Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please present all proposed solutions and proposed problems according to formatting requirements delineated near the end of this document. Also, please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

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Solutions to the problems published in this issue should be submitted *before* March 1, 2024.

• **5751** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu," Drobeta Turnu - Severin, Romania.*

Show that if $0 < a \leqslant b < \dfrac{\pi}{2}$, then

$$6\log\left|\frac{\cos(b)}{\cos(a)}\right| + 3(b^2-a^2) + 2\int_a^b x^2\tan(x)\,dx \leqslant 0.$$

• **5752** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Show that if $x, y, z > 0$, then

$$\sum_{cyc}\frac{(kx^3+y^3)z}{x^3y^3(1+nz)} \geqslant \frac{3(k+1)}{8}\left(\frac{15}{x^2+y^2+z^2} - n^2\right).$$

• **5753** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1, \ldots, x_n > 0$ and set $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1+x_i)^{x_i}.$$

When does equality occur?

• **5754** *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate $S = \sum_{n=1}^{\infty}(2n-1)\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(n+k)^2}\right)^2.$

• **5755** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy .*

Formally assuming that $(\sin 0)/0 = 1$, prove

$$\forall x \in \left[0, \pi/2\right]: \quad \frac{\sin x}{x} + \frac{(\sin x)^4}{x^4} \geqslant 2\cos x.$$

• **5756** *Proposed by Toyesh Prakash Sharma (Undergraduate Student) Agra College, India.*

Calculate $T = \int_0^{\infty} \frac{dx}{x^2\left(\tan^2 x + \cot^2 x\right)}.$

Solutions

To Formerly Published Problems

• **5727** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

If $f : (0, \infty) \to (0, \infty)$ is a continuous function and $\int_a^b f(x)dx = 5(b-a)$ where $0 < a \leqslant b$, then

$$\int_a^b \left(\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11}\right) dx \leqslant 9(b-a).$$

Solution 1 by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.

Note that

$$\frac{5f(x)+3}{f(x)+7} + \frac{6f(x)+4}{f(x)+9} + \frac{7f(x)+5}{f(x)+11} = 18 - 2\left(\frac{4^2}{f(x)+7} + \frac{5^2}{f(x)+9} + \frac{6^2}{f(x)+11}\right) \quad ....(1)$$

By Titu's lemma,

$$2\left(\frac{4^2}{f(x)+7} + \frac{5^2}{f(x)+9} + \frac{6^2}{f(x)+11}\right) \geqslant 2\left(\frac{15^2}{3f(x)+27}\right) \geqslant \frac{144}{f(x)+9} \quad ....(2)$$

By AM-GM inequality,

$$\frac{144}{f(x)+9}+f(x)+9\geqslant 2\sqrt{\left(\frac{144}{f(x)+9}\right)(f(x)+9)}=24 \quad ....(3)$$

Combining (1), (2), and (3) gives us

$$\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\leqslant 18-(15-f(x))=f(x)+3.$$

Then,

$$\int_a^b\left(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\right)dx\leqslant\int_a^b f(x)\,dx+\int_a^b 3\,dx=8(b-a)\leqslant 9(b-a),$$

proven. Equality holds if and only if $a=b$.

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The function

$$g(x):=\frac{5x+3}{x+7}+\frac{6x+4}{x+9}+\frac{7x+5}{x+11}=18-\frac{32}{x+7}-\frac{50}{x+9}-\frac{72}{x+11}$$

is concave on $(0,\infty)$ since $g''(x)<0$. Therefore, it can be estimated from above by its tangent in the point $\big(5,g(5)\big)$, i.e., we have

$$g(x)\leqslant g(5)+g'(5)(x-5).$$

We infer that

$$\int_a^b\left(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\right)dx=\int_a^b g\big(f(x)\big)\,dx$$
$$\leqslant \int_a^b\Big(g(5)+g'(5)\big(f(x)-5\big)\Big)\,dx=g(5)(b-a),$$

since by assumption $\int_a^b\big(f(x)-5\big)\,dx=0$. Now the inequality follows since $g(5)=305/42\approx 7.2619<9$.

Remark: The inequality shown above is sharp. Equality occurs if $f(x)=5$ on $(0,\infty)$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We will prove the stronger inequality

$$\int_a^b\left(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\right)dx\leqslant\frac{305}{42}(b-a).$$

Clearly,

$$\int_a^b \left(\frac{5f(x)+3}{f(x)+7}+\frac{6f(x)+4}{f(x)+9}+\frac{7f(x)+5}{f(x)+11}\right)dx =$$
$$= \int_a^b \left(5-\frac{32}{f(x)+7}+6-\frac{50}{f(x)+9}+7-\frac{72}{f(x)+11}\right)dx =$$
$$= 18(b-a)-\int_a^b \left(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\right)dx.$$

We need to prove that

$$\frac{451}{42}(b-a) \leqslant \int_a^b \left(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\right)dx. \qquad (*)$$

Let r>0. By the Cauchy-Schwarz inequality for integrals,

$$(b-a)^2 = \left(\int_a^b dx\right)^2 \leqslant \int_a^b (f(x)+r)\,dx \int_a^b \left(\frac{1}{f(x)+r}\right)dx = (5+r)(b-a)\int_a^b \left(\frac{1}{f(x)+r}\right)dx$$

which implies

$$\int_a^b \left(\frac{1}{f(x)+r}\right)dx \geqslant \frac{b-a}{5+r}.$$

We conclude that

$$\int_a^b \left(\frac{32}{f(x)+7}+\frac{50}{f(x)+9}+\frac{72}{f(x)+11}\right)dx \geqslant (b-a)\left(\frac{32}{5+7}+\frac{50}{5+9}+\frac{72}{5+11}\right) = \frac{451}{42}(b-a)$$

which is (*).

Solution 4 by Michel Bataille, Rouen, France.

Let $g(x) = \frac{5x+3}{x+7}$, $h(x) = \frac{6x+4}{x+9}$, $k(x) = \frac{7x+5}{x+11}$. It is easily checked that g, h, k are non-decreasing and concave on $(0,\infty)$.

We want to prove that $\int_a^b \phi(x)\,dx \leqslant 9(b-a)$ where $\phi(x) = g(f(x)) + h(f(x)) + k(f(x))$. .

Let m and M be the minimum and the maximum of the continuous function f on the interval $[a,b]$. Then, $0 < m \leqslant M$ and since $m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$, the hypothesis gives $m \leqslant 5 \leqslant M$.

From the concavity of g on the interval $[m, M]$, the curve $y = g(x)$ is under its tangent at $(5, g(5))$. The equation of this tangent is $y - \frac{7}{3} = \frac{2}{9}(x-5)$ (note that $g'(x) = \frac{32}{(x+7)^2}$), that is, $y = \frac{2x}{9} + \frac{11}{9}$ and therefore $g(f(x)) \leqslant \frac{2f(x)}{9} + \frac{11}{9}$ for $x \in [a,b]$.

Similar calculations lead to $h(f(x)) \leqslant \frac{25f(x)}{98} + \frac{113}{98}$ and $k(f(x)) \leqslant \frac{9f(x)}{32} + \frac{35}{32}$ and we deduce that for $x \in [a,b]$,

$$\phi(x) \leqslant \left(\frac{2}{9}+\frac{25}{98}+\frac{9}{32}\right)\cdot f(x) + \frac{11}{9}+\frac{113}{98}+\frac{35}{32} = \frac{10705}{14112}\cdot f(x) + \frac{48955}{14112}.$$

Integrating yields

$$\int_a^b \phi(x)\,dx \leqslant \frac{10705}{14112}\int_a^b f(x)\,dx + \frac{48955}{14112}(b-a),$$

that is,

$$\int_a^b \phi(x)\,dx \leqslant \left(\frac{53525}{14112}+\frac{48955}{14112}\right)(b-a) = \frac{6405}{882}(b-a).$$

Since $\dfrac{6405}{882} < 9$, we obtain a sharper result than the required one.

Also solved by the problem proposer.

• **5728** *Proposed by Florică Anastase, "Alexandru Odobescu" high school, Lehliu-Gară, Călăraşi, Romania.*

Define the sequences $(a_n)_{n\geqslant 1}$, $(b_n)_{n\geqslant 1}$ as follows:

$$a_n = \int_1^n \left\lfloor \frac{n^2}{x} \right\rfloor dx \text{ and } \quad b_1 > 1,\, b_{n+1} = 1 + \log(b_n)$$

where $\lfloor \cdot \rfloor$ denotes greatest integer (i.e., floor) function. Find the limit

$$\Omega = \lim_{n\to\infty} \frac{a_n \cdot \log\sqrt[n]{b_n}}{\log n}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\int_1^n \left\lfloor \frac{n^2}{x} \right\rfloor dx = \sum_{m=n}^{n^2-1} \int_{\frac{n^2}{m+1}}^{\frac{n^2}{m}} \left\lfloor \frac{n^2}{x} \right\rfloor dx = \sum_{m=n}^{n^2-1} m\left(\frac{n^2}{m} - \frac{n^2}{m+1}\right) = n^2 \sum_{m=n}^{n^2-1} \frac{1}{m+1} = n^2\left(H_{n^2} - H_n\right),$$

where H_n denotes the n^{th} harmonic number. It is well-known that the asymptotics of the harmonic numbers is

$$H_n = \log n + \gamma + O\left(\frac{1}{n}\right), \quad n \to \infty.$$

Thus

$$a_n = \int_1^n \left\lfloor \frac{n^2}{x} \right\rfloor dx = n^2 \log n + O(n), \quad n \to \infty.$$

