_0^1\int_0^1 \sqrt{xyz}\,dx\,dy\,dz$$

The triple integral yields

$$I = \frac{1}{6}\left(\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1\right)^3 = \frac{1}{6}\left(\frac{2}{3}\right)^3 = \frac{4}{81}$$

Solution 4 by Michel Bataille, Rouen, France.

Let $f(x,y,z) = \dfrac{\sqrt{x^3y^2z}}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}$ and let $I = \int_0^1\int_0^1\int_0^1 f(x,y,z)\,dxdydz$.

Note that $f(x,y,z) \geqslant 0$ for $(x,y,z) \in [0,1]^3$ and that $f(x,y,z)$ is defined a.e. on $[0,1]^3$.
If we permute x,y,z in some way in the expression of $f(x,y,z)$, the integral remains unchanged. Since the denominator of $f(x,y,z)$ is symmetric in x,y,z, it follows that

$$6I = \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}+\sqrt{x^3yz^2}+\sqrt{x^2y^3z}+\sqrt{x^2yz^3}+\sqrt{xy^2z^3}+\sqrt{xy^3z^2}}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}\,dxdydz.$$

Here, the numerator is $\sqrt{xyz}(x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y}))$, hence

$$6I = \int_0^1\int_0^1\int_0^1 \sqrt{xyz}\,dxdydz = \left(\int_0^1 \sqrt{x}\,dx\right)\left(\int_0^1 \sqrt{y}\,dy\right)\left(\int_0^1 \sqrt{z}\,dz\right) = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{8}{27}$$

and therefore $I = \dfrac{4}{81}$.

Solution 5 by Moti Levy, Rehovot, Israel.

Since the integrand and the domain of integration has symmetry, then the value of the integral remains fixed after any of the six permutations of (x,y,z). It follows that

$$\begin{aligned} I &= \frac{1}{6}\int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}+\sqrt{xy^3z^2}+\sqrt{x^2yz^3}+\sqrt{x^3yz^2}+\sqrt{xy^2z^3}+\sqrt{x^2y^3z}}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}dxdydz \\ &= \frac{1}{6}\int_0^1\int_0^1\int_0^1 \frac{\sqrt{xyz}\left(x\sqrt{y}+y\sqrt{z}+z\sqrt{x}+x\sqrt{z}+z\sqrt{y}+y\sqrt{x}\right)}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}dxdydz \\ &= \frac{1}{6}\int_0^1\int_0^1\int_0^1 \sqrt{xyz}dxdydz = \frac{1}{6}\left(\int_0^1 \sqrt{x}dx\right)^3 = \frac{4}{81}. \end{aligned}$$

Solution 6 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$
\begin{aligned}
6I &= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{x^3y^2z}+\sqrt{x^3y^2z}+\sqrt{x^3y^2z}+\sqrt{x^3y^2z}+\sqrt{x^3y^2z}+\sqrt{x^3y^2z}}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}dxdydz\\
&= \int_0^1\int_0^1\int_0^1 \frac{\sqrt{xyz}\left[x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})\right]}{x(\sqrt{y}+\sqrt{z})+y(\sqrt{z}+\sqrt{x})+z(\sqrt{x}+\sqrt{y})}dxdydz\\
&= \int_0^1\int_0^1\int_0^1 \sqrt{x}\sqrt{y}\sqrt{z}dxdydz\\
&= \left(\int_0^1 \sqrt{x}dx\right)^3\\
&= \left(\frac{2}{3}x^{\frac{3}{2}}\right)^3\Bigg|_0^1\\
&= \left(\frac{2}{3}\right)^3 = \frac{8}{81}\\
\therefore\quad I &= \frac{4}{243}.
\end{aligned}
$$

Solution 7 by Albert Stadler, Herrliberg, Switzerland.

The expression $x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)$ is symmetric with respect to the variables x, y, and z. Hence

$$
I=\frac{1}{6}\int_0^1\int_0^1\int_0^1 \frac{\sum_{symm}\sqrt{x^3y^2z}}{x\left(\sqrt{y}+\sqrt{z}\right)+y\left(\sqrt{z}+\sqrt{x}\right)+z\left(\sqrt{x}+\sqrt{y}\right)}dxdydz=\frac{1}{6}\int_0^1\int_0^1\int_0^1\sqrt{xyz}dxdydz=\frac{4}{81}.
$$

Also solved by the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ **Thank You!** ♣ ♣ ♣