Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please present all proposed solutions and proposed problems according to formatting requirements delineated near the end of this document. Also, please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

To propose problems, email them to: problems4ssma@gmail.com

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Solutions to the problems published in this issue should be submitted *before* April 1, 2024.

• **5757** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Suppose $f:[0,1]\to\mathbb{R}$ is continuous and $\displaystyle\int_0^1 f(x)dx = 1/2$. Show that

$$2+\int_0^1 f^2(x)dx \geqslant 6\int_0^1 xf(x)dx.$$

• **5758** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose $P,Q\in Int(\triangle ABC)$ such that $\beta\overrightarrow{AB}+\gamma\overrightarrow{BP}+\overrightarrow{PC}=0$ and $\overrightarrow{AQ}+\alpha\overrightarrow{QB}+\overrightarrow{BC}=0$ with $\alpha,\beta,\gamma\in\mathbb{R};\alpha,\beta\neq 1$. Prove that A,P,Q are collinear if and only if $\alpha+\gamma=\beta+1$.

• **5759** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all pairs (a,b) of non-negative integers satisfying the equation $a^b-b^a=a+b$.

• **5760** *Proposed by Michel Bataille, Rouen, France.*

Let a,b be real numbers such that $0<a<b$. Prove that

$$(a+b)^{a+b}(b-a)^{b-a}>(a^2+b^2)^b.$$

• **5761** *Proposed by Narendra Bhandari and Yogesh Joshi, Nepal.*

$$\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(H_{\left[\frac{n}{2}\right]}-H_{\left[\frac{n-1}{2}\right]}\right)}{4^n(6n+3)}+\int_0^{\frac{\pi}{4}}\frac{4y\sec y dy}{\sqrt{9\cos 2y}}=\zeta(2)$$

where $H_{[n]}=\int_0^1\frac{1-x^n}{1-x}dx$ and $\zeta(n)=\sum_{k=1}^{\infty}\frac{1}{k^n}$ is Riemann zeta function for $n>1$.

• **5762** *Proposed by Paolo Perfetti, dipartimento di matematica Universitą di "Tor Vergata", Rome, Italy.*

Let $f\colon[-1,1]\to\mathbb{R}$ be a three-times continuously differentiable such that $f(-1)=f'(-1)=f''(1)=0$. Prove that

$$\left(\int_{-1}^{1}f(x)dx\right)^2\leqslant\frac{34}{315}\int_{-1}^{1}\left(f'''(x)\right)^2dx.$$

Solutions

To Formerly Published Problems

• **5733** *Proposed by Ivan Hadinata, Senior High School 1 Jember, Jember, Indonesia.*

Find all strictly increasing function(s) $f:\mathbb{N}\to\mathbb{N}$ so that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{f(i)^8}=\left(\frac{f(6)-f(5)}{5}+\frac{f(7)-6}{7}\right)\cdot\zeta(2)^2\cdot\zeta(4)$$

where ζ is Riemann zeta function.

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\left(f(i)\right)^8}=\sum_{i=1}^{\infty}\frac{1}{\left(f(i)\right)^8}\leqslant\sum_{i=1}^{\infty}\frac{1}{i^8}=(8)=\frac{\pi^8}{9450}=\left(\frac{1}{5}+\frac{1}{7}\right)\left(\frac{\pi^2}{6}\right)^2\frac{\pi^4}{90}=$$

$$=\left(\frac{1}{5}+\frac{1}{7}\right)\left((2)\right)^2\left((4)\right)\leqslant\left(\frac{f(6)-f(5)}{5}+\frac{f(7)-6}{7}\right)\left((2)\right)^2\left((4)\right)$$

with equality if and only if f(i)=i, i=1, 2, 3,. . . .

Solution 2 by Michel Bataille, Rouen, France.

Since $\zeta(2) = \dfrac{\pi^2}{6}$ and $\zeta(4) = \dfrac{\pi^4}{90}$, a simple calculation yields

$$\left(\frac{1}{5} + \frac{1}{7}\right) \cdot \zeta(2)^2 \cdot \zeta(4) = \frac{\pi^8}{9450} = \sum_{i=1}^{\infty} \frac{1}{i^8},$$

showing that the function $id_{\mathbb{N}} : n \mapsto n$ is a solution.
Conversely, suppose that $f : \mathbb{N} \to \mathbb{N}$ is stricly increasing, so that $f(i) \geqslant i$ for all $i \in \mathbb{N}$ (easily proved by induction). Assume that $f \neq id_{\mathbb{N}}$, so that $f(m) > m$ for some positive integer m. Then, we have $f(6) - f(5) \geqslant 1$, $f(7) \geqslant 7$ and therefore the right-hand side of the equality is greater than or equal to $\left(\dfrac{1}{5} + \dfrac{1}{7}\right) \cdot \zeta(2)^2 \cdot \zeta(4) = \displaystyle\sum_{i=1}^{\infty} \frac{1}{i^8}$. However, $\dfrac{1}{f(m)^8} < \dfrac{1}{m^8}$ and $\dfrac{1}{f(i)^8} \leqslant \dfrac{1}{i^8}$ for $i \neq m$, hence $\displaystyle\sum_{i=1}^{\infty} \frac{1}{f(i)^8} < \sum_{i=1}^{\infty} \frac{1}{i^8}$, contradicting the equality.
We conclude that $id_{\mathbb{N}}$ is the only solution for f.

Solution 3 by Moti Levy, Rehovot, Israel.

If the function f is strictly increasing $f : \mathbb{N} \to \mathbb{N}$ then the following facts are true:

$$f(n) \geqslant n, \tag{1}$$

$$f(n+1) - f(n) \geqslant 1, \tag{2}$$

$$f(n+1) - n \geqslant 1. \tag{3}$$

The zeta function at integer values $2, 4, 8$ satisfies,

$$\zeta(8) = \sum_{i=1}^{\infty} \frac{1}{i^8} = \frac{12}{35} \zeta^2(2) \zeta(4) = \frac{\pi^8}{9450}. \tag{4}$$

It follows from (1) that

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{f^8(i)} \leqslant \sum_{i=1}^{\infty} \frac{1}{i^8} = \zeta(8). \tag{5}$$

By (2) and (3) , we have

$$\left(\frac{f(6) - f(5)}{5} + \frac{f(7) - 6}{7}\right) \zeta^2(2) \zeta(4) \geqslant \left(\frac{1}{5} + \frac{1}{7}\right) \zeta^2(2) \zeta(4) = \zeta(8) \tag{6}$$

Combining (5) and (6) we get

$$\zeta(8) \leqslant \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{f^8(i)} \leqslant \zeta(8).$$

Hence

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{f^8(i)} = \zeta(8) = \sum_{i=1}^{\infty} \frac{1}{i^8}$$

It follows that $f(i) = i$ is the only strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ that satisfies the equation in the problem statement.

Also solved by the problem proposer.

• **5734** *Proposed by Narendra, Bhandari, Bajura, Nepal.*

Prove

$$\int_0^{\frac{\pi}{4}} \frac{\log\left(2\tanh^{-1}(\tan x)\right)\tanh^{-1}(\tan x)}{\tan 2x}\,dx = \frac{\pi^2}{96}\log\left(\frac{2e^3\pi^3}{A^{36}}\right),$$

where A is Glaisher- Kinkelin constant and e is Euler's number.

Solution 1 by Moti Levy, Rehovot, Israel.

We will show that

$$I := \int_0^{\frac{\pi}{4}} \frac{\ln\left(2\tanh^{-1}\left(\tan(x)\right)\right)\tanh^{-1}\left(\tan(x)\right)}{\tan(2x)}dx = \frac{\pi^2}{96}\left(\ln 2 + 3\ln\pi - 36\ln(A) + 3\right).$$

By a series of integration variable changes we get

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{4}} \frac{\ln\left(2\tanh^{-1}\left(\tan(x)\right)\right)\tanh^{-1}\left(\tan(x)\right)}{\tan(2x)}dx \\
&\overset{x=\tan(u)}{=} \int_0^1 \frac{\left(1-u^2\right)\operatorname{arctanh} u}{2u}\frac{\ln\left(2\operatorname{arctanh} u\right)}{u^2+1}\,du \\
&= \int_0^1 \frac{1-u^2}{4u\left(1+u^2\right)}\ln\left(\frac{1+u}{1-u}\right)\ln\left(\ln\left(\frac{1+u}{1-u}\right)\right)du \\
&\overset{w=\ln\left(\frac{1+u}{1-u}\right)}{=} \int_0^\infty we^{-2w}\left(\ln w\right)\frac{1}{1-e^{-4w}}\,dw.
\end{aligned}$$

Using the geometric series $\frac{1}{1-e^{-4w}} = \sum_{k=0}^{\infty} e^{-4kw}$, and after interchanging the order of summation and integration , we get

$$I = \sum_{k=0}^{\infty}\int_0^\infty we^{-2w(1+2k)}\ln(w)\,dw.$$

We simplify by integration by parts, to get

$$I = \sum_{k=0}^{\infty}\frac{1}{2(1+2k)}\int_0^\infty \left(1+\ln(w)\right)e^{-2w(1+2k)}dw.$$

Now

$$I=\sum_{k=0}^{\infty}\frac{1}{2\left(1+2k\right)}\int_0^{\infty}e^{-2w(1+2k)}dw+\sum_{k=0}^{\infty}\frac{1}{2\left(1+2k\right)}\int_0^{\infty}\ln\left(w\right)e^{-2w(1+2k)}dw.$$

$$\int_0^{\infty}e^{-2w(1+2k)}dw=\frac{1}{2\left(2k+1\right)}$$

$$\int_0^{\infty}\ln\left(w\right)e^{-2w(1+2k)}dw=-\frac{1}{2\left(2k+1\right)}\Big(\gamma+\ln\big(2\left(2k+1\right)\big)\Big).$$

$$I=\sum_{k=0}^{\infty}\frac{1-\gamma-\ln\left(2\right)}{4\left(1+2k\right)^2}-\frac{1}{4}\sum_{k=0}^{\infty}\frac{\ln\left(1+2k\right)}{\left(1+2k\right)^2}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\ln\left(1+2k\right)}{\left(1+2k\right)^2}&=-\frac{d}{ds}\sum_{k=0}^{\infty}\frac{1}{\left(1+2k\right)^s}\Bigg|_{s=2}=-\frac{d\left(1-2^{-s}\right)\zeta\left(s\right)}{ds}\Bigg|_{s=2}\\&=-2^{-s}\ln\left(2\right)\zeta\left(s\right)+\left(1-2^{-s}\right)\zeta'\left(s\right)\Big|_{s=2}\\&=-\frac{\pi^2}{24}\ln\left(2\right)-\frac{3}{4}\zeta'\left(2\right).\end{aligned}$$