Let $c_n = \log b_n$. Then $c_1>0$ and $c_{n+1} = \log(1+c_n) \leqslant c_n$. So $(c_n)_{n1}$ is a monotonically decreasing sequence of positive numbers which tends to a limit $c\geqslant 0$. That limit equals 0, for $c = \lim\limits_{n\to\infty} c_{n+1} =$

$\lim\limits_{n\to\infty} \log(1+c_n) = \log(1+c)$ implies c=0. We have

$$\frac{1}{2} - \frac{x}{12} \leqslant \frac{x - \log(1+x)}{x\log(1+x)} \leqslant \frac{1}{2}, \quad x > 0.$$

To prove these two inequalities we replace x by e^y-1 and use Taylor's expansion of the exponential function. We then see that

$$x - \log(1+x) - x\log(1+x)\left(\frac{1}{2} - \frac{x}{12}\right) = e^y - 1 - y - \frac{7}{12}y(e^y-1) + \frac{1}{12}y\left(e^{2y} - e^y\right) =$$

$$= -1 + e^y - \frac{5y}{12} - \frac{2e^y y}{3} + \frac{1}{12}e^{2y}y = \sum_{k=4}^{\infty} \frac{y^k}{k!}\left(1 + \frac{\left(2^{k-1} - 8\right)k}{12}\right) \geqslant 0$$

and

$$x\log(1+x) - 2\left(x - \log(1+x)\right) = y(e^y-1) - 2(e^y - 1 - y) = 2 + y - 2e^y + ye^y =$$

$$= \sum_{k=3}^{\infty} \frac{y^k}{k!}(-2+k) \geqslant 0.$$

Hence

$$\frac{1}{2} - \frac{c_k}{12} \leqslant \frac{c_k - \log(1+c_k)}{c_k\log(1+c_k)} = \frac{c_k - c_{k+1}}{c_k c_{k+1}} = \frac{1}{c_{k+1}} - \frac{1}{c_k} \leqslant \frac{1}{2}.$$

We sum over k from k=1 to k=n-1 and get

$$\frac{1}{2}(n-1) - \frac{1}{12}\sum_{k=1}^{n-1} c_k \leqslant \frac{1}{c_n} - \frac{1}{c_1} \leqslant \frac{1}{2}(n-1)$$

which is equivalent to

$$\frac{1}{\frac{n}{2} - \frac{1}{2} + \frac{1}{c_1}} \leqslant c_n \leqslant \frac{1}{\frac{n}{2} - \frac{1}{2} + \frac{1}{c_1} - \frac{1}{12}\sum_{k=1}^{n-1} c_k}.$$

However $\sum\limits_{k=1}^{n-1} c_k = o(n)$, since c_n tends to zero. Thus $c_n = \dfrac{2}{n}\left(1 + o(1)\right)$ and we conclude that

$$\lim_{n\to\infty} \frac{a_n \cdot \log\left(\sqrt[n]{b_n}\right)}{\log n} = \lim_{n\to\infty} \frac{a_n c_n}{n\log n} = \lim_{n\to\infty} \frac{\left(n^2\log n + O(n)\right)\frac{2}{n}\left(1 + o(1)\right)}{n\log n} = 2.$$

Solution 2 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

For any real number $b_1 > 1$, $\Omega = 2$.

For each integer k with $n \leqslant k < n^2$, the function $\left\lfloor \frac{n^2}{x} \right\rfloor$ has constant value k on the interval $\left(\frac{n^2}{k+1}, \frac{n^2}{k} \right)$, so that

$$\int_{\frac{n^2}{k+1}}^{\frac{n^2}{k}} \left\lfloor \frac{n^2}{x} \right\rfloor \, dx = \left(\frac{n^2}{k} - \frac{n^2}{k+1} \right) k = \frac{n^2}{k+1},$$

and

$$a_n = \sum_{k=n}^{n^2-1} \frac{n^2}{k+1} = n^2 \sum_{j=n+1}^{n^2} \frac{1}{j}.$$

Since

$$\int_{n+1}^{n^2+1} \frac{1}{x} \, dx \leqslant \sum_{j=n+1}^{n^2} \frac{1}{j} \leqslant \int_{n}^{n^2} \frac{1}{x} \, dx,$$

then

$$\log \left(\frac{n^2+1}{n+1} \right) \leqslant \sum_{j=n+1}^{n^2} \frac{1}{j} \leqslant \log n$$

and

$$n^2 \log \left(\frac{n^2+1}{n+1} \right) \leqslant a_n \leqslant n^2 \log n.$$

Notice that

$$\frac{a_n \cdot \log \sqrt[n]{b_n}}{\log n} = \frac{a_n \cdot \log b_n}{n \cdot \log n} = \frac{a_n \left(b_{n+1} - 1 \right)}{n \log n},$$

so that

$$\frac{\log \left(\frac{n^2+1}{n+1} \right)}{\log n} \cdot n \left(b_{n+1} - 1 \right) \leqslant \frac{a_n \cdot \log \sqrt[n]{b_n}}{\log n} \leqslant n \left(b_{n+1} - 1 \right).$$

L'hospital's Rule shows that

$$\lim_{n \to \infty} \frac{\log \left(\frac{n^2+1}{n+1} \right)}{\log n} = 1,$$

so it only remains to show that

$$\lim_{n \to \infty} n \left(b_{n+1} - 1 \right) = 2.$$

Let $c_n = b_n - 1$. Then $c_1 > 0$ and $c_{n+1} = \log \left(1 + c_n \right)$ for all positive integers n. We prove that if A and B are any two real numbers with $0 < A < 2 < B$, then

$$A \leqslant \liminf nc_n \leqslant \limsup nc_n \leqslant B.$$

Lemma 1 *If A and B are any two real numbers with $0 < A < 2 < B$, then there are positive integers N_A and $M_B > B$ such that*

$$\frac{1}{n} - \frac{B}{2n^2} + \frac{B^2}{2n^3} < \frac{1}{n+1} < \frac{1}{n} - \frac{A}{2n^2}$$

for all integers $n > \max\{N_A, M_B\}$.

Proof: For positive integers n,

$$\frac{1}{n} - \frac{B}{2n^2} + \frac{B^2}{2n^3} < \frac{1}{n+1}$$

if and only if

$$(3B-6)n^2 + (3B-2B^2)n - 2B^2 > 0.$$

Since $3B-6 > 0$, there exists a positive integer $M_B > B$ such that $(3B-6)n^2+(3B-2B^2)n-2B^2 > 0$ for all $n > M_B$. Similarly, for positive integers n,

$$\frac{1}{n} - \frac{A}{2n^2} > \frac{1}{n+1}$$

if and only if $-A + (2-A)n > 0$. Since $2 - A > 0$, there exists an integer N_A such that $-A + (2-A)n > 0$ for all integers $n > N_A$.

Lemma 2 *If $x_1 > 0$ and $x_{n+1} = \log(1 + x_n)$ for each positive integer n, then (x_n) is decreasing and converges to 0.*

Proof: The sequence is decreasing since $x_{n+1} = \log(1+x_n) < x_n$ and $x_n > 0$ for all positive integers n, so by the Monotone Convergence Theorem, the sequence converges to some real number L. Since $L = \log(1 + L)$, then $L = 0$.

Lemma 3 *Let $(x_n)_{n\geqslant 1}$ be a sequence of positive real numbers such that $\liminf nx_n = S$ and $\limsup nx_n = T$. Then for any positive integer N,*

$$\liminf nx_{n-N} = S \text{ and } \limsup nx_{n-N} = T.$$

Proof: The result follows from the fact that

$$nx_{n-N} = \frac{n}{n-N}(n-N)x_{n-N}.$$

Lemma 4 *Let A and B be two real numbers such that $0 < A < 2 < B$, and let $c_n = b_n - 1$. Then*

$$A \leqslant \liminf nc_n \leqslant \limsup nc_n \leqslant B.$$

Proof: Let N_A and M_B be as described in Lemma 1. Since $c_n \to 0$, there are positive integers N and J such that $c_N < B/M_B$, $c_N \geqslant A/J$ and $J > N_A$. Let $x_n = c_{n-M_B+N}$ for $n \geqslant M_B$ and $y_n = c_{n-J+N}$ for $n \geqslant J$, so that $x_{M_B} = c_N \leqslant B/M_B$ and $y_J = c_N > A/J$. Notice that the sequences (x_n) and (y_n) also satisfy the recurrence relation in Lemma 2.

We use induction to prove the inequalities $x_n < B/n$ for all $n \geqslant M_B$ and $y_n > A/n$ for all $n \geqslant J$. For the first inequality, the basis case is true because $x_{M_B} = c_N$. Suppose that $x_n < B/n$ for some $n \geqslant M_B$. By the cubic approximation of $\log(1+x)$, which is increasing for $x > 0$, and Lemma 1, we have

$$x_{n+1} = \log\left(1 + x_n\right) < x_n - \frac{x_n^2}{2} + \frac{x_n^3}{3} < \frac{B}{n} - \frac{B^2}{2n^2} + \frac{B^3}{2n^3} < \frac{B}{n+1}.$$

For the second inequality, the basis case is true since $y_J = c_N$. Suppose that $y_n > A/n$ for some $n \geqslant J$. By the quadratic approximation of $\log(1+x)$ for $0 < x < 1$, which is increasing, and Lemma 1, we have

$$y_{n+1} = \log\left(1 + y_n\right) > y_n - \frac{y_n^2}{2} > \frac{A}{n} - \frac{A^2}{2n^2} > \frac{A}{n+1}.$$

Thus, if $n \geqslant \max\{M_B, J\}$, then $nx_n \leqslant B$ and $ny_n \geqslant A$, and hence,

$$\limsup nx_n \leqslant B \text{ and } \liminf ny_n \geqslant A.$$

By Lemma 3,

$$\limsup nc_n = \limsup nx_{n+M_B-N} = \limsup nx_n \leqslant B$$

and

$$\liminf nc_n = \liminf ny_{n+J-N} = \liminf ny_n \geqslant A.$$

Since the values of $\liminf nc_n$ and $\limsup nc_n$ are independent of our choice of A and B, A and B may be chosen arbitrarily close to 2, so that Lemma 4 implies that

$$\lim_{n\to\infty} n\left(b_{n+1} - 1\right) = \lim_{n\to\infty} nc_n = 2.$$

Solution 3 by Michel Bataille, Rouen, France.

We claim that $\Omega = 2$.

If $x \in (1, n]$, then $\left\lfloor \dfrac{n^2}{x} \right\rfloor \in \{n, n+1, \ldots, n^2 - 1\}$ and for k in this set, we have

$$\left\lfloor \frac{n^2}{x} \right\rfloor = k \;\Leftrightarrow\; \frac{n^2}{k+1} < x \leqslant \frac{n^2}{k}.$$

It follows that

$$a_n = \sum_{k=n}^{n^2-1} \int_{n^2/(k+1)}^{n^2/k} k\,dx = \sum_{k=n}^{n^2-1} k\left(\frac{n^2}{k} - \frac{n^2}{k+1}\right) = n^2 \sum_{k=n}^{n^2-1} \frac{1}{k+1} = n^2(H_{n^2} - H_n)$$

where $H_m = \sum_{k=1}^{m} \frac{1}{k}$.
We know that $H_n = \log(n) + \gamma + o(1)$ as $n \to \infty$ (where γ is the Euler constant). Therefore,

$$a_n = n^2(\log(n^2) + \gamma - (\log(n) + \gamma) + o(1)) = n^2 \log(n) + o(n^2)$$

and we obtain $a_n \sim n^2 \log(n)$ as $n \to \infty$.
Now we consider the sequence (b_n). An easy induction shows that $b_n > 1$ for any positive integer n. In addition, $b_{n+1} - b_n = \log(b_n) - (b_n - 1) < 0$ (since $\log(x) < x - 1$ for $x > 0, x \neq 1$). We deduce that (b_n) is bounded below and non-increasing, hence convergent. Its limit ℓ satisfies $\ell = 1 + \log(\ell)$, hence $\ell = 1$.
To go further, we introduce the sequence (c_n) defined by $c_n = b_n - 1$. Then $c_{n+1} = \log(1 + c_n)$ for any positive integer n and $\lim_{n\to\infty} c_n = 0$. Using $\log(1 + x) = x - \frac{x^2}{2} + o(x^2)$ as $x \to 0$, we obtain

$$\frac{1}{c_{n+1}} - \frac{1}{c_n} = \frac{1}{2} \cdot \frac{c_n^2 + o(c_n^2)}{c_n c_{n+1}} = \frac{1}{2} \cdot \frac{c_n}{\log(1 + c_n)} \cdot (1 + o(1))$$

so that $\lim_{n\to\infty} \left(\frac{1}{c_{n+1}} - \frac{1}{c_n}\right) = \frac{1}{2}$.
From the Stolz-Cesaro theorem, we deduce that $\frac{1}{c_n} \sim \frac{n}{2}$ as $n \to \infty$. Thus, $c_n \sim \frac{2}{n}$, that is, $b_n = 1 + \frac{2}{n} + o(1/n)$, and $\log(b_n) \sim \frac{2}{n}$, which leads to $\log(\sqrt[n]{b_n}) = \frac{1}{n}\log(b_n) \sim \frac{2}{n^2}$ as $n \to \infty$.
We conclude

$$\frac{a_n \cdot \log(\sqrt[n]{b_n})}{\log n} \sim n^2 \log(n) \cdot \frac{2}{n^2} \cdot \frac{1}{\log(n)} = 2$$

as $n \to \infty$ and the claim follows.

Also solved by Paolo Perfetti, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy; and the problem proposer.

• **5729** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $0 < a < b$ be real numbers. Prove the following inequality

$$(a + b)^{a+b}(b - a)^{b-a} < \left(\frac{a^2 + b^2}{b}\right)^{2b}.$$

Solution 1 by Paolo Perfetti, dipartimento di matematica, Universitą di "Tor Vergata", Roma, Italy.

$$\frac{a + b}{2b} \ln(a + b) + \frac{b - a}{2b} \ln(b - a) < \ln\left(\frac{a^2 + b^2}{2}\right)$$

$$\frac{a+b}{2b} + \frac{b-a}{2b} = 1$$

and the concavity of the logarithm yield

$$\frac{a+b}{2b}\ln(a+b) + \frac{b-a}{2b}\ln(b-a) \leqslant \ln\left(\frac{a+b}{2b}(a+b) + \frac{b-a}{2b}(b-a)\right) = \ln\left(\frac{a^2+b^2}{2}\right).$$

Solution 2 by Ivan Hadinata, Department of mathematics, Gadjah Mada University, Yogyakarta, Indonesia.

Let $x = a+b$ and $y = b-a$ then $x, y \in \mathbb{R}^+$. By weighted AM-GM inequality,

$$\left(\frac{x^2+y^2}{x+y}\right)^{x+y} = \left(\frac{x\cdot x + y \cdot y}{x+y}\right)^{x+y} \geqslant x^x y^y \tag{1}$$

Equality of (1) holds if and only if $x = y$, then $a = 0$ which is impossible. Therefore,

$$\left(\frac{x^2+y^2}{x+y}\right)^{x+y} > x^x y^y \quad \Longrightarrow \quad \left(\frac{a^2+b^2}{b}\right)^{2b} > (a+b)^{a+b}(b-a)^{b-a}.$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

The desired inequality is an immediate consequence of the weighted AGM inequality for two variables:

$$x_1^{w_1/W} \cdot x_2^{w_2/W} \leqslant \frac{w_1}{W}\cdot x_1 + \frac{w_2}{W}\cdot x_2, \text{ where } W = w_1 + w_2.$$

In this problem, $x_1 = a+b$, $w_1 = a+b$; $x_2 = b-a$, $w_2 = b-a$; $W = 2b$. Equality holds in the AGM if and only if $x_1 = x_2$. Since $0 < a < b$, the inequality in this problem is strict.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

With $0 < a < b$, it follows that $a+b$ and $b-a$ are both positive numbers with $a+b \neq b-a$. Now, by the weighted arithmetic mean - geometric mean inequality,

$$(a+b)^{a+b}(b-a)^{b-a} < \left(\frac{(a+b)^2+(b-a)^2}{2b}\right)^{2b} = \left(\frac{a^2+b^2}{b}\right)^{2b}.$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland.

Put u:=a+b, v:=b-a. By assumption u>v>0. The stated inequality is equivalent to

$u^u v^v < \left(\frac{u^2+v^2}{u+v}\right)^{u+v}$ and to $\frac{u}{u+v}\ln u + \frac{v}{u+v}\ln v < \ln\left(\frac{u^2+v^2}{u+v}\right).$

The function x→ln(x) is concave for x>0. Hence, by Jensen's inequality,

$$\frac{u}{u+v}\ln u \ + \frac{v}{u+v}\ln v \ \leqslant \ln\left(\frac{u}{u+v}u + \frac{v}{u+v}v\right) \ = \ln\left(\frac{u^2+v^2}{u+v}\right).$$

Equality in Jensen's inequality holds if and only if ln(x) is linear or u=v. But ln(x) is nowhere piecewise linear (since the second derivative is nowhere 0) and u>v. So we have strict inequality.

Solution 6 by Michel Bataille, Rouen, France.

We prove the equivalent inequality

$$(a+b)^{\frac{a+b}{2b}}(b-a)^{\frac{b-a}{2b}} < \frac{a^2+b^2}{b}. \tag{1}$$

Since $\frac{a+b}{2b}, \frac{b-a}{2b}$ are positive and sum to 1, the weighted AM-GM gives

$$(a+b)^{\frac{a+b}{2b}}(b-a)^{\frac{b-a}{2b}} < \frac{a+b}{2b}\cdot(a+b) + \frac{b-a}{2b}\cdot(b-a)$$

(strict inequality because $a+b \neq b-a$).
The inequality (1) follows since $(a+b)^2 + (b-a)^2 = 2(a^2+b^2)$.

Solution 7 by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.

We modify the problem taking ln on both sides,

$$(a+b)\ln(a+b) + (b-a)\ln(b-a) < 2b\ln(\frac{a^2+b^2}{b}) \tag{1}$$

Now, we will focus on proving this statement.

Then, consider a function $f(x) = \ln(x)$. Then f''(x) = $-\frac{1}{x^2} < 0\ \forall x > 0$. So, the function is concave $\forall x \neq 0$. Also notice that both a+b and b-a are strictly positive here.
Now, using Jensen's inequality,

$$\frac{(a+b)\ln(a+b) + (b-a)\ln(b-a)}{a+b+b-a} \leqslant \ln(\frac{(a+b)(a+b) + (b-a)(b-a)}{a+b+b-a})$$

which on simplification gives

$$(a+b)\ln(a+b) + (b-a)\ln(b-a) \leqslant 2b\ln(\frac{a^2+b^2}{b})$$

And the equality holds when $a+b = b-a \Rightarrow$ a=0. But since $0 < a < b$, the equality does not hold. Hence, we have proved (1).