It is known that

$$\zeta'\left(2\right)=-2\pi^2\ln\left(A\right)+\frac{\pi^2}{6}\ln\left(2\pi\right)+\frac{\gamma\pi^2}{6}.$$

Hence

$$\sum_{k=0}^{\infty}\frac{\ln\left(1+2k\right)}{\left(1+2k\right)^2}=-\frac{1}{24}\pi^2\left(3\gamma+4\ln 2+3\ln\pi-36\ln A\right).$$

It is known that

$$\sum_{k=0}^{\infty}\frac{1}{\left(1+2k\right)^2}=\frac{\pi^2}{32}.$$

$$\begin{aligned}I&=\frac{1}{32}\pi^2\left(-\ln 2-\gamma+1\right)-\frac{1}{4}\left(-\frac{1}{24}\pi^2\left(3\gamma+4\ln 2+3\ln\pi-36\ln A\right)\right)\\&=\frac{\pi^2}{96}\left(\ln 2+3\ln\pi-36\ln A+3\right).\end{aligned}$$

Solution 2 by Yunyong Zhang, Chinaunicom, Yunnan, China.

let $t=\tan x,\ \mathrm{d}t=\left(1+t^2\right)\mathrm{d}x$

$$I=\int_0^1\frac{\ln\left(2\tanh^{-1}t\right)\tanh^{-1}t}{\frac{2t}{1-t^2}}\frac{\mathrm{d}t}{1+t^2}$$

let $y = \tanh^{-1} t,\ \mathrm{d}y = \dfrac{1}{1-t^2}\ \mathrm{d}t, t = \tanh y$

$$I = \int_0^\infty \frac{\ln(2y)\cdot y\left(1-t^2\right)^2}{2t\left(1+t^2\right)}\mathrm{d}y = \int_0^\infty \ln(2y)y\cdot 2\operatorname{csch}(4y)\mathrm{d}y = 2\int_0^\infty \frac{y\ln(2y)}{\sinh(4y)}\mathrm{d}y$$

$\because \quad \sinh(4y) = \dfrac{2}{\mathrm{e}^{4y}-\mathrm{e}^{-4y}}$

$\therefore \quad I = 4\left[\displaystyle\int_0^\infty \frac{y\ln(2y)}{\mathrm{e}^{4y}-1}\mathrm{d}y - \int_0^\infty \frac{y\ln(2y)}{\mathrm{e}^{8y}-1}\mathrm{d}y\right]$

let $x = 2y, y = \dfrac{x}{2}, \mathrm{d}y = \dfrac{1}{2}\mathrm{d}x$

$$I = \int_0^\infty \frac{x\ln x}{\mathrm{e}^{2x}-1}\mathrm{d}x - \int_0^\infty \frac{x\ln x}{\mathrm{e}^{4x}-1}\mathrm{d}x$$

let $I(s) = \displaystyle\int_0^\infty \frac{x^{s-1}}{\mathrm{e}^{cx}-1}\mathrm{d}x = \frac{1}{c^s}\int_0^\infty \frac{x^{s-1}}{\mathrm{e}^{x}-1}\mathrm{d}x = \frac{1}{c^s}\int_0^\infty \frac{\mathrm{e}^{-x}x^{s-1}}{1-\mathrm{e}^{-x}}\mathrm{d}x$

$$= \frac{1}{c^s}\sum_{k=1}^\infty\int_0^\infty x^{s-1}\mathrm{e}^{-kx}\mathrm{d}x = \frac{1}{c^s}\sum_{k=1}^\infty\frac{1}{k^s}\int_0^\infty x^{s-1}\mathrm{e}^{-x}\mathrm{d}x = \frac{\Gamma(s)\xi(s)}{c^s}$$

Taking the derivative of s on both sides

$$\int_0^\infty \frac{x^{s-1}\ln x}{\mathrm{e}^{cx}-1}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\Gamma(s)\xi(s)}{c^s}\right]$$

$$= \frac{c^s\left[\Gamma'(s)\xi(s)+\Gamma(s)\xi'(s)\right] - c^s\ln c\cdot\left[\Gamma(s)\xi(s)\right]}{\left(c^s\right)^2}$$

$$= \frac{\left[\Gamma(s)\psi(s)\xi(s)+\Gamma(s)\xi'(s)\right] - \ln c\cdot\left[\Gamma(s)\xi(s)\right]}{c^s}$$

when $s = 2$

$$\int_0^\infty \frac{x\ln x}{\mathrm{e}^{cx}-1}\mathrm{d}x = \frac{\left[\Gamma(2)\psi(2)\xi(2)+\Gamma(2)\xi'(2)\right] - \ln c\cdot\left[\Gamma(2)\xi(2)\right]}{c^s}$$

$$= \frac{\left[(1-\gamma)\xi(2)+\xi(2)[\ln(2\pi)+\gamma-12\ln A]\right] - \xi(2)\ln c}{c^2}$$

$$= \xi(2)\left[\frac{1-\ln c+\ln(2\pi)-12\ln A}{c^2}\right]$$

when $c = 2$

$$\int_0^\infty \frac{x\ln x}{\mathrm{e}^{2x}-1}\mathrm{d}x = \xi(2)\left[\frac{1-\ln 2+\ln(2\pi)-12\ln A}{4}\right]$$

$c = 4$

$$\int_0^\infty \frac{x\ln x}{\mathrm{e}^{4x}-1}\mathrm{d}x = \xi(2)\left[\frac{1-\ln 4+\ln(2\pi)-12\ln A}{16}\right]$$

$$\therefore \quad I=\frac{\xi(2)}{16}[4-4\ln 2+4\ln(2\pi)-48\ln A-1+\ln 4-\ln(2\pi)+12\ln A]$$

$$=\frac{\xi(2)}{16}[3-2\ln 2+3\ln(2\pi)-36\ln A]=\frac{\pi^2}{96}\left[3+\ln\left(\frac{2\pi^3}{A^{36}}\right)\right]=\frac{\pi^2}{96}\ln\left(\frac{2e^3\pi^3}{A^{36}}\right)$$

Appendix: calculation of $\xi'(-1)$ and $\xi'(2)$

$$\xi(s)=\lim_{m\to\infty}\left[\sum_{k=1}^{m}k^{-s}-\frac{m^{1-s}}{1-s}-\frac{m^{-s}}{2}+\frac{sm^{-s-1}}{12}\right],\mathrm{Re}\{s\}>-3$$

$$\xi'(-1)=\frac{1}{12}-\lim_{m\to\infty}\left[\sum_{k=1}^{m}k\ln k-\left(\frac{m^2}{2}+\frac{m}{2}+\frac{1}{12}\right)\ln m+\frac{m^2}{4}\right]=\frac{1}{12}-\ln A$$

similarly

$$\xi(s)=\frac{1}{\pi}(2\pi)^s\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\xi(1-s)$$

$$\xi'(s)=\frac{1}{\pi}\ln(2\pi)(2\pi)^s\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\xi(1-s)$$

$$+\frac{1}{2}(2\pi)^s\left(\frac{\pi s}{2}\right)\Gamma(1-s)\xi(1-s)$$

$$-\frac{1}{\pi}(2\pi)^s\sin\left(\frac{\pi s}{2}\right)\Gamma'(1-s)\xi(1-s)$$

$$-\frac{1}{\pi}(2\pi)^s\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\xi'(1-s)$$

let $s=-1$, then

$$\xi'(-1)=-\frac{1}{2\pi^2}\ln(2\pi)\xi(2)+\frac{1}{2\pi^2}(1-\gamma)\xi(2)+\frac{1}{2\pi^2}\xi'(2)$$

$\because \quad \Gamma'(2)=\Gamma(2)\psi(2)=\psi(2)=\psi(1)+1=1-\gamma$, then

$$\xi'(2)=2\pi^2\xi'(-1)+\xi(2)[\ln(2\pi)+\gamma-1]$$

$$=2\pi^2\left(\frac{1}{12}-\ln A\right)+\xi(2)[\ln(2\pi)+\gamma-1]$$

$$=\xi(2)-\{2\xi(2)\ln A+\xi(2)[\ln(2\pi)+\gamma-1]\}$$

$$=\xi(2)[-12\ln A+\gamma+\ln(2\pi)]$$

Solution 3 by Péter Fülöp, Gyömrő, Hungary.

1. Substitution: $u = \tanh^{-1}(\tan(x))$,

where $\dfrac{dx}{du} = \dfrac{1}{\sinh^2(u) + \cosh^2(u)} = \dfrac{1}{\cosh(2u)}$,

and $\tan(2x) = \dfrac{2\tanh(u)}{1 - \tanh^2(u)} = \sin(2u)$.