Also solved by the problem proposer.

• **5730** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $\forall a,b,c\in\mathbb{N}:\ f(a)^{f(b)^{f(c)}}+a^{b^{f(c)}}=f(a)^{b^{c}}+a^{f(b)^{c}}$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

To show impossibility (or contradiction), suppose that $f(1)\neq 1$. If $a=b=c=1$, then $f(1)^{f(1)^{f(1)}}=f(1)$. So $f(1)^{f(1)}=1$. However this is impossible given our supposition that $f(1)\neq 1$. So, we must conclude that $f(1)=1$ when $a=b=c=1$.

Consider the case $c=1, a=b$. This implies $f(a)^{f(a)}-a^{f(a)}=f(a)^{a}-a^{a}$.

Suppose $a>1$. To show impossibility (or contradiction), suppose that $f(a)>a$. We divide both sides of the latter equation by $f(a)-a$ and get

$$\sum_{k=1}^{f(a)}\left(f(a)\right)^{f(a)-k}a^{k-1}=\sum_{k=1}^{a}\left(f(a)\right)^{a-k}a^{k-1}.$$

However this is impossible since $\left(f(a)\right)^{f(a)-k}>\left(f(a)\right)^{a-k}$ for $1\leqslant k\leqslant a$, and the left-hand side has more (positive) terms than the right-hand side.

To show impossibility (or contradiction), suppose that $f(a)<a$. As before we have

$$\sum_{k=1}^{f(a)}a^{f(a)-k}\left(f(a)\right)^{k-1}=\sum_{k=1}^{a}a^{a-k}\left(f(a)\right)^{k-1}.$$

Again this is impossible since $a^{f(a)-k}<a^{a-k}$ for $1\leqslant k\leqslant f(a)$, and the left-hand side has fewer (positive) terms than the right-hand side.

So $f(x)=x$ for all x and this is the only function with the required property.

Also solved by ; and the problem proposer.

• **5731** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve for real x: $\sqrt{6-11x+6x^2-x^3}+\sqrt{12-19x+8x^2-x^3}=\sqrt{15-23x+9x^2-x^3}$.

Solution 1 by Michael C. Faleski, Delta College, University Center, MI .

We start by factoring the polynomials. The expression given can be rewritten as

$$\sqrt{(x-2)(x-3)(1-x)} + \sqrt{(x-4)(x-3)(1-x)} = \sqrt{(x-5)(x-3)(1-x)}$$

By bringing all of the tems to one side of the equation, we have

$$\left(\sqrt{x-3}\right)\left(\sqrt{1-x}\right)\left(\sqrt{x-2} + \sqrt{x-4} - \sqrt{x-5}\right) = 0$$

This leads to three possible solutions: $x = 3$, $x = 1$, and $\sqrt{x-2} + \sqrt{x-4} - \sqrt{x-5} = 0$.

The third condition cannot provide a real solution because for all $x \geqslant 5$ (which makes all of the terms real), $\sqrt{x-2} > \sqrt{x-5}$ and $\sqrt{x-4} > \sqrt{x-5}$ meaning that $\sqrt{x-2} + \sqrt{x-4} - \sqrt{x-5} > 0$.

The only real solutions are $x = 1, 3$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

We note that

$$6 - 11x + 6x^2 - x^3 = -(-3+x)(-2+x)(-1+x),$$
$$12 - 19x + 8x^2 - x^3 = -(-4+x)(-3+x)(-1+x),$$
$$15 - 23x + 9x^2 - x^3 = -(-5+x)(-3+x)(-1+x).$$

So x=1 and x=3 are solutions of the given equation.

If x≠1 and x≠3 we may divide both sides by $\sqrt{-(-3+x)(-1+x)}$ and get

$$\sqrt{x-2} + \sqrt{x-4} = \sqrt{x-5}. \qquad (*)$$

Squaring both sides gives

$$2\sqrt{x-2}\sqrt{x-4} = 1 - x.$$

Squaring both sides again gives

$$31 - 22x + 3x^2 = 0$$

which has the two roots $\left(11 \pm 2\sqrt{7}\right)/3$, with one root less than 2 and the other greater 5.

If we define $\sqrt{x} = i\sqrt{-x}$ if $x < 0$, then $x = \left(11 - 2\sqrt{7}\right)/3$ is a solution of (*), but $x = (11 + 2\sqrt{7})/3$ is not.

So the set of real solutions of the given equation is

$$\left\{1,\ 3, \left(11 - 2\sqrt{7}\right)/3\right\}.$$

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

$6-11x+6x^2-x^3=-(x-1)(x-2)(x-3)$, $12-19x+8x^2-x^3=-(x-1)(x-3)(x-4)$, and $15-23x+9x^2-x^3=-(x-1)(x-3)(x-5)$, it follows that $x=1$ and $x=3$ are real solutions of the proposed equation, and the given equation simplifies to $\sqrt{2-x}+\sqrt{3-x}=\sqrt{5-x}$, where $x\leqslant 2$. Solving this last equation, since $x\leqslant 2$ it follows that other real solution of the problem is $x=\left(11-2\sqrt{7}\right)/3$.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Because

$$\begin{aligned}6-11x+6x^2-x^3 &= (1-x)(6-5x+x^2)=(1-x)(2-x)(3-x),\\ 12-19x+8x^2-x^3 &= (1-x)(12-7x+x^2)=(1-x)(3-x)(4-x),\\ 15-23x+9x^2-x^3 &= (1-x)(15-8x+x^2)=(1-x)(3-x)(5-x),\end{aligned}$$

it follows that

$$\sqrt{6-11x+6x^2-x^3}+\sqrt{12-19x+8x^2-x^3}=\sqrt{15-23x+9x^2-x^3}$$

is equivalent to

$$\sqrt{(1-x)(3-x)}\left(\sqrt{2-x}+\sqrt{4-x}-\sqrt{5-x}\right)=0.$$

Thus, either

$$\sqrt{(1-x)(3-x)}=0 \quad \text{or} \quad \sqrt{2-x}+\sqrt{4-x}-\sqrt{5-x}=0.$$

The equation $\sqrt{(1-x)(3-x)}=0$ has roots $x=1$ and $x=3$. For the remaining equation, transpose the $\sqrt{5-x}$ to the right side, square both sides, and then combine like terms to obtain

$$2\sqrt{(2-x)(4-x)}=x-1.$$

Again, square both sides and combine like terms to obtain

$$3x^2-22x+31=0,$$

an equation whose roots are

$$x=\frac{11}{3}\pm\frac{2}{3}\sqrt{7}.$$

With $x=\frac{11}{3}+\frac{2}{3}\sqrt{7}$,

$$2-x<4-x<5-x<0,$$

so

$$\operatorname{Im}\left(\sqrt{2-x}+\sqrt{4-x}\right)>\operatorname{Im}\sqrt{5-x},$$

and $x = \dfrac{11}{3} + \dfrac{2}{3}\sqrt{7}$ is thus an extraneous root. Therefore, the real roots of

$$\sqrt{6 - 11x + 6x^2 - x^3} + \sqrt{12 - 19x + 8x^2 - x^3} = \sqrt{15 - 23x + 9x^2 - x^3}$$

are

$$x = 1, \quad x = 3, \quad \text{and} \quad x = \frac{11}{3} - \frac{2}{3}\sqrt{7}.$$

If the intent of this problem is to determine the real roots of the given equation using only real arithmetic, then $x = \dfrac{11}{3} - \dfrac{2}{3}\sqrt{7}$ must be excluded as $6 - 11x + 6x^2 - x^3 < 0$ for $x = \dfrac{11}{3} - \dfrac{2}{3}\sqrt{7}$. In fact, all three radicands are negative for $x = \dfrac{11}{3} - \dfrac{2}{3}\sqrt{7}$.

Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.

The three cubic polynomials $-x^3 + 6x^2 - 11x + 6$, $-x^3 + 8x^2 - 19x + 12$, and $-x^3 + 9x^2 - 23x + 15$ have leading coefficient -1 and constant terms 6, 12, and 15, respectively. So the sets of possible rational roots for the three poynomials are, respectively, $\{\pm 1, \pm 2, \pm 3, \pm 6\}$, $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$, and $\{\pm 1, \pm 3, \pm 5, \pm 15\}$.

Synthetic division yields $x = 1$ as a root of each of the three polynomials, so that $-x^3 + 6x^2 - 11x + 6 = -(x-1)(x^2 - 5x + 6) = -(x-1)(x-2)(x-3)$, $-x^3 + 8x^2 - 19x + 12 = -(x-1)(x^2 - 7x + 12) = -(x-1)(x-3)(x-4)$, and $-x^3 + 9x^2 - 23x + 15 = -(x-1)(x^2 - 8x + 15) = -(x-1)(x-3)(x-5)$.