Integral equals to

$$I = \int_0^\infty \frac{2u\ln(2u)}{\sinh(4u)}du$$

2. Substitution: $t = 4u$

$$I = \frac{1}{4}\int_0^\infty \frac{te^{-t}(\ln(t) - \ln(2))}{1 - e^{-2t}}dt$$

3. We can write that: $\displaystyle\sum_{k=0}^\infty e^{-2kt} = \frac{1}{1 - e^{-2t}}$

$$I = \frac{1}{4}\int_0^\infty \sum_{k=0}^\infty te^{-(2k+1)t}\ln(t)dt - \frac{1}{4}\int_0^\infty \sum_{k=0}^\infty te^{-(2k+1)t}\ln(2)dt$$

4. Substitution: $t = \dfrac{r}{2k+1}$ and replace the order of the integrals and sums (where ψ - digamma and Γ - Gamma functions)

$$I = \frac{1}{4}\sum_{k=0}^\infty \frac{1}{(2k+1)^2}\underbrace{\int_0^\infty re^{-r}\ln(r)dr}_{\psi(2)} - \frac{1}{4}\sum_{k=0}^\infty \frac{\ln(2k+1)}{(2k+1)^2}\underbrace{\int_0^\infty re^{-r}dr}_{\Gamma(2)}$$

$$-\frac{\ln 2}{4}\sum_{k=0}^\infty \frac{1}{(2k+1)^2}\underbrace{\int_0^\infty re^{-r}dr}_{\Gamma(2)}$$

5. Calculating $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$, $\psi(2) = \psi(1) + 1 = 1 - \gamma$ and $\Gamma(2) = 1$ we get:

$$I = \frac{\pi^2}{32}\Big(1 - \gamma - \ln(2)\Big) - \frac{1}{4}\sum_{k=0}^{\infty} \frac{\ln(2k+1)}{(2k+1)^2}$$

6. Determination of the sum: $S = \sum_{k=0}^{\infty} \frac{\ln(2k+1)}{(2k+1)^2}$

$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ can be separated for even and odd parts:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} = \sum_{k=0}^{\infty} \frac{\ln(2k+1)}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{\ln(2k)}{(2k)^2}$$

where the odd part is equal to S.

$$S = \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} - \frac{1}{4}\sum_{k=1}^{\infty} \frac{\ln(k) + \ln(2)}{k^2} = \frac{3}{4}\sum_{k=1}^{\infty} \frac{\ln(k)}{k^2} - \frac{\pi^2 \ln(2)}{24}$$

7. The first derivative of the Zeta function at a value of 2 is known from the literature:

$$\zeta(2) = -\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} = -\frac{\pi^2}{6}\Big(12\ln(A) - \gamma - \ln(2\pi)\Big)$$

Using these results in the calculation of I integral in 5. point we get:

$$I = \frac{\pi^2}{32}\Big(1 - \gamma - \ln(2)\Big) + \frac{\pi^2 \ln(2)}{96} - \frac{\pi^2}{32}\Big(12\ln(A) - \gamma - \ln(2\pi)\Big)$$

After performing the possible cancellations:

$$I = \frac{\pi^2}{96}\ln\left(\frac{2e^3\pi^3}{A^{36}}\right)$$

The statement is proved!

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

We have

$$\tanh^{-1}(x) = \frac{1}{2}\log\frac{1+x}{1-x}, \quad \tan(2x) = \frac{2\tan x}{1-\tan^2 x},$$

$$'(2) = \frac{\pi^2}{6}\left(\gamma + \log(2\pi) - 12\log A\right),$$

where ζ(s) denotes the Riemann zeta function and γ the Euler-Mascheroni constant (see for instance https://en.wikipedia.org/wiki/Hyperbolic_functions and https://en.wikipedia.org/wiki/Glaisher%E2%80%93Kinke
Hence

$$\frac{\pi^2}{96}\log\left(\frac{2e^3\pi^3}{A^{36}}\right) = \frac{\pi^2}{32}\left(\log(2\pi) + 1 - 12\log A - \frac{2}{3}\log 2\right) = \frac{3}{16}{}'(2) + \frac{\pi^2}{32}(1-\gamma) - \frac{\pi^2}{48}\log 2$$

and

$$\int_0^{\frac{\pi}{4}} \frac{\log\left(2\tanh^{-1}(\tan x)\right)\tanh^{-1}(\tan x)}{\tan(2x)}dx \underbrace{=}_{x=\arctan y\ ,\ dx=\frac{dy}{1+y^2}} \int_0^1 \frac{\log\left(\log\frac{1+y}{1-y}\right)\log\frac{1+y}{1-y}}{4y}\frac{1-y^2}{1+y^2}dy =$$

$$\underbrace{=}_{y=\frac{1-z}{1+z},\quad z=\frac{1-y}{1+y},\quad dy=-\frac{2}{(1+z)^2}dz} -\int_0^1 \frac{\log(-\log z)\log z}{4\frac{1-z}{1+z}}\frac{1-\left(\frac{1-z}{1+z}\right)^2}{1+\left(\frac{1-z}{1+z}\right)^2}\frac{2}{(1+z)^2}dz =$$

$$= -\int_0^1 z\frac{\log(-\log z)\log z}{1-z^4}dz \underbrace{=}_{z=e^{-t},dz=-e^{-t}dt} \int_0^\infty \frac{e^{-2t}t\log t}{1-e^{-4t}}dt.$$

Let s be a complex number with Re(s)>0 and put

$$f(s) = \int_0^\infty \frac{e^{-2t}t^s}{1-e^{-4t}}dt.$$

f(s) is analytic in Re(s)>0. We evaluate f(s):

$$f(s) = \int_0^\infty \frac{e^{-2t}t^s}{1-e^{-4t}}dt = \sum_{k=0}^\infty \int_0^\infty e^{-(4k+2)t}t^s dt = (s+1)\sum_{k=0}^\infty \frac{1}{(4k+2)^{s+1}} =$$

$$= \frac{(s+1)}{2^{s+1}}\sum_{k=0}^\infty \frac{1}{(2k+1)^{s+1}} = \frac{(s+1)}{2^{s+1}}\left(1-\frac{1}{2^{s+1}}\right)(s+1).$$

(This representation is the analytic continuation of f(s) to the whole complex plane.)
Finally,

$$\int_0^\infty \frac{e^{-2t}t\log t}{1-e^{-4t}}dt = f'(1) =$$

$$=' (2)\left(\frac{1}{4}-\frac{1}{16}\right)(2) + (2)\left(\frac{-\log 2}{4}+\frac{2\log 2}{16}\right)(2) + (2)\left(\frac{1}{4}-\frac{1}{16}\right)'(2) =$$

$$= (1-\gamma)\frac{3}{16}\cdot\frac{\pi^2}{6} - \frac{\log 2}{8}\cdot\frac{\pi^2}{6} + \frac{3}{16}{}'(2) = \frac{\pi^2}{96}\log\left(\frac{2e^3\pi^3}{A^{36}}\right).$$

Solution 5 by G. C. Greubel, Newport News, VA.

For the integral

$$I = \int_0^{\pi/4} \frac{\ln\left(2\,\tanh^{-1}(\tan x)\right)\,\tanh^{-1}(\tan x)}{\tan(2x)}\,dx$$

make the change of variable $x = \tan^{-1}(\tanh(u))$. This leads the integral to the form

$$I = \int_0^{\infty} \frac{u\,\ln(2u)\,du}{\sinh(2u)\,\cosh(2u)}$$

or

$$I = 2\int_0^{\infty} u\,\ln(2u)\,\operatorname{csch}(4u)\,du.$$

Now make the change of variable $y = 4u$ to obtain

$$\begin{aligned}
I &= \frac{1}{8}\int_0^{\infty} y\,\ln\left(\frac{y}{2}\right)\,\operatorname{csch}(y)\,dy \\
&= \frac{1}{8}\left[\int_0^{\infty} y\,\ln(y)\,\operatorname{csch}(y)\,dy - \ln 2\int_0^{\infty} y\,\operatorname{csch}(y)\,dy\right] \\
&= \frac{1}{8}\left[-\frac{3}{2}\,\ln 2\,\zeta(2) + \int_0^{\infty} y\,\ln(y)\,\operatorname{csch}(y)\,dy\right] \\
&= \frac{1}{8}\left[-\frac{3\,\zeta(2)}{2}\,\ln 2 + \frac{\zeta(2)}{2}\,(4\,\ln 2 + 3\,\ln\pi + 36\,\zeta^{'}(-1))\right] \\
&= \frac{\zeta(2)}{16}\,\left(\ln 2 + 3\,\ln\pi + 36\,\zeta^{'}(-1)\right) \\
I &= \frac{\zeta(2)}{16}\,\ln\left(2\,\pi^3\,e^{36\,\zeta^{'}(-1)}\right).
\end{aligned}$$

By using $A = \exp(\frac{1}{12} - \zeta^{'}(-1))$ then

$$I = \frac{\zeta(2)}{16}\,\ln\left(\frac{2\,e^3\,\pi^3}{A^{36}}\right).$$

This can be stated as

$$\int_0^{\pi/4} \frac{\ln\left(2\,\tanh^{-1}(\tan x)\right)\,\tanh^{-1}(\tan x)}{\tan(2x)}\,dx = \frac{\zeta(2)}{16}\,\ln\left(\frac{2\,e^3\,\pi^3}{A^{36}}\right)$$

which is the desired result.

Also solved by the problem proposer.

• **5735** *Proposed by Mihaly Bencze Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve the system of equations for real x, y and z:

$$\sqrt{3x+1} = z^2+1, \ \sqrt{3z+1} = y^2+1, \ \sqrt{3y+1} = x^2+1.$$

Solution 1 by Hossaena Tedla, ADA University, Baku, Azerbaijan.

Lemma: $x = y = z$ if and only if $\sqrt{3x+1} = z^2+1, \ \sqrt{3z+1} = y^2+1, \ \sqrt{3y+1} = x^2+1$.

Proof:

Suppose that $x \neq y \neq z$

$\sqrt{3x+1} = z^2+1 \qquad (1)$

$\sqrt{3z+1} = y^2+1 \qquad (2)$

$\sqrt{3y+1} = x^2+1 \qquad (3)$

Since $x \neq y$ let's subtract (1) from (2) $\sqrt{3z+1} - \sqrt{3x+1} = y^2 - z^2$ (4) Let's simplify (4) by multiplying by its conjugate

$$\left(\sqrt{3z+1} - \sqrt{3x+1}\right)\left(\sqrt{3z+1} + \sqrt{3x+1}\right) = \left(y^2 - z^2\right)\left(\sqrt{3z+1} + \sqrt{3x+1}\right)$$

$$(3z+1) - (3x+1) = \left(y^2 - z^2\right)\left(\sqrt{3z+1} + \sqrt{3x+1}\right)$$

$$3z - 3x = \left(y^2 - z^2\right)\left(\sqrt{3z+1} + \sqrt{3x+1}\right) \qquad (5)$$

Similarly, subtract (2) from (3)

$$\sqrt{3y+1} - \sqrt{3z+1} = x^2 - y^2 \qquad (6)$$

And let's simplify (6) by multiplying by its conjugate,

$$\left(\sqrt{3y+1} - \sqrt{3z+1}\right)\left(\sqrt{3y+1} + \sqrt{3z+1}\right) = \left(x^2 - y^2\right)\left(\sqrt{3y+1} + \sqrt{3z+1}\right)$$

$$(3y+1) - (3z+1) = \left(x^2 - y^2\right)\left(\sqrt{3y+1} + \sqrt{3z+1}\right)$$

$$3y - 3z = \left(x^2 - y^2\right)\left(\sqrt{3y+1} + \sqrt{3z+1}\right) \qquad (7)$$

Now subtract (7) from (5) and simplify,

$$(3z-3x)-(3y-3z) = \left(\left(y^2 - z^2\right)\left(\sqrt{3z+1} + \sqrt{3x+1}\right)\right) - \left(\left(x^2 - y^2\right)\left(\sqrt{3y+1} + \sqrt{3z+1}\right)\right)$$

$$6z-3x-3y=\left(\left(y^2-z^2\right)\left(\sqrt{3z+1}+\sqrt{3x+1}\right)\right)-\left(\left(x^2-y^2\right)\left(\sqrt{3y+1}+\sqrt{3z+1}\right)\right)$$

$$6z–3x–3y=\left(y^2-z^2\right)\left(\sqrt{3z+1}\right)+\left(y^2-z^2\right)\left(\sqrt{3x+1}\right)-\left(x^2-y^2\right)\left(\sqrt{3y+1}\right)\left(x^2-y^2\right)\left(\sqrt{3z+1}\right)$$

$$6z–3x–3y=\left(\left(y^2-z^2\right)\sqrt{3z+1}-\left(x^2-y^2\right)\sqrt{3y+1}\right)+\left(\left(y^2-z^2\right)\sqrt{3x+1}-\left(x^2-y^2\right)\sqrt{3z+1}\right)$$

By rearranging we will get,

$$6z-3x-3y=\left(y^2-z^2\right)\left(\sqrt{3z+1}-\sqrt{3x+1}\right)+\left(x^2-y^2\right)\left(\sqrt{3y+1}-\sqrt{3z+1}\right) \qquad (8)$$

Now let's substitute (4) and (6) in (8)

$$6z-3x-3y=\left(y^2-z^2\right)\left(y^2-z^2\right)+\left(x^2-y^2\right)\left(x^2-y^2\right)$$

$$6z-3x-3y=\left(y^2-z^2\right)^2+\left(x^2-y^2\right)^2$$

Hence, right-hand side equations are square and always a non-negative number. So,

$$6z-3x-3y\geqslant 0$$

But this is a contradiction to our assumption that $x\neq y\neq z$ as it implies $6z-3x-3y=0$ Therefore, $x=y=z$ Now we can continue to the problem,

$$\sqrt{3x+1}=z^2+1,\quad \sqrt{3z+1}=y^2+1,\quad \sqrt{3y+1}=x^2+1$$

Hence $x=y=z$, we can substitute one variable with the other to find the real value of $x,\ \ y,\ \ z$. So,

$\sqrt{3x+1}=z^2+1$, when we replace z by x and simplify

$$\sqrt{3x+1}=x^2+1$$

$$3x+1=\left(x^2+1\right)^2$$

$$3x+1=x^4+2x^2+1$$

$$x^4+2x^2+3x=0$$

$$x(x-1)\left(x^2+x+3\right)=0$$

Now we can split it,

$$x=0\ or\ x-1=0\ or\ x^2+x+3=0$$

We can get the zeros of $\left(x^2+x+3\right)$ by using quadratic formula ($\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ for ax^2+bx+c)

Therefore, the solutions are

$$x = 0 \text{ or } x = 1 \text{ or } x = \frac{-1}{2} + \frac{\sqrt{-11}}{2} \text{ or } x = \frac{-1}{2} - \frac{\sqrt{-11}}{2}$$

But we are only interested on real values so, $x = 0$ *or* $x = 1$. As we mentioned earlier $x = y = z$ then the real values of $x, \ y, \ z$ are

$$x = 0, \ \ y = 0, \ \ z = 0$$

$$x = 1, \ \ y = 1, \ \ z = 1$$

Check:

When $x = 0, \ \ y = 0, \ \ z = 0$

$$\sqrt{3x+1} = z^2 + 1, \quad \sqrt{3z+1} = y^2 + 1, \quad \sqrt{3y+1} = x^2 + 1$$

$$\sqrt{0+1} = 0 + 1, \qquad \sqrt{0+1} = 0 + 1, \qquad \sqrt{0+1} = 0 + 1$$

$$1 = 1, \qquad 1 = 1, \qquad 1 = 1$$

Correct

When $x = 1, \ \ y = 1, \ \ z = 1$

$$\sqrt{3x+1} = z^2 + 1, \quad \sqrt{3z+1} = y^2 + 1, \quad \sqrt{3y+1} = x^2 + 1$$

$$\sqrt{3+1} = 1 + 1, \qquad \sqrt{3+1} = 1 + 1, \quad \sqrt{3+1} = 1 + 1$$

$$2 = 2, \qquad 2 = 2, \qquad 2 = 2$$

Correct

Answer:

$$x = 0, \ \ y = 0, \ \ z = 0$$

$$x = 1, \ \ y = 1, \ \ z = 1$$

Solution 2 by Michel Bataille, Rouen, France.

Obvious solutions for (x, y, z) are $(0, 0, 0)$ and $(1, 1, 1)$. We show that there are no other solutions.
Let (x, y, z) be a solution. Since $\sqrt{3x+1} \geqslant 1$, we must have $x \geqslant 0$. Similarly, $y \geqslant 0$ and $z \geqslant 0$.
Squaring, we see that x, y, z satisfy

$$3x = z^4 + 2z^2, \quad 3z = y^4 + 2y^2, \quad 3y = x^4 + 2x^2. \tag{1}$$

We consider two mutually exclusive cases:
• if one among x, y, z is 0, then (1) shows that $x = y = z = 0$.

• if $x, y, z > 0$, then (1) gives for example $3(x-y) = (z^2 - x^2)(z^2 + x^2 + 2)$ and similar equalities. Assuming without loss of generality that $x = \min(x, y, z)$, we obtain $3(x-y) \leqslant 0$ and $(z^2 - x^2)(z^2 + x^2 + 2) \geqslant 0$, so that $x = y = z$. Then, we have $x^4 + 2x^2 - 3x = 0$, that is, $x(x-1)(x^2 + x + 3) = 0$ and therefore $x = 1$ and $x = y = z = 1$.
Thus $(x, y, z) = (0, 0, 0)$ or $(1, 1, 1)$ and we are done.

Solution 3 by Ivan Hadinata, Department of mathematics, Gadjah Mada University, Yogyakarta, Indonesia.

Since all of $x^2 + 1$, $y^2 + 1$, and $z^2 + 1$ are greater than or equal to 1, then so are $3x + 1$, $3y + 1$, and $3z + 1$. Therefore, $x, y, z \geqslant 0$. WLOG, let $x = min\{x, y, z\}$. Consequently,

$$x^2 + 1 = min\{x^2 + 1, y^2 + 1, z^2 + 1\} = min\{\sqrt{3x+1}, \sqrt{3y+1}, \sqrt{3z+1}\} = \sqrt{3x+1}$$

$$\Longrightarrow \quad x = 0 \quad \text{or} \quad x = 1.$$

If $x = 0$, checking it to the original system of equations gives us $x = y = z = 0$.
If $x = 1$, checking it to the original system of equations gives us $x = y = z = 1$.
Hence the solutions are $(x, y, z) = (0, 0, 0), (1, 1, 1)$.

Solution 4 proposed by G. C. Greubel, Newport News, VA.

The set of equations can be placed into the form

$$\begin{aligned} 3\,x &= z^4 + 2\,z^2 \\ 3\,z &= y^4 + 2\,y^2 \\ 3\,y &= x^4 + 2\,x^2. \end{aligned}$$

For x, y, or z, the general equation to solve is given by

$$3^{21}\,x = x^{16}\,(x^2+2)^8\,(18 + x^4\,(x^2+2)^2)^4 + 2\cdot 3^{10}\,x^8\,(x^2+2)^4\,(18 + x^4\,(x^2+2)^2)^2.$$

By inspection it is seen that $x = 0$ is a solution. The equation can be factored into the form

$$3^{21}\,(x-1) = (x-1)(x+1)(x^2+3)\,(x^4 + 2\,x^2 + 3)\,(x^8 + 4\,x^6 + 4\,x^4 + 27)\,p_1(x)\,p_2(x)$$

where $p_1(x) = x^{16} + 8\,x^{14} + 24\,x^{12} + 32\,x^{10} + 34\,x^8 + 72\,x^6 + 72\,x^4 + 243$ and $p_2(x) = x^{32} + 16\,x^{30} + 112\,x^{28} + 448\,x^{26} + 1156\,x^{24} + 2224\,x^{22} + 3952\,x^{20} + 6784\,x^{18} + 9220\,x^{16} + 9504\,x^{14} + 10080\,x^{12} + 10368\,x^{10} + 5184\,x^8 + 177147$. From this it can be seen that $x = 1$ is a solution. The remaining roots that can be determined are complex in nature. This leads to $x = \{0, 1\}$ being the only real roots. The same applies for y and z.

A shorter method can be used. This is the assumption that x, y, and z are equal for which the equation set reduces to $\sqrt{3x+1} = x^2 + 1$ which factors to $x\,(x-1)\,(x^2+x+3) = 0$ which yields the real roots as $x = \{0, 1\}$.

By making use of either method the solutions of the set of equations, for real values, is $(x, y, z) =$

$\{(0,0,0),(1,1,1)\}$.

Solution 5 proposed by David A. Huckaby, Angelo State University, San Angelo, TX.

Squaring both sides of each equation and then subtracting 1 from both sides of each resulting equation yields $3x = z^4+2z^2$, $3z = y^4+2y^2$, and $3y = x^4+2x^2$. By inspection, $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$ are solutions.