So the original equation is

$$\sqrt{-(x-1)(x-2)(x-3)} + \sqrt{-(x-1)(x-3)(x-4)} = \sqrt{-(x-1)(x-3)(x-5)}. \qquad (2)$$

It is clear that $x = 1$ and $x = 3$ are solutions to the equation. Squaring both sides of the equation, isolating the remaining radicals on one side, and then squaring both sides again yields

$$\begin{aligned}
-(x-1)(x-2)(x-3) + 2\sqrt{-(x-1)(x-2)(x-3)}\sqrt{-(x-1)(x-3)(x-4)}& \\
-(x-1)(x-3)(x-4)& \\
&= -(x-1)(x-3)(x-5) \\
2\sqrt{-(x-1)(x-2)(x-3)}\sqrt{-(x-1)(x-3)(x-4)}& \\
&= (x-1)(x-3)[(x-2)+(x-4)-(x-5)] \\
2\sqrt{-(x-1)(x-2)(x-3)}\sqrt{-(x-1)(x-3)(x-4)} &= (x-1)^2(x-3) \\
4(x-1)^2(x-3)^2(x-2)(x-4) &= (x-1)^4(x-3)^2 \\
(x-1)^2(x-3)^2[4(x-2)(x-4) - (x-1)^2] &= 0.
\end{aligned}$$

The solutions $x = 1$ and $x = 3$ are again obvious. Expanding $4(x-2)(x-4)-(x-1)^2$ and collecting like terms, we have the quadratic equation $3x^2 - 22x - 31 = 0$, whose solutions are $x = \dfrac{11 \pm 2\sqrt{7}}{3}$.

Since $x = \dfrac{11-2\sqrt{7}}{3} \approx 1.9$ and $x = \dfrac{11+2\sqrt{7}}{3} \approx 5.4$, it is clear from (2) that for each of these values of x, all three cubic polynomials are negative, and therefore all three radical expressions are imaginary. If we allow this, then checking $x = \dfrac{11-2\sqrt{7}}{3}$ and $x = \dfrac{11+2\sqrt{7}}{3}$ in the original equation, we find that the former is a solution, whereas the latter is not.

So $x = 1$ and $x = 3$ are solutions to the original equation. If we are restricting solutions—but not operations—to the real numbers, then $x = \dfrac{11-2\sqrt{7}}{3}$ is also a solution.

Solution 6 by Perfetti Paolo, dipartimento di matematica Universitą di "Tor Vergata", Roma, Italy.

$$\begin{aligned} 6-11x+6x^2-x^3 &= (1-x)(x-2)(x-3), \\ 12-19x+8x^2-x^3 &= (1-x)(x-3)(x-4) \\ 15-23x+9x^2-x^3 &= (1-x)(x-3)(x-5) \end{aligned}$$

It follows that the equation is defined for $x \leqslant 1$, $x = 3$ and $x = 1$, $x = 3$ are solutions. Let's square

$$\left(\sqrt{6-11x+6x^2-x^3} + \sqrt{12-19x+8x^2-x^3}\right)^2 = 15-23x+9x^2-x^3$$

$$2\sqrt{6-11x+6x^2-x^3}\sqrt{12-19x+8x^2-x^3} = -3+7x-5x^2+x^3 = (x-3)(x-1)^2 \leqslant 0$$

It follows that there are no other solution apart of those found thus the only solutions are $x = 1$, $x = 3$.

Solution 7 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The solutions are the zeros of the function

$$f(x) := \sqrt{6-11x+6x^2-x^3} + \sqrt{12-19x+8x^2-x^3} - \sqrt{15-23x+9x^2-x^3}.$$

Observe that

$$\begin{aligned} 6-11x+6x^2-x^3 &= (1-x)(2-x)(3-x), \\ 12-19x+8x^2-x^3 &= (1-x)(3-x)(4-x), \\ 15-23x+9x^2-x^3 &= (1-x)(3-x)(5-x). \end{aligned}$$

We distinguish two cases.

Case 1: Only square roots of nonnegative real numbers are accepted.

We have

$$f\left(x\right)=\begin{cases}\sqrt{\left(1-x\right)\left(3-x\right)}\left(\sqrt{2-x}+\sqrt{4-x}-\sqrt{5-x}\right), & |x-2|\geqslant 1,\\ \sqrt{-\left(1-x\right)\left(3-x\right)}\left(\sqrt{x-2}+\sqrt{x-4}-\sqrt{x-5}\right), & |x-2|\leqslant 1.\end{cases}.$$

Since $\sqrt{x-5}$ is defined only for $x\geqslant 5$ there is no real solution of $f\left(x\right)=0$ if $|x-2|<1$. Since $\sqrt{2-x}$ is defined only for $x\leqslant 2$ there is no real solution of $f\left(x\right)=0$ if $x>3$. In the case $x<1$, we obviously have $\sqrt{2-x}<\sqrt{4-x}+\sqrt{5-x}$, which implies that $f\left(x\right)<0$. Summarizing, the only real solutions are $x=1$ and $x=3$.

Case 2: Complex roots are accepted.

Solutions different from $x=1$ and $x=3$ satisfy $\sqrt{x-2}+\sqrt{x-4}=\sqrt{x-5}$. We consider different intervals.

- If $x<1$ or $1<x<2$, the equation is equivalent to $\sqrt{2-x}+\sqrt{4-x}=\sqrt{5-x}$. Taking squares on both sides yields $2\sqrt{\left(2-x\right)\left(4-x\right)}=x-1$. Taking again squares leads to the equation $3x^2-22x+31=0$, whose unique solution in $(1,2)$ is $x=\left(11-2\sqrt{7}\right)/3$.

- If $2<x<3$ or $3<x<4$, the equation is equivalent to $\sqrt{x-2}+i\sqrt{4-x}=i\sqrt{5-x}$, which has no real solution.

- If $4<x<5$, the equation is equivalent to $\sqrt{x-2}+\sqrt{x-4}=i\sqrt{5-x}$, which has no real solution.

- If $x>5$, the equation is equivalent to $\sqrt{x-2}+\sqrt{x-4}=\sqrt{x-5}$, Taking squares on both sides yields $2\sqrt{\left(x-2\right)\left(x-4\right)}=1-x$. Taking again squares leads to the equation $3x^2-22x+31=0$, whose unique solution in $(5,+\infty)$ is $x=\left(11+2\sqrt{7}\right)/3$. Interpreting $\sqrt{z}\geqslant 0$ and $\sqrt{-z}=i\sqrt{z}$, for $z\geqslant 0$, this solution does not solve the original equation.

Summarizing, the (real) solution set to the equation is $\left\{1,3,\left(11-2\sqrt{7}\right)/3\right\}$.

Solution 8 by Brian D. Beasley, Simpsonville, SC.

We let $f(x)=6-11x+6x^2-x^3$, $g(x)=12-19x+8x^2-x^3$, and $h(x)=15-23x+9x^2-x^3$. This yields:

$$f(x)=-(x-1)(x-2)(x-3)$$
$$g(x)=-(x-1)(x-3)(x-4)$$
$$h(x)=-(x-1)(x-3)(x-5)$$

Then the domain of $\sqrt{f(x)}$ is $(-\infty, 1] \cup [2, 3]$, of $\sqrt{g(x)}$ is $(-\infty, 1] \cup [3, 4]$, and of $\sqrt{h(x)}$ is $(-\infty, 1] \cup [3, 5]$. Hence any real solution of the given equation must be in $(-\infty, 1] \cup \{3\}$.

It is straightforward to verify that both $x = 1$ and $x = 3$ satisfy the equation. If $x < 1$, then $(x - 1)(x - 3) > 0$, so the equation becomes

$$\sqrt{2 - x} + \sqrt{4 - x} = \sqrt{5 - x}.$$

But the only real solution of the latter equation is $x = (11 - 2\sqrt{7})/3 \approx 1.903$. Hence the solution set of the original equation over the real numbers is $\{1, 3\}$.

Solution 9 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX.

If we use the Rational Root Theorem and some trial and error, we obtain that

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3),$$
$$x^3 - 8x^2 + 19x - 12 = (x-1)(x-3)(x-4),$$
and
$$x^3 - 9x^2 + 23x - 15 = (x-1)(x-3)(x-5).$$

Hence, the equation to solve may be considered in the form

$$\sqrt{(1-x)(x-2)(x-3)} + \sqrt{(1-x)(x-3)(x-4)} = \sqrt{(1-x)(x-3)(x-5)}. \quad (1)$$

Equation (1) implies that $x = 1$, $x = 3$, or

$$\sqrt{x-2} + \sqrt{x-4} = \sqrt{x-5}. \quad (2)$$

By squaring both sides of (2), we get

$$(x-2) + (x-4) + 2\sqrt{x-2}\sqrt{x-4} = x - 5$$

which reduces to

$$2\sqrt{x-2}\sqrt{x-4} = 1 - x. \quad (3)$$

If we now square both sides of (3), the situation reduces again to

$$4(x-2)(x-4) = (1-x)^2 = x^2 - 2x + 1$$

or

$$3x^2 - 22x + 31 = 0. \quad (4)$$

Finally, if we apply the quadratic formula to (4), we obtain

$$x = \frac{11 \pm 2\sqrt{7}}{3}. \quad (5)$$

It remains for us to test the expressions given in (5) to see if they satisfy equation (2).

First of all, we note that if

$$x = \frac{11 + 2\sqrt{7}}{3} \approx 5.4305009,$$

then $2\sqrt{x-2}\sqrt{x-4} > 0$ while $1 - x < 0$. This violates equation (3) and thus this choice will not satisfy equation (2) This makes $x = \dfrac{11 + 2\sqrt{7}}{3}$ extraneous as a possible solution of equation (1).

On the other hand,

$$x = \frac{11 - 2\sqrt{7}}{3} \approx 1.9028325 \tag{6}$$

and this leads to $\sqrt{x-2} \approx .3117171i$, $\sqrt{x-4} \approx 1.4481601i$, and $\sqrt{x-5} \approx 1.759871i$. We conclude here that

$$\sqrt{x-2} + \sqrt{x-4} \approx \sqrt{x-5}$$

and hence, (6) is a plausible solution to equation (2).