From the first equation, $x = \frac{1}{3}(z^4 + 2z^2)$, so that the third equation becomes

$$3y = \left[\frac{1}{3}(z^4 + 2z^2)\right]^4 + \left[\frac{1}{3}(z^4 + 2z^2)\right]^2. \tag{7}$$

From the second equation, $z = \frac{1}{3}(y^4 + y^2)$, so that equation (11) becomes

$$3y = \left[\frac{1}{3}\left(\left[\frac{1}{3}(y^4+y^2)\right]^4 + 2\left[\frac{1}{3}(y^4+y^2)\right]^2\right)\right]^4 + \left[\frac{1}{3}\left(\left[\frac{1}{3}(y^4+y^2)\right]^4 + 2\left[\frac{1}{3}(y^4+y^2)\right]^2\right)\right]^2. \tag{8}$$

The right-hand side of equation (8) is a polynomial $r(y)$ with all coefficients positive and all exponents even. Define polynomial $p(y) = r(y) - 3y$, whose roots are obviously the solutions to equation (8). Since all of the coefficients of $r(y)$ are positive, by Descartes' rule of signs, $p(y)$ has one positive root. Since all of the exponents of $r(y)$ are even, $r(-y) = r(y)$, so that $p(-y) = r(y) + 3y$. So by Descartes' rule of signs, $p(y)$ has no negative roots. So the $y = 0$ and $y = 1$ solutions noted earlier are the only real solutions.

So the two real solutions to the system are $x = 0, y = 0, z = 0$ and $x = 1, y = 1, z = 1$.

Solution 6 proposed by Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania.

Consider

$$A \overset{\text{not}}{=} \{x, y, z\}.$$

It is clear that

$$t^2 + 1 \geqslant 0, (\forall)\, t \in A.$$

It follows that $x, y, z \geqslant 0$.

We find

$$\sqrt{3x+1} - \sqrt{3z+1} = \left(z^2+1\right) - \left(y^2+1\right) \Leftrightarrow \sqrt{3x+1} - \sqrt{3z+1} = (z-y)(z+y).$$

It follows that

$$x < z \Leftrightarrow z < y.$$

From

$$\sqrt{3z+1} - \sqrt{3y+1} = \left(y^2+1\right) - \left(x^2+1\right) \Leftrightarrow \sqrt{3z+1} - \sqrt{3y+1} = (y-x)(y+x).$$

We obtain

$$x < z < y < x,$$

a contradiction. We obtain $x = y = z$.

The equation of problem is

$$\sqrt{3x+1} = x^2+1 \Leftrightarrow \left(\sqrt{3x+1}\right)^2 = \left(x^2+1\right)^2 \Leftrightarrow 3x+1 = x^4+2x^2+1;$$

we obtain $x^4 + 2x^2 - 3x = 0 \Leftrightarrow x\left(x^3+2x-3\right) = 0 \Leftrightarrow x\left(x^3 - x^2 + x^2 - x + 3x - 3\right) = 0.$

We have

$$x^4 + 2x^2 - 3x = 0 \Leftrightarrow x(x-1)\left(x^2+x+3\right) = 0.$$

We obtain

$$x \in \{0, 1\}.$$

Solution 7 by Brian D. Beasley, Simpsonville, SC.

We show that the two real solutions are $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (1, 1, 1)$.

Any real solution (x, y, z) of the given system must satisfy $x \geqslant 0$, $y \geqslant 0$, and $z \geqslant 0$ with

$$3x = z^4 + 2z^2,\ \ 3z = y^4 + 2y^2,\ \text{ and }\ 3y = x^4 + 2x^2.$$

For each positive integer n, let $S_n = x^n + y^n + z^n$. Then adding the three previous equations yields

$$3S_1 = S_4 + 2S_2.$$

(i) If $x = 0$, then $y = 0$ and hence $z = 0$. It is straightforward to verify that $(x, y, z) = (0, 0, 0)$ satisfies the original system.

(ii) If $x = 1$, then $y = 1$ and hence $z = 1$. It is straightforward to verify that $(x, y, z) = (1, 1, 1)$ satisfies the original system.

(iii) If $0 < x < 1$, then $1 < x^2 + 1 < 2$, so

$$1 < 3y+1 < 4 \quad \Longrightarrow \quad 0 < y < 1.$$

Similarly, $0 < z < 1$ as well. But this would imply $x > x^2 > x^4$, $y > y^2 > y^4$, and $z > z^2 > z^4$, which yields the contradiction

$$S_1 > S_2 > S_4 \quad \Longrightarrow \quad 3S_1 > S_4 + 2S_2.$$

(iv) If $x > 1$, then $x^2 + 1 > 2$, so

$$3y + 1 > 4 \quad \Longrightarrow \quad y > 1.$$

Similarly, $z > 1$ as well. But this would imply $x < x^2 < x^4$, $y < y^2 < y^4$, and $z < z^2 < z^4$, which yields the contradiction

$$S_1 < S_2 < S_4 \quad \Longrightarrow \quad 3S_1 < S_4 + 2S_2.$$

Solution 8 by Albert Stadler, Herrliberg, Switzerland.

Clearly, x=y=z=0 and x=y=z=1 are solutions to the given system of equations. We will show that these are the only ones.
We have x⩾-1/3 , y⩾-1/3, z⩾-1/3, since the square root of a negative number is not real. The given equations are then equivalent to

$$x=\frac{z^2\left(z^2+2\right)}{3},\quad z=\frac{y^2\left(y^2+2\right)}{3},\quad y=\frac{x^2\left(x^2+2\right)}{3}.$$

Hence x,y,z⩾0, and we may even assume that x,y,z>0, as x=y=z=0 was already identified as a solution.
Let $f(t)=\dfrac{t^2\left(t^2+2\right)}{3}$. The function f(t) is monotonically increasing for t>0 and f(**??**)=1. Clearly,

$$x=f\Big(f\left(f(x)\right)\Big),\quad y=f\Big(f\left(f(y)\right)\Big),\quad z=f\Big(f\left(f(z)\right)\Big).$$

We have f(x)>x if x>1 and f(x)<x if x<1, since

$$f(x)-x=\frac{1}{3}(x-1)\,x\left(x^2+x+3\right).$$

Suppose that (x,y,z) is a solution to x=f(f(f(x))), y=f(f(f(y))), z=f(f(f(z))) (under the assumption x,y,z>0). Then x=y=z=1, for suppose if possible that x>1. Then x=f(f(f(x)))>f(f(x))>f(x)>x which is absurd, and similarly the assumption that x<1 leads to x=f(f(f(x)))<f(f(x))<f(x)<x which is absurd as well.

Also solved by the problem proposer.

• **5736** *Proposed by Seán M. Stewart, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia.*

If *m* is a non-negative integer, evaluate

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{k+2}{m+n}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2n+2i+1}\right]^n\right\}.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Since $\binom{k+2}{m+n} = 0$ if $k \leqslant m$ and $n > 2$, the left-hand side of the equation is equal to

$$\begin{aligned}
\text{LHS} &= \sum_{n=0}^{2}\sum_{k=0}^{m} (-1)^k \binom{m}{k}\binom{k+2}{m+n}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2n+2i+1}\right]^n \\
&= \sum_{k=m-2}^{m} (-1)^k \binom{m}{k}\binom{k+2}{m}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+1}\right]^0 \\
&\quad + \sum_{k=m-1}^{m} (-1)^k \binom{m}{k}\binom{k+2}{m+1}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+3}\right]^1 \\
&\quad + \sum_{k=m}^{m} (-1)^k \binom{m}{k}\binom{k+2}{m+2}\left[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+5}\right]^2 .
\end{aligned}$$

Taking advantage of the Gregory-Leibniz series $\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1} = \pi/4$, we have

$$\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+3} = \frac{\pi}{4} - 1 + \frac{1}{3} \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+5} = -\frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5}.$$

Hence,

$$\begin{aligned}
(-1)^m\,\text{LHS} &= \binom{m}{m-2} - \binom{m}{m-1}\binom{m+1}{m} + \binom{m+2}{m} \\
&\quad + \left[-\binom{m}{m-1} + \binom{m+2}{m+1}\right]\left(\frac{\pi}{4} - \frac{2}{3}\right) + \left(-\frac{\pi}{4} + \frac{13}{15}\right)^2 \\
&= 1 + 2\left(\frac{\pi}{4} - \frac{2}{3}\right) + \left(\frac{\pi}{4} - \frac{13}{15}\right)^2 .
\end{aligned}$$

After a simplification we obtain

$$\text{LHS} = (-1)^m\left[\left(\frac{\pi}{4} + \frac{2}{15}\right)^2 + \frac{2}{5}\right].$$

Solution 2 by Péter Fülöp, Gyömrő, Hungary.

If $m < k$ or $k + 2 < n + m$ then the sum equals to zero.

It means that the value of the sum different from zero in the following case:

$$\boxed{m \geqslant k \geqslant m+n-2}$$

$$m \geqslant m+n-2$$

$$2 \geqslant n \geqslant 0$$

1. Let's calculate the sum in $n=0$ case, $\underline{m \geqslant k \geqslant m-2}$:

$$S_{0,k=m-2} = \left\{(-1)^{m-m+2}\binom{m}{m-2}\binom{m-2+2}{m}\Big[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+1}\Big]^0\right\} = \frac{m(m-1)}{2}$$

$$S_{0,k=m-1} = \left\{(-1)^{m-m+1}\binom{m}{m-1}\binom{m-1+2}{m}\Big[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+1}\Big]^0\right\} = -m(m+1)$$

$$S_{0,k=m} = \left\{(-1)^{m-m}\binom{m}{m}\binom{m+2}{m}\Big[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+1}\Big]^0\right\} = \frac{(m+1)(m+2)}{2}$$

After cancelling the three terms we get that:

$$S_0 = 1$$

2. $n=1$ case, $\underline{m \geqslant k \geqslant m-1}$:

$$S_{1,k=m-1} = \left\{(-1)^{m-m+1}\binom{m}{m-1}\binom{m-1+2}{m+1}\Big[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+3}\Big]^1\right\} = -m(\frac{\pi}{4}-\frac{2}{3})$$

Where $\sum\limits_{i=0}^{\infty}\frac{(-1)^i}{2i+1} = \beta(1)$ - Dirichlet beta function at $s=1$ place.

$$S_{1,k=m} = \left\{(-1)^{m-m}\binom{m}{m}\binom{m+2}{m+1}\Big[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+3}\Big]^1\right\} = (m+2)(\frac{\pi}{4}-\frac{2}{3})$$

After cancelling the two terms we get that:

$$S_1 = \frac{\pi}{2}-\frac{4}{3}$$

3. $n=2$ case, $\underline{k=m}$:

$$S_{2,k=m} = \left\{(-1)^{m-m}\binom{m}{m}\binom{m+2}{m+2}\Big[\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i+5}\Big]^2\right\} = \Big[-\frac{\pi}{4}+\frac{13}{15}\Big]^2$$

Finally the original sum equals to $S = S_0 + S_1 + S_2$

$$S = \frac{\pi^2}{16}+\frac{\pi}{15}+\frac{94}{225}$$

Solution 3 by Moti Levy, Rehovot, Israel.

The sum can be rewritten as

$$S := \sum_{n=0}^{\infty}\left(-\frac{1}{2}\Phi\left(-1,1,\frac{2n+1}{2}\right)\right)^{n} d\left(m,n\right), \tag{9}$$

where

$$d\left(m,n\right) := (-1)^{m}\sum_{k=0}^{m}(-1)^{k}\binom{m}{k}\binom{k+2}{m+n},$$

and $\Phi\left(z,s,\alpha\right)$ is the Lerch transcendent,

$$\Phi\left(z,s,\alpha\right) = \sum_{i=0}^{\infty}\frac{z^{i}}{\left(i+\alpha\right)^{s}}.$$

We begin by simplifying $d\left(m,n\right)$.

$$\begin{aligned} d\left(m,n\right) &:= (-1)^{m}\sum_{k=0}^{m}(-1)^{k}\binom{m}{k}\binom{k+2}{m+n} \\ &= (-1)^{m}\sum_{k=m+n-2}^{m}(-1)^{k}\binom{m}{k}\binom{k+2}{m+n} \\ &= (-1)^{n}\sum_{l=0}^{2-n}(-1)^{l}\binom{m}{m+n-2+l}\binom{l+m+n}{m+n} \end{aligned}$$

$$\begin{aligned} d\left(m,0\right) &= \sum_{l=0}^{2}(-1)^{l}\binom{m}{m-2+l}\binom{l+m}{m} = 1 \\ d\left(m,1\right) &= (-1)\sum_{l=0}^{1}(-1)^{l}\binom{m}{m-1+l}\binom{l+m+1}{m+1} = 2 \\ d\left(m,2\right) &= 1 \end{aligned}$$

We conclude that

$$d\left(m,n\right) = \begin{cases} 0 & if \quad n>2 \\ 1 & if \quad n=2 \\ 2 & if \quad n=1 \\ 1 & if \quad n=0 \end{cases}. \tag{10}$$

Applying (10) to (9) , we get

$$S = 1-\left(\Phi\left(-1,1,\frac{3}{2}\right)-\frac{2}{3}\right)+\frac{1}{4}\left(\Phi\left(-1,1,\frac{5}{2}\right)-\frac{2}{5}\right)^{2}.$$

The integral representation of the Lerch transcendent is

$$\Phi(z,s,\alpha) = \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-\alpha t}}{1-ze^{-t}} dt.$$

$$\Phi\left(-1,1,\frac{3}{2}\right) = \int_0^{\infty} \frac{e^{-\frac{3}{2}t}}{1+e^{-t}} dt = \int_0^1 \frac{\sqrt{u}}{u+1}\, du = \int_0^1 \frac{2w^2}{w^2+1}\, dw = 2 - \frac{1}{2}\pi.$$

$$\Phi\left(-1,1,\frac{5}{2}\right) = \int_0^{\infty} \frac{e^{-\frac{5}{2}t}}{1+e^{-t}} dt = \int_0^1 \frac{\left(\sqrt{u}\right)^3}{u+1}\, du = \int_0^1 \frac{2w^4}{w^2+1}\, dw = \frac{1}{2}\pi - \frac{4}{3}.$$

$$S = 1 - \left(2 - \frac{1}{2}\pi - \frac{2}{3}\right) + \frac{1}{4}\left(\frac{1}{2}\pi - \frac{4}{3} - \frac{2}{5}\right)^2 = \frac{94}{225} + \frac{\pi}{15} + \frac{\pi^2}{16} \cong 1.244\,1.$$

Solution 4 by Michel Bataille, Rouen, France.

The required sum writes as $S_m = \sum_{n=0}^{\infty} U_n \cdot A_{m,n}$ where

$$U_n = \left[\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2n+2i+1}\right]^n, \qquad A_{m,n} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k+2}{m+n}.$$

We observe that for $n \geqslant 3$ and $0 \leqslant k \leqslant m$, we have $k+2 < m+n$ so that $\binom{k+2}{m+n} = 0$ and $A_{m,n} = 0$. It follows that $S_m = U_0 A_{m,0} + U_1 A_{m,1} + U_2 A_{m,2}$.
We calculate

$$\begin{aligned} A_{m,0} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k+2}{m} &= \binom{m}{m-2}\binom{m}{m} - \binom{m}{m-1}\binom{m+1}{m} + \binom{m}{m}\binom{m+2}{m} \\ &= \frac{m(m-1)}{2} - m(m+1) + \frac{(m+2)(m+1)}{2} = 1 \end{aligned}$$

$$A_{m,1} = -\binom{m}{m-1}\binom{m+1}{m+1} + \binom{m}{m}\binom{m+2}{m+1} = 2, \qquad A_{m,2} = \binom{m}{m}\binom{m+2}{m+2} = 1$$

and deduce that $S_m = U_0 + 2U_1 + U_2$.
Now, we obtain $U_0 = 1, \quad U_1 = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i+3} = \frac{\pi}{4} - 1 + \frac{1}{3} = \frac{\pi}{4} - \frac{2}{3}$ and

$$U_2 = \left[\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i+5}\right]^2 = \left(\frac{\pi}{4} - 1 + \frac{1}{3} - \frac{1}{5}\right)^2 = \frac{\pi^2}{16} + \frac{169}{225} - \frac{13\pi}{30}$$

and therefore

$$S_m = 1 + \frac{\pi}{2} - \frac{4}{3} + \frac{\pi^2}{16} + \frac{169}{225} - \frac{13\pi}{30} = \frac{\pi^2}{16} + \frac{\pi}{15} + \frac{94}{225}.$$

Solution 5 by by G. C. Greubel, Newport News, VA.

Let the series in question be

$$S_m = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k+2}{m+n} \left[\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2n+2i+1} \right]^n \right\}.$$

By using the series

$$\sum_{k=0}^{m} \binom{m}{k} \binom{k+2}{m+n} (-1)^{m-k} = \frac{(-1)^m \, (m+n)! \, (n-3)!}{n! \, (m+n-3)!} \binom{2}{m+n} = \binom{2}{n}$$

then

$$\begin{aligned} S_m &= \sum_{n=0}^{\infty} \binom{2}{n} \left[\sum_{i=0}^{\infty} \frac{(-1)^i}{2n+2i-1} \right]^n = 1 + 2 \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} + \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{2i+3} \right)^2 \\ &= 1 - \frac{\pi}{2} + \left(\frac{\pi - 4}{4} \right)^2 \\ &= 2 - \pi + \frac{3\,\zeta(2)}{8}. \end{aligned}$$

This is the result for all $m \geqslant 0$.

Solution 6 by by Albert Stadler, Herrliberg, Switzerland.

We start with

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} t^k = (-1)^m (1-t)^m.$$

Hence

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k+2}{m+n} = \frac{1}{(m+n)!} \frac{d^{m+n}}{dt^{m+n}} \left(t^2 \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} t^k \right) \Bigg|_{t=1} =$$

$$= \frac{(-1)^m}{(m+n)!} \frac{d^{m+n}}{dt^{m+n}} \left(\left((1-t)^2 - 2\,(1-t) + 1 \right) (1-t)^m \right) \Bigg|_{t=1} =$$

$$= \frac{1}{(m+n)!} \frac{d^{m+n}}{dt^{m+n}} \left((t-1)^{m+2} + 2(t-1)^{m+1} + (t-1)^m \right) \Bigg|_{t=1} = \frac{1}{(m+n)!} f_{m,n}$$

where $f_{m,0} = m!$, $f_{m,1} = 2\,(m+1)!$, $f_{m,2} = (m+2)!$, $f_{m,n} = 0$, $n \geqslant 3$.

We conclude that

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k+2}{m+n} \left[\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2n+2i+1} \right]^n \right\} = 1 + 2 \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i+3} + \left[\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i+5} \right]^2 =$$

$$= 1 + \frac{1}{6}(3\pi - 8) + \left(\frac{13}{15} - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{16} + \frac{\pi}{15} + \frac{94}{225}.$$

Also solved by ; and the problem proposer.

• **5737** *Proposed by Rafael Jakimczuk, Departamento de Ciencias Básicas, División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Find the limit

$$\ell = \lim_{n\to\infty}\left[\prod_{k=1}^{n-1}\left(1 + \frac{k}{n}\right)\right]^{1/n}.$$

Solution 1 by Rob Downes, Henderson, NV.

Taking the natural logarithm of both sides gives:

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\ln\left[\prod_{k=1}^{n-1}\left(1 + \frac{k}{n}\right)\right] = \lim_{n\to\infty}\frac{1}{n}\left[\sum_{k=1}^{n-1}\ln\left(1 + \frac{k}{n}\right)\right]$$

Adding and subtracting $\ln\left(1 + \frac{n}{n}\right)$:

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\left[\left(\sum_{k=1}^{n}\ln\left(1 + \frac{k}{n}\right)\right) - \ln(1 + \frac{n}{n})\right] = \lim_{n\to\infty}\left[-\frac{\ln 2}{n} + \sum_{k=1}^{n}\ln\left(1 + \frac{k}{n}\right)\frac{1}{n}\right] =$$

$$\int_0^1 \ln(1 + x)\,dx = -1 + \ln 4$$

Exponentiating both sides yields the desired result:

$$L = \frac{4}{e}.$$

Solution 2 by Albert Stadler, Herrliberg, Switzerland.