As a result, we see that $x = 1$ and $x = 3$ are solutions of equation (1) and $x = \dfrac{11 - 2\sqrt{7}}{3}$ is at least a plausible approximate solution of equation (1).□ equation over the real numbers is $\{1, 3\}$.

Solution 10 by the Eagle Problem Solvers, Georgia Southern University, Savannah, GA and Statesboro, GA.

There are three real solutions: $x = 1$, $x = 3$, and $x = \dfrac{11 - 2\sqrt{7}}{3}$.

Notice that the three radicands may be factored as $(x-1)(x-3)(2-x)$, $(x-1)(x-3)(4-x)$, and $(x-1)(x-3)(5-x)$, respectively, so that the original equation can be rewritten as

$$\sqrt{(x-1)(x-3)}\left(\sqrt{2-x} + \sqrt{4-x} - \sqrt{5-x}\right) = 0.$$

Thus, $x = 1$ and $x = 3$ are solutions. In addition,

$$\begin{aligned}
\sqrt{2-x} + \sqrt{4-x} &= \sqrt{5-x} \\
2 - x + 2\sqrt{(2-x)(4-x)} + 4 - x &= 5 - x \\
2\sqrt{x^2 - 6x + 8} &= x - 1 \\
4(x^2 - 6x + 8) &= x^2 - 2x + 1 \\
3x^2 - 22x + 31 &= 0
\end{aligned}$$

gives solutions $x = \dfrac{11 \pm 2\sqrt{7}}{3}$, but of these, only $x = \dfrac{11 - 2\sqrt{7}}{3}$ results in positive radicands. Thus the three real solutions are $x = 1$, $x = 3$, and $x = \dfrac{11 - 2\sqrt{7}}{3}$.

Solution 11 by Michel Bataille, Rouen, France.

The solutions of the proposed equation are 1 and 3.
Since the equation rewrites as

$$\sqrt{(1-x)(2-x)(3-x)} + \sqrt{(1-x)(3-x)(4-x)} = \sqrt{(1-x)(3-x)(5-x)} \qquad (1)$$

1 and 3 are obvious solutions. We show that there are no other solutions.
For the purpose of a contradiction, assume that x is a solution and that $x \notin \{1,3\}$. Since $(1-x)(2-x)(3-x)$, $(1-x)(3-x)(4-x)$ and $(1-x)(3-x)(5-x)$ are nonnegative, x must satisfy $x < 1$. Squaring (1) then readily leads to

$$(1-x)(3-x)(1-x+2\sqrt{(2-x)(4-x)}) = 0,$$

a contradiction since $x \neq 1,3$ and $1-x > 0, 2\sqrt{(2-x)(4-x)} > 0$. This completes the proof.

Solution 12 by Péter Fülöp, Gyömrő, Hungary.

Let's group the quantities under the root as follows:

$$\sqrt{\underbrace{5-11x+6x^2}+1-x^3} + \sqrt{\underbrace{11-19x+8x^2}+1-x^3} = \sqrt{\underbrace{14-23x+9x^2}+1-x^3}$$

The underbraced parts are quadratic expressions, solve them for zero we get the roots of them.

$$5-11x+6x^2 = 0 \qquad x_1 = 1, x_2 = \frac{5}{6}$$

$$11-19x+8x^2 = 0 \qquad x_1 = 1, x_2 = \frac{11}{8}$$

$$14-23x+9x^2 = 0 \qquad x_1 = 1, x_2 = \frac{14}{9}$$

And konwn that $1-x^3 = (1-x)(1+x+x^2)$ the $\sqrt{1-x}$ term can be highlighted on the both sides.

$$\sqrt{1-x}\sqrt{x^2+\frac{11}{6}} + \sqrt{1-x}\sqrt{x^2+\frac{19}{8}} = \sqrt{1-x}\sqrt{x^2+\frac{23}{9}}$$

It means that $\underline{x=1}$ is a real root of the original equation.

It can be shown that the remaining equation cannot have any additional real roots:

$$\sqrt{x^2+\frac{11}{6}}+\sqrt{x^2+\frac{19}{8}}=\sqrt{x^2+\frac{23}{9}}$$

Let's introduce $z^2 = x^2 + \dfrac{23}{9}$. Substitute back to the previous equation we get:

$$\underbrace{\sqrt{z^2+\frac{39}{54}}}_{>z}+\underbrace{\sqrt{z^2+\frac{13}{72}}}_{>z}=\underbrace{\sqrt{z^2}}_{=z}$$

It can be seen that $LHS > RHS$ for all real values.

The only real root is 1.

Also solved by Bruno Salgueiro Fanego, Viveiro, Lugo; Hossaena Tedla, ADA University, Baku, Azerbaijan; and by the problem proposer.

• **5732** *Proposed by Seán M. Stewart, Physical Sciences and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia..*

If $0 < s < 1$, prove
$$\int_1^\infty \frac{t\,dt}{(t^2+1)\,(t-1)^s}=\frac{\pi}{2^{s/2}}\csc(\pi s)\cos\left(\frac{\pi s}{4}\right).$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

First, make the change of variable $t \to \dfrac{1}{t}$. to obtain
$$\int_1^\infty \frac{t\,dt}{(t^2+1)(t-1)^s}=\int_0^1 \frac{1}{t^2+1}t^{s-1}(1-t)^{-s}\,dt.$$

Next, replace $\dfrac{1}{1+t^2}$ by its geometric series expansion and interchange the order of integration and summation to obtain
$$\int_1^\infty \frac{t\,dt}{(t^2+1)(t-1)^s}=\sum_{k=0}^{\infty}(-1)^k\int_0^1 t^{s+2k-1}(1-t)^{-s}\,dt.$$

Now,
$$\int_0^1 t^{s+2k-1}(1-t)^{-s}\,dt=\frac{\Gamma(s+2k)\Gamma(1-s)}{\Gamma(2k+1)}=\frac{\Gamma(s+2k)\Gamma(1-s)}{(2k)!},$$
where $\Gamma(z)$ denotes the gamma function. By Euler's reflection formula,
$$\Gamma(1-s)=\frac{1}{\Gamma(s)}\pi\csc(\pi s),$$

so

$$\int_1^\infty \frac{t\,dt}{(t^2+1)(t-1)^s} = \frac{1}{\Gamma(s)}\pi\csc(\pi s)\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(s+2k)}{(2k)!}.$$

Recall

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \quad \text{so} \quad \Gamma(s+2k) = \int_0^\infty t^{s+2k-1}e^{-t}\,dt.$$

It then follows that

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(s+2k)}{(2k)!} &= \int_0^\infty t^{s-1}e^{-t}\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\,dt = \int_0^\infty t^{s-1}e^{-t}\cos t\,dt \\ &= \text{Re}\int_0^\infty t^{s-1}e^{-(1-i)t}\,dt = \text{Re}\left\{\frac{\Gamma(s)}{(1-i)^s}\right\} \\ &= \frac{\Gamma(s)}{2^s}\,\text{Re}\{(1+i)^s\} = \frac{\Gamma(s)}{2^{s/2}}\cos\left(\frac{\pi s}{4}\right).\end{aligned}$$

Finally,

$$\int_1^\infty \frac{t\,dt}{(t^2+1)(t-1)^s} = \frac{\pi}{2^{s/2}}\csc(\pi s)\cos\left(\frac{\pi s}{4}\right).$$

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

The integral is

$$\int_0^\infty \frac{(t+1)dt}{(1+(t+1)^2)t^s}.$$

We pass to the complex function $f(z) = \dfrac{z+1}{(1+(1+z)^2)z^s}$ compute the contour integral on the path ($\varepsilon > 0$)

$$\gamma_1(t) = t + i\varepsilon,\ 0 \leqslant t \leqslant R, \qquad \gamma_2(t) = \sqrt{R^2+\varepsilon^2}e^{it},\ \varphi_0 \leqslant t \leqslant 2\pi - \varphi_0,\ \varphi_0 = \arctan(\varepsilon/R)$$

$$\gamma_3(t) = (-t - i\varepsilon)e^{2i\pi},\ -R \leqslant t \leqslant 0, \qquad \gamma_4(t) = \varepsilon e^{-it},\ -3\pi/2 \leqslant t \leqslant -\pi/2.$$

Let's show that

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\gamma_2} f(z)dz = \lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\gamma_4} f(z)dz = 0 \qquad (1).$$

If R is large enough and $\varepsilon < 1$ we have

$$|z+1|_{z\in\gamma_2} = |1 + \sqrt{R^2+\varepsilon^2)}e^{it}| \leqslant 2\sqrt{R^2+\varepsilon^2} \leqslant 2\sqrt{2R^2} = 2\sqrt{2}R$$

$$|1+(z+1)^2|_{z\in\gamma_2} = |(\gamma_2+1)|^2 - 1 \geqslant (|\gamma_2|-1)^2 - 1 \geqslant (R-1)^2 - 1 \geqslant R^2/2.$$

$$|z^s|_{z\in\gamma_2} = (R^2+\varepsilon^2)^{s/2} \geqslant R^s$$

We get

$$0 \leqslant \int_{\gamma_2} |f(z)| \cdot |dz| \leqslant \int_{\varphi_0}^{2\pi-\varphi_0} \frac{2\sqrt{2}R\sqrt{R^2+\varepsilon^2}dt}{\frac{R^2}{2}R^s} < 2\pi\frac{8R^2}{R^{2+s}}.$$

The limit $R \to \infty$ shows that the integral tends to zero.

As for the second integral in (1) we have

$$|z+1|_{z\in\gamma_4} = |1+\varepsilon e^{-it}| \leqslant 1+\varepsilon < 2$$

$$|1+(z+1)^2|_{z\in\gamma_4} = |2+2\gamma_2+\gamma_2^2| \geqslant 2-2|\gamma_2|-|\gamma_2|^2 \geqslant 2-2\varepsilon-\varepsilon^2 \geqslant 1$$

$$|z^s||_{z\in\gamma_4} = \varepsilon^s$$

$$0 \leqslant \int_{\gamma_4} |f(z)| \cdot |dz| \leqslant \int_{\frac{-3\pi}{2}}^{\frac{-\pi}{2}} \frac{2\varepsilon|(-i)e^{-it}|dt}{\varepsilon^s} < 2\pi\varepsilon^{1-s}$$

and tends to zero as $\varepsilon \to 0$.

Of course

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\gamma_1} f(z)dz = \int_0^\infty \frac{(t+1)dt}{(1+(t+1)^2)t^s}$$

$$\int_{\gamma_3} f(z)dz = \int_{-\infty}^{0} \frac{(1+(-t-i\varepsilon)e^{2i\pi})(-dte^{2i\pi})}{(1+(1+(-t-i\varepsilon)e^{2i\pi})^2)(-t-i\varepsilon)^s e^{2i\pi s}} \underbrace{=}_{t=-\tau}$$

$$= -\int_0^\infty \frac{(1+(\tau-i\varepsilon))d\tau}{(1+(1+(\tau-i\varepsilon))^2)(\tau-i\varepsilon)^s e^{2i\pi s}} \doteq -e^{-2\pi is}\int_0^\infty h_\varepsilon(t)dt.$$

We perform the limit $\varepsilon \to 0$ under the integral and a sufficient condition is that the integrand is bounded in modulus with a function, say $g(t)$, *independent* by ε and such that $\int_0^\infty g(t)dt < \infty$. To this end we break the integral as

$$\int_0^\infty h_\varepsilon(t)dt = \int_0^1 h_\varepsilon(t)dt + \int_1^\infty h_\varepsilon(t)dt.$$

If $0 \leqslant \tau \leqslant 1$ we have

$$|1+\tau-i\varepsilon| \leqslant 2+\varepsilon \leqslant 3$$

$$|1+(1+\tau-i\varepsilon)^2| = |1+(1+\tau)^2-2i\varepsilon(1+\tau)-\varepsilon^2| \geqslant 1+(1+\tau)^2-4\varepsilon-\varepsilon^2 \geqslant \frac{1}{2}(2+2\tau+\tau^2) \geqslant 1$$

$$|\tau-i\varepsilon|^s = (\tau^2+s^2)^s \geqslant \tau^{s/2}$$

yielding

$$|h_\varepsilon(t)| \leqslant \frac{3}{\tau^{s/2}} \doteq g(t), \quad 0 \leqslant \tau \leqslant 1, \quad \int_0^1 g(t)dt < \infty$$

hence we can perform

$$\lim_{\varepsilon\to 0}\int_0^1 h_\varepsilon(t)dt = \int_0^1 \frac{(t+1)dt}{(1+(t+1)^2)t^s}.$$

Moreover if $\tau \geqslant 1$

$$|1+\tau-i\varepsilon| \leqslant 1+\tau+\varepsilon \leqslant 2+\tau \leqslant$$

$$|1+(1+\tau-i\varepsilon)^2| = |1+(1+\tau)^2-2i\varepsilon(1+\tau)-\varepsilon^2| \geqslant 1+(1+\tau)^2-4\varepsilon-\varepsilon^2 \geqslant \frac{1}{2}\tau^2)$$

thus

$$|h_\varepsilon(t)| \leqslant \frac{2+\tau}{\frac{\tau^2}{2}\tau^{s/2}} \doteq g(t), \quad \tau \geqslant 1, \quad \int_0^1 g(t)dt < \infty$$

hence

$$\lim_{\varepsilon\to 0}\int_1^\infty h_\varepsilon(t)dt = \int_1^\infty \frac{(t+1)dt}{(1+(t+1)^2)t^s}.$$

We have obtained

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz\right) = (1-e^{-2\pi i s})\int_0^\infty \frac{(t+1)dt}{(1+(t+1)^2)t^s}$$

The zeroes of the denominators are $t=-1+e^{i\frac{\pi}{2}}$, $t=-1+e^{i\frac{3\pi}{2}}$, . The residues theorem yields

$$(1-e^{-2\pi i s})\int_0^\infty \frac{(t+1)dt}{(1+(t+1)^2)t^s} = 2\pi i\left(\frac{i}{2i(-1+i)^s} + \frac{-i}{2(-i)(-1-i)^s}\right)$$

$$\int_0^\infty \frac{(t+1)dt}{(1+(t+1)^2)t^s} = \frac{2\pi i}{1-e^{-2\pi i s}}\left(2^{-s/2}e^{\frac{-i3\pi}{4}} + 2^{-s/2}e^{\frac{-i5\pi}{4}}\right) =$$
$$= \frac{2\pi i}{i\sin(\pi s)}\left(2^{-s/2}e^{\frac{i\pi}{4}} + 2^{-s/2}e^{\frac{-i\pi}{4}}\right) = \frac{\pi}{2^{s/2}}\csc(\pi s)\cos\left(\frac{\pi s}{4}\right).$$

Solution 3 by Yunyong Zhang, Chinaunicom, Yunnan, China.

Solution 1

According to the following theorem:

Let function $f(x)$ defined holomorphic on $\mathbb{C}\backslash[a,b]$ except $a_1, a_2, \cdots, a_n$, continuous to line segment $[a,b]$. If $0<|m|,|n|<1, m+n\in\mathbb{Z}$, and

$$\lim_{z\to\infty} z^{m+n+1}f(z) = A \in \mathbb{C}.$$

Then

$$\int_a^b (x-a)^m(b-x)^n f(x)\mathrm{d}x = -\frac{A\pi}{\sin\pi n} + \frac{\pi}{e^{-n\pi i}\sin\pi n}\sum_{k=1}^p Res(F,a_k)$$

in which $F(z) = (z-a)^m(b-z)^n f(z)$.

So

$$I = \int_1^{\infty} \frac{t\mathrm{d}t}{(t^2+1)(t-1)^s} = \int_0^1 \frac{\frac{1}{x}(\frac{1}{x^2})}{(\frac{1}{x^2}+1)(\frac{1}{x}-1)^s}\mathrm{d}x = \int_0^1 \frac{x^{s-1}}{(1-x)^s(1+x^2)}\mathrm{d}x.$$

Let $a = 0, b = 1, m = s-1, n = -s$, and $f(z) = \dfrac{1}{1+x^2}, \lim\limits_{z\to\infty} \dfrac{1}{1+z^2} = A = 0$. Then $I = \dfrac{\pi}{e^{-n\pi i}\sin \pi n} \sum\limits_{k=1}^{p} Res\left(\dfrac{z^{s-1}}{(1-z)^s(1+z^2)}, a_k\right)$. Now evaluate

$$\sum_{k=1}^{p} Res\left(\frac{z^{s-1}}{(1-z)^s(1+z^2)}, a_k\right) = \sum_{k=1}^{p} Res\left(\frac{1}{(1-z)^s(1+z^2)z^{1-s}}, a_k\right) = \underset{z=i}{Res}F(z) + \underset{z=-i}{Res}F(z).$$

When $z = i, \arg z = \dfrac{\pi}{2}, \arg(1-z) = -\dfrac{\pi}{4}$: $\underset{z=i}{Res}F(z) = \lim\limits_{z\to i} \dfrac{1}{(1-z)^s z^{1-s}(z+i)} = \dfrac{1}{(2i)2^{\frac{s}{2}} e^{-\frac{i\pi}{4}s} e^{-\frac{i\pi}{2}(1-s)}} = \dfrac{e^{(\frac{i3}{4}\pi s - \frac{i\pi}{2})}}{2^{\frac{s}{2}} \times 2i}$. When $z = -i, \arg z = \dfrac{3\pi}{2}, \arg(1-z) = \dfrac{\pi}{4}$: $\underset{z=-i}{Res}F(z) = \lim\limits_{z\to i} \dfrac{1}{(1-z)^s z^{1-s}(z-i)} = \dfrac{1}{(-2i)2^{\frac{s}{2}} e^{\frac{i3\pi}{4}s} e^{\frac{i3\pi}{2}(1-s)}} = \dfrac{e^{(\frac{i5}{4}\pi s - \frac{i3\pi}{2})}}{2^{\frac{s}{2}} \times (-2i)}$.

$$\therefore \quad I = \frac{\pi}{-\sin(\pi s)} \frac{e^{-i\pi s}}{2^{\frac{s}{2}} \times 2}\left[e^{\frac{i3}{4}\pi s} - e^{\frac{i5}{4}\pi s}\right] = \frac{\pi}{\sin(\pi s)} \frac{1}{2^{\frac{s}{2}} \times 2}\left[e^{\frac{i}{4}\pi s} + e^{\frac{-i}{4}\pi s}\right]$$

$$= \frac{\pi}{\sin(\pi s)} \frac{1}{2^{\frac{s}{2}} \times 2} \times 2\cos\frac{\pi s}{4} = \frac{\pi}{\sin(\pi s)} \frac{\cos\frac{\pi s}{4}}{2^{\frac{s}{2}}}.$$

Solution 2

Let $x = t-1, t = x+1, y = \dfrac{1}{x}$.