By the definition of a Riemann sum,

$$\log(l) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\log\left(1 + \frac{k}{n}\right) = \int_0^1 \log(1 + x)\,dx = \log 4 - 1.$$

Hence $l = 4/e$.

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By taking logarithms, we get

$$\begin{aligned}
\ln \ell &= \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n-1} \ln\left(1+\frac{k}{n}\right) \text{ (which is a Riemamm sum)} \\
&= \int_0^1 \ln(1+x)\, \mathrm{d}x \\
&= (x+1)\ln(x+1) - x \Big]_0^1 \\
&= \ln 4 - 1 = \ln\frac{4}{e}
\end{aligned}$$

and, consequently, $\ell = \dfrac{4}{e}$.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Note

$$\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right) = \frac{1}{n^{n-1}}\prod_{k=1}^{n-1}(n+k) = \frac{(2n-1)!}{n^{n-1}n!}.$$

By Stirling's approximation,

$$n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$$

and

$$(2n-1)! \sim \sqrt{2\pi}(2n-1)^{2n-1/2}e^{-2n-1},$$

so

$$\frac{(2n-1)!}{n^{n-1}n!} \sim \left(\frac{2n-1}{n}\right)^{2n-1/2} e^{1-n},$$

and

$$\left[\frac{(2n-1)!}{n^{n-1}n!}\right]^{1/n} \sim \left(2-\frac{1}{n}\right)^{2-1/n} e^{-1+1/n}.$$

Finally,

$$\ell = \lim_{n\to\infty}\left[\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n} = \lim_{n\to\infty}\left[\left(2-\frac{1}{n}\right)^{2-1/n} e^{-1+1/n}\right] = \frac{4}{e}.$$

Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.

The product can be rewritten

$$\begin{aligned}\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right) &= \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n-1}{n}\right)\\ &= \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\left(\frac{n+3}{n}\right)\cdots\left(\frac{2n-1}{n}\right)\\ &= \frac{(2n-1)!}{n^n(n-1)!}.\end{aligned}$$

Using Stirling's approximation for the two factorials gives

$$\begin{aligned}\ell = \lim_{n\to\infty}\left[\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n} &= \lim_{n\to\infty}\left[\frac{(2n-1)!}{n^n(n-1)!}\right]^{1/n}\\ &= \lim_{n\to\infty}\left[\frac{\sqrt{2\pi(2n-1)}\left(\frac{2n-1}{e}\right)^{2n-1}}{n^n\sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}}\right]^{1/n}\\ &= \lim_{n\to\infty}\left[\frac{(2n-1)^{2n}}{n^n(n-1)^n e^n}\sqrt{\frac{n-1}{2n-1}}\right]^{1/n}\\ &= \lim_{n\to\infty}\frac{(2n-1)^2}{n(n-1)e}\sqrt[2n]{\frac{n-1}{2n-1}}. \qquad (11)\end{aligned}$$

To find the limit of the second factor, consider the following limit calculated using L'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty}\ln\left[\sqrt[2n]{\frac{n-1}{2n-1}}\right] &= \lim_{n\to\infty}\frac{1}{2n}[\ln(n-1)-\ln(2n-1)]\\ &= \lim_{n\to\infty}\left[\frac{\ln(n-1)}{2n}-\frac{\ln(2n-1)}{2n}\right]\\ &= \lim_{n\to\infty}\left[\frac{1}{2(n-1)}-\frac{1}{2n-1}\right]\\ &= 0.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{n\to\infty}\sqrt[2n]{\frac{n-1}{2n-1}} &= \lim_{n\to\infty} e^{\ln\left[\sqrt[2n]{\frac{n-1}{2n-1}}\right]}\\ &= e^{\lim_{n\to\infty}\ln\left[\sqrt[2n]{\frac{n-1}{2n-1}}\right]}\\ &= e^0 = 1.\end{aligned}$$

So from equation (11), the desired limit is

$$\begin{aligned}\ell &= \lim_{n\to\infty} \frac{(2n-1)^2}{n(n-1)e} \sqrt[2n]{\frac{n-1}{2n-1}} \\ &= \lim_{n\to\infty} \frac{(4n^2-4n+1)}{(n^2-n)e} \cdot \lim_{n\to\infty} \sqrt[2n]{\frac{n-1}{2n-1}} \\ &= \frac{4}{e}\cdot 1 = \frac{4}{e}.\end{aligned}$$

Solution 6 by G. C. Greubel, Newport News, VA.

The limit can be seen as

$$L = \lim_{n\to\infty} \left[\frac{\Gamma(2n+1)}{2\, n^n\, \Gamma(n+1)}\right]^{1/n}.$$

By using Stirling's approximation for the Gamma function then it can be found that

$$\frac{\Gamma(2x+1)}{2\, x^x\, \Gamma(x+1)} \approx e^{-x}\, 2^{2x-1/2} \left(1 - \frac{1}{24\, x} + \frac{1}{1152\, x^2} + O\left(\frac{1}{x^3}\right)\right)$$

which gives

$$\begin{aligned}\left(\frac{\Gamma(2x+1)}{2\, x^x\, \Gamma(x+1)}\right)^{1/x} &\approx \frac{4}{e}\, 2^{-1/(2x)} \left(1 - \frac{1}{24\, x} + \frac{1}{1152\, x^2} + O\left(\frac{1}{x^3}\right)\right)^{1/x} \\ &\approx \frac{4}{e}\, 2^{-1/(2x)} \left(1 - \frac{1}{24\, x^2} + O\left(\frac{1}{x^4}\right)\right).\end{aligned}$$

Taking the desired limit then

$$\lim_{n\to\infty} \left[\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n} = \lim_{n\to\infty} \left[\frac{\Gamma(2n+1)}{2\, n^n\, \Gamma(n+1)}\right]^{1/n} = \frac{4}{e}.$$

Solution 7 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Letting $P_n = \left[\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n}$, we have

$$\ln P_n = \frac{1}{n}\sum_{k=1}^{n-1} \ln\left(1+\frac{k}{n}\right) = \sum_{k=0}^{n-1} \ln\left(1+\frac{k}{n}\right)\cdot\frac{1}{n},$$

which is a left Riemann sum for the integral $\int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$, so that

$$\lim_{n\to\infty} P_n = e^{\lim_{n\to\infty} \ln P_n} = e^{2\ln 2 - 1} = \frac{4}{e}.$$

Solution 8 by Ivan Hadinata, Department of mathematics, Gadjah Mada University, Yogyakarta, Indonesia.

Let log and e be respectively the natural logarithm and Euler's number. By Riemann Sum-Integral,

$$\begin{aligned}\log \ell &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} \log\left(1+\frac{k}{n}\right)\\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\left(1+\frac{k}{n}\right) - \lim_{n\to\infty}\frac{\log 2}{n}\\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \log\left(1+\frac{k}{n}\right)\\ &= \int_0^1 \log(1+x)\,dx\\ &= \quad \log\left(\frac{4}{e}\right).\end{aligned}$$

Then,

$$\ell = \frac{4}{e}.$$

Solution 9 by Michel Bataille, Rouen, France.

We have

$$\ln\left[\left(\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right)^{1/n}\right] = \frac{1}{n}\sum_{k=1}^{n} \ln\left(1+\frac{k}{n}\right) - \frac{\ln 2}{n}.$$

Now, from $\lim_{n\to\infty} \frac{\ln 2}{n} = 0$ and

$$\lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\right] = \int_0^1 \ln(1+x)\,dx = [(x+1)\ln(1+x) - x]_0^1 = 2\ln 2 - 1 = \ln\frac{4}{e}$$

we deduce that $\ell = \frac{4}{e}$.

Solution 10 by Moti Levy, Rehovot, Israele.

$$\begin{aligned}\ln(l) &= \lim_{n\to\infty}\left[\ln\left(\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right)\right]^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\ln\left(1+\frac{k}{n}\right)\\ &= \int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1.\end{aligned}$$

$$l = e^{2\ln 2 - 1} = \frac{4}{e}.$$

Solution 11 by Péter Fülöp, Gyömrő, Hungary.

Let's transform the product as follows:

$$p = \Big[\prod_{k=1}^{n-1}\Big(1+\frac{k}{n}\Big)\Big]^{\frac{1}{n}} = \Big(\frac{1}{n}\Big)^{\frac{1}{n}}(1+n)^{\frac{1}{n}}\Big(\frac{1}{n}\Big)^{\frac{1}{n}}(2+n)^{\frac{1}{n}}\ldots\ldots\Big(\frac{1}{n}\Big)^{\frac{1}{n}}(n-1+n)^{\frac{1}{n}}$$

$$p = \underbrace{\Big[\Big(\frac{1}{n}\Big)^{\frac{1}{n}}\Big]^{n-1}}_{\frac{(n)^{\frac{1}{n}}}{n}}(1+n)^{\frac{1}{n}}(2+n)^{\frac{1}{n}}\ldots\ldots(n-1+n)^{\frac{1}{n}} = \frac{1}{n}\Big[\frac{(2n-1)!}{(n-1)!}\Big]^{\frac{1}{n}}$$

Applying the Stirling's approximation to the factorials:

$$p = \frac{1}{n}\Big[\frac{(2n-1)!}{(n-1)!}\Big]^{\frac{1}{n}} = \frac{1}{n}\Big[\frac{(2n-1)^{2n-\frac{1}{2}}}{(n-1)^{n-\frac{1}{2}}}\frac{e^{n-1}}{e^{2n-1}}\Big]^{\frac{1}{n}}$$

Performing further transformations:

$$p = \frac{2n-1}{n}\Big[\Big(\frac{2n-1}{n-1}\Big)^{n-\frac{1}{2}}\Big]^{\frac{1}{n}}e^{-1} = (2-\frac{1}{n})\Big(1+\frac{n}{n-1}\Big)\Big(1+\frac{n}{n-1}\Big)^{-\frac{1}{2n}}e^{-1}$$

Finally we get the limit values:

$$l = \lim_{n\to\infty}\underbrace{(2-\frac{1}{n})}_{\to 2}\underbrace{\Big(1+\frac{n}{n-1}\Big)}_{\to 2}\underbrace{\Big(1+\frac{n}{n-1}\Big)^{-\frac{1}{2n}}}_{\to 1}e^{-1} = \frac{4}{e}$$

Solution 12 by Toyesh Prakash Sharma (Student) Agra College, Agra, India.