Then $I = \displaystyle\int_1^{\infty} \frac{t\mathrm{d}t}{(t^2+1)(t-1)^s} = \int_0^{\infty} \frac{(1+x)}{x^3(x^2+2x+2)}\mathrm{d}x$

$= \displaystyle\int_0^{\infty} \frac{(\frac{1}{y}+1)\frac{1}{y^2}}{\frac{1}{y^s}(\frac{1}{y^2}+\frac{2}{y}+2)}\mathrm{d}y = \int_0^{\infty} \frac{y^s + y^{s+1}}{2y^3+2y^2+y}\mathrm{d}y.$

According to Laplace Transform and Inverse Laplace Transform,

$$L^{-1}\left(\frac{1}{2y^3+2y^2+y}\right) = 1 - e^{-\frac{t}{2}}\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right)$$

$$L^{-1}\left(\frac{1}{2y^2+2y+1}\right) = -e^{-\frac{t}{2}}\sin\frac{t}{2}$$

$$L(y^s) = \frac{\Gamma(s+1)}{t^{s+1}}.$$

$$\therefore \quad I = \Gamma(s+1)\int_0^\infty \frac{e^{-\frac{t}{2}}\sin\frac{t}{2} + 1 - e^{-\frac{t}{2}}\cos\frac{t}{2} - e^{-\frac{t}{2}}\sin\frac{t}{2}}{t^{s+1}}\mathrm{d}t$$

$$= \Gamma(s+1)\int_0^\infty \frac{1-e^{-\frac{t}{2}}\cos\frac{t}{2}}{t^{s+1}}\mathrm{d}t \quad (\text{let } x = \frac{t}{2}, t = 2x)$$

$$= \Gamma(s+1)\int_0^\infty \frac{1-e^{-x}\cos x}{2^{s+1}x^{s+1}}2\mathrm{d}x = \frac{\Gamma(s+1)}{2^s}\int_0^\infty \frac{1-e^{-x}\cos x}{x^{s+1}}\mathrm{d}x$$

$$= \frac{\Gamma(s+1)}{2^s}\int_0^\infty \frac{e^{-x}(e^x-\cos x)}{x^{s+1}}\mathrm{d}x.$$

If $F(s)$ is the Laplace Transform of $f(t)$, then $\int_0^\infty f(t)e^{-xt}\mathrm{d}t = F(x)$

$$\because \quad L(\frac{e^x-\cos x}{x^{s+1}}) = \frac{1}{2}\Gamma(-s)\left[2(t-1)^s - (t-i)^s - (t+i)^a\right].$$

When $t = 1$, $e^{-xt} = e^{-x}$:

$$L(\frac{e^x-\cos x}{x^{s+1}}) = -\frac{1}{2}\Gamma(-s)\left[(1-i)^s + (1+i)^a\right]$$

$$= -\frac{1}{2}\Gamma(-s)(\sqrt{2})^3\cos\frac{\pi s}{4}\times 2.$$

$$\therefore \quad \frac{\Gamma(s+1)}{2^s}\int_0^\infty e^{-x}\frac{e^x-\cos x}{x^{s+1}}\mathrm{d}x = \frac{\Gamma(s+1)}{2^s}\left[\Gamma(-s)\cos\frac{\pi s}{4}\right](\sqrt{2})^s$$

$$\because \quad \Gamma(s+1)\Gamma(-s) = -\frac{\pi}{\sin\pi s}$$

$$\therefore \quad I = \frac{\pi}{\sin\pi s}\frac{\cos\frac{\pi s}{4}}{2^{\frac{s}{2}}} = \frac{\pi}{2^{s/2}}\csc(\pi s)\cos(\frac{\pi s}{4}).$$

Solution 4 by Michel Bataille, Rouen, France.

Let I denote the integral. The substitution $t = 1 + \frac{1}{u}$ readily leads to $I = \int_0^\infty \frac{R(u)}{u^\alpha}\,du$ where $R(u) = \frac{u+1}{2u^2+2u+1}$ and $\alpha = 1 - s$.

We apply the formula established in **[1]** p. 106-107, which gives: $(1 - e^{-2\pi i\alpha})I = 2\pi i\sigma$ where σ denotes the sum of the residus of $\frac{R(u)}{u^\alpha}$ at the poles $\frac{-1+i}{2} = \frac{1}{\sqrt{2}}e^{3\pi i/4}$ and $\frac{-1-i}{2} = \frac{1}{\sqrt{2}}e^{5\pi i/4}$.

We calculate

$$\sigma = \frac{(1+i)/2}{2i}\cdot\frac{2^{\alpha/2}}{e^{3\pi\alpha i/4}} + \frac{(1-i)/2}{-2i}\cdot\frac{2^{\alpha/2}}{e^{5\pi\alpha i/4}} = \frac{2^{\alpha/2}}{4}\left((1-i)e^{-3\pi\alpha i/4} + (1+i)e^{-5\pi\alpha i/4}\right).$$

Since $1 - e^{-2\pi i\alpha} = 2i\sin(\pi\alpha)e^{-\pi i\alpha}$, we deduce that

$$I = \frac{\pi}{\sin(\pi\alpha)}e^{\pi i\alpha}\sigma = \frac{\pi}{\sin(\pi\alpha)} \cdot \frac{2^{\alpha/2}}{4}\left((1-i)e^{\pi i\alpha/4} + (1+i)e^{-\pi i\alpha/4}\right)$$
$$= \frac{\pi 2^{\alpha/2}}{4\sin(\pi\alpha)}\left(\sqrt{2}e^{-i\pi/4}e^{\pi i\alpha/4} + \sqrt{2}e^{i\pi/4}e^{-\pi i\alpha/4}\right)$$

or, back to s,

$$I = \frac{1}{2}\cdot\frac{\pi}{2^{s/2}\sin(\pi s)}\left(e^{-\pi i s/4} + e^{\pi i s/4}\right) = \frac{\pi}{2^{s/2}\sin(\pi s)}\cos\left(\frac{\pi s}{4}\right),$$

as desired.

[1] H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Dover, 1995

Also solved by Albert Stadler, Herrliberg, Switzerland; Péter Fülöp, Gyömrő, Hungary; and the problem proposer.

Belated Acknowledgement: As the Section Editor, I failed to publish The Eagle Problem Solvers' solution to Problem 5722 due to my regrettable error in failing to download their sent solution into the download folder on my computer. I express my sincerest apologies for this shortcoming. Despite my systematic process for downloading and acknowledging the submission of solutions, I am surprised at myself that I let this happen. I will be much more cautious from now on. This error is the first of its kind. Hopefully such an error will not repeat. Below is the statement of the Problem 5722, followed by Eagle Problem Solvers' solution:

Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.

Let p_n be the n-th prime number. Prove the following inequality

$$p_{n+1} < 3p_{\lfloor \frac{n}{2} \rfloor + 1} \text{ for } n \geqslant 1,$$

where $\lfloor . \rfloor$ denotes the integer part function.

Hint: Use the Rosser-Schoenfeld inequalities $p_n < n\log n + n\log\log n - \dfrac{n}{2}$ for $n \geqslant 20$ and $p_n > n\log n$ for $n \geqslant 1$, along with a small table of primes.

Solution of the Problem: If n is odd, then $n = 2k+1$ for an integer k, and $\left\lfloor \dfrac{n}{2} \right\rfloor = k = \left\lfloor \dfrac{n-1}{2} \right\rfloor$. Since $p_{2k+1} < p_{2k+2}$, then it suffices to show $p_{2k+2} < 3p_{k+1}$, or equivalently, $p_{2m} < 3p_m$ for each positive integer m.

If $f(x) = (\log 2x)^2$, then $f'(x) = \dfrac{2\log 2x}{x}$ and $f''(x) = \dfrac{2 - 2\log 2x}{x^2} < 0$ for $x > \dfrac{e}{2}$, so that $f(x)$ is concave down for $x \geqslant 2$. Since $f(21) < \dfrac{21e}{4}$, and $f'(21) < \dfrac{e}{4}$, then the graph of f lies below the line $y = \dfrac{ex}{4}$ for $x \geqslant 21$. Therefore, for integers $m \geqslant 21$,

$$\begin{aligned}
(\log 2m)^2 &< \left(\frac{e}{4}\right) m \\
\frac{4m^2 (\log 2m)^2}{m^3} &< e \\
\log\left(4m^2 (\log 2m)^2\right) - \log m^3 &< 1 \\
2 \log (2m \log 2m) - 3 \log m &< 1 \\
2 (\log 2m + \log\log 2m) - 1 &< 3 \log m \\
2m \log 2m + 2m \log\log 2m - \frac{2m}{2} &< 3m \log m.
\end{aligned}$$

By the Rosser-Schoenfeld inequalities, since $m \geqslant 10$,

$$p_{2m} < 2m \log 2m + 2m \log\log 2m - \frac{2m}{2} < 3m \log m < 3p_m.$$

To complete the proof, the following table verifies that $p_{2m} < 3p_m$ for $1 \leqslant m \leqslant 20$.

m	p_{2m}	$3p_m$	m	p_{2m}	$3p_m$
1	3	6	11	79	93
2	7	9	12	89	111
3	13	15	13	101	123
4	19	21	14	107	129
5	29	33	15	113	141
6	37	39	16	131	159
7	43	51	17	139	177
8	53	57	18	151	183
9	61	69	19	163	201
10	71	87	20	173	213

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

Please adopt the following structure, in the order shown, for the presentation of your proposal:

1. On the top of first page of your proposal, begin with the phrase:

"Problem proposed to SSMJ"

2. On the second line, write

"Problem proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s) if any.

3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