Given

$$l = \lim_{n\to\infty}\left[\prod_{k=`1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n} \Rightarrow \ln l = \lim_{n\to\infty}\frac{1}{n}\ln\prod_{k=`1}^{n-1}\left(1+\frac{k}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\left[-\ln 2 + \sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\right]$$

$$= -\lim_{n\to\infty}\frac{\ln 2}{n} + \lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right) = \lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$$

Using Definite integrals as limit of the sum,

$$\lim_{n\longrightarrow\infty}\frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right) = \int_0^1 \ln(1+x)\mathrm{dx}$$

Now applying Integration by parts gives

$$\int_0^1 \ln(1+x)\mathrm{dx} = \left[\ln(1+x)\int \mathrm{dx} - \int\left(\frac{d}{\mathrm{dx}}\ln(1+x)\int \mathrm{dx}\right)\mathrm{dx}\right]_0^1$$

$$= \left[x\ln(1+x) - \int \frac{x}{1+x}\mathrm{dx}\right]_0^1 = \left[x\ln(1+x) - x + \ln(1+x)\right]_0^1 = 2\ln 2 - 1$$

Now,

$$\ln l = 2\ln 2 - 1 \Rightarrow l = e^{2\ln 2 - 1} = \frac{e^{\ln 4}}{e} = \frac{4}{e}.$$

Solution 13 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

We have

$$\log\left[\prod_{k=1}^{n-1}\left(1+\frac{k}{n}\right)\right]^{1/n} = \frac{1}{n}\sum_{k=1}^{n-1}\log\left(1+\frac{k}{n}\right) \to \int_0^1 \log(1+x)\,dx = \log 4 - 1$$

as $n\to\infty$, since the second expression is a Riemannian sum of $f(x) = \log(1+x)$. It follows that $\ell = 4/e$.

Solution 14 by Yunyong Zhang, Chinaunicom, Yunnan, China.

$$\ln(\ell) = \lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\ln(1+\frac{k}{n})}{n}$$

According to Riemann Sum

$$A = \lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\ln(1+\frac{k}{n})}{n} = \int_0^1 \ln(1+x)\mathrm{d}x$$

So

$$\ln(\ell) = \lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\ln(1+\frac{k}{n})}{n} = \int_0^1 \ln(1+x)\mathrm{d}x$$

$$= x\ln(1+x)\Big|_0^1 - \int_0^1 \frac{x}{1+x}\mathrm{d}x$$

$$= \ln(2) - \left[1 - \int_0^1 \frac{1}{1+x}\mathrm{d}x\right]$$

$$= \ln(2) - 1 + \lim_{n\to\infty}(1+x)\Big|_0^1$$

$$= 2\ln(2) - 1$$

$\therefore \quad \ell = e^{2\ln(2)-1} = 4/e.$

Also solved by Etisha Sharma (Student) Agra college, Agra, India; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; Daniel Văcaru, National Economical College "Maria Teiuleanu", Pitești, Romania; and the problem proposer.

• **5738** *Proposed by Goran Conar, Varaždin, Croatia.*

Let $x_1, \ldots, x_n > 0$ be real numbers and $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n} (1+x_i)^{x_i}.$$

When does equality occur?

Solution 1 by Prakash Pant, Mathematics Initiatives in Nepal, Bardiya, Nepal.

We first rearrange the problem as follows:

$$\prod_{i=1}^{n} \left(\frac{x_i}{1+x_i}\right)^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s$$

We further modify the problem taking log on both sides,

$$\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant s \ln\left(\frac{s}{n+s}\right) \qquad \text{...........(1)}$$

Now, we will focus on proving this statement.

Then, consider a function $f(x) = x \ln\left(\frac{x}{1+x}\right)$. Then f''(x) $= \frac{1}{x(1+x)^2} > 0 \forall x \geqslant 0$. So, the function is convex in our required interval.
Now, using Jensen's inequality,

$$\frac{\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right)}{n} \geqslant \frac{\sum_{i=1}^{n} x_i}{n} \ln\left(\frac{\frac{\sum_{i=1}^{n} x_i}{n}}{1+\frac{\sum_{i=1}^{n} x_i}{n}}\right)$$

which, upon simplification, gives

$$\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant s \ln\left(\frac{s}{n+s}\right)$$

which is what was to be proved from (1). And the equality holds when $x_1 = x_2 = ... = x_n$.

Solution 2 by Toyesh Prakash Sharma (Student) Agra College, Agra, India.

Let $f(x) = x\ln\left(\dfrac{x}{1+x}\right)$ then $f''(x) = \dfrac{1}{x(1+x)^2}$ which is positive for all positive integer x respectively. From this we can say that given function is convex in nature then using Jensen's Inequality we have

$$\sum_{i=1}^{n}\frac{f(x_i)}{n} \geqslant f\left(\sum_{i=1}^{n}\frac{x_i}{n}\right) \Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant \left(\sum_{i=1}^{n}\frac{x_i}{n}\right)\ln\left(\frac{\sum_{i=1}^{n}\frac{x_i}{n}}{1+\sum_{i=1}^{n}\frac{x_i}{n}}\right)$$

$$\Rightarrow \sum_{i=1}^{n}\ln\left(\frac{x_i}{1+x_i}\right)^{x_i} \geqslant \ln\left(\frac{\sum_{i=1}^{n} x_i}{n+\sum_{i=1}^{n} x_i}\right)^{\left(\sum_{i=1}^{n} x_i\right)}$$

$$\Rightarrow \ln\left(\prod_{i=`1}^{n}\left(\frac{x_i}{1+x_i}\right)^{x_i}\right) \geqslant \ln\left(\frac{\sum_{i=1}^{n} x_i}{n+\sum_{i=1}^{n} x_i}\right)^{\left(\sum_{i=1}^{n} x_i\right)}$$

Using $s = \sum_{i=1}^{n} x_i$

$$\prod_{i=`1}^{n}\left(\frac{x_i}{1+x_i}\right)^{x_i} \geqslant \left(\frac{s}{n+s}\right)^{s} \Rightarrow \prod_{i=`1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^{s}\prod_{i=`1}^{n}(1+x_i)^{x_i}$$

And equality occur when $x_1 = x_2 = \cdots = x_n$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

The stated inequality is equivalent to the inequality

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)^{\frac{x_i}{s}} \leqslant 1+\frac{n}{s}$$

which follows from the weighted AM-GM inequality:

$$\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)^{\frac{x_i}{s}} \leqslant \sum_{i=1}^{n}\frac{x_i}{s}\left(1+\frac{1}{x_i}\right) = 1+\frac{n}{s}.$$

Equality occurs if and only if $x_1 = \ldots = x_n$.

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Let
$$f(x) = x\ln\left(1+\frac{1}{x}\right) = x\left(\ln(1+x) - \ln x\right).$$
Then
$$f''(x) = -\frac{1}{x(1+x)} + \frac{1}{(1+x)^2} = -\frac{1}{x(1+x)^2} < 0$$
for $x > 0$. It then follows from Jensen's inequality that
$$\sum_{i=1}^{n} f(x_i) \leqslant nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = nf\left(\frac{s}{n}\right);$$
that is,
$$\sum_{i=1}^{n} x_i \ln\left(1+\frac{1}{x_i}\right) \leqslant n\cdot\frac{s}{n}\ln\left(1+\frac{n}{s}\right) = \ln\left(\frac{n+s}{s}\right)^s,$$
or
$$\ln\prod_{i=1}^{n}\left(\frac{1+x_i}{x_i}\right)^{x_i} \leqslant \ln\left(\frac{n+s}{s}\right)^s.$$
Exponentiation and rearrangement of factors yields
$$\left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1+x_i)^{x_i} \leqslant \prod_{i=1}^{n} x_i^{x_i}.$$
Equality holds if and only if $x_i = \dfrac{s}{n}$ for each $i = 1, 2, \ldots, n$.

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Since the function $f(x) = x\ln(x/(1+x))$ is convex for $x > 0$—$f''(x) = 1/((1+x)^2 x)$—we can apply Jensen's inequality to see that
$$\sum_{i-1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant n\cdot\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)\ln\left(\frac{\sum_{i=1}^{n} x_i/n}{1+\sum_{i=1}^{n} x_i/n}\right),$$
$$\text{or}\quad \sum_{i=1}^{n} x_i \ln\left(\frac{x_i}{1+x_i}\right) \geqslant s\cdot\ln\left(\frac{s}{n+s}\right).$$
Exponentiating yields
$$\prod_{i=1}^{n}\left(\frac{x_i}{1+x_i}\right)^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s, \ \text{ or } \ \prod_{i=1}^{n} x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1+x_i)^{x_i}.$$
Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

Solution 6 by Ivan Hadinata, Department of mathematics, Gadjah Mada University, Yogyakarta, Indonesia.

By weighted AM-GM inequality, observe that

$$\left(\frac{n+s}{s}\right)^s = \left(\frac{\sum_{i=1}^{n}\left(1+\frac{1}{x_i}\right)x_i}{\sum_{i=1}^{n}x_i}\right)^{\sum_{i=1}^{n}x_i} \geqslant \prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)^{x_i} = \frac{\prod_{i=1}^{n}(1+x_i)^{x_i}}{\prod_{i=1}^{n}x_i^{x_i}},$$

then

$$\prod_{i=1}^{n}x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1+x_i)^{x_i} \tag{12}$$

Equality of (12) holds if and only if $x_1 = x_2 = \cdots = x_n = \dfrac{s}{n}$.

Solution 7 by Michel Bataille, Rouen, France.

We introduce the function $f(x) = x\ln\left(\dfrac{x}{1+x}\right)$. For $x > 0$, we easily obtain

$$f''(x) = \frac{1}{x(1+x)^2} > 0,$$

hence f is strctly convex on $(0,\infty)$. Jensen's inequality gives $\dfrac{1}{n}\sum_{i=1}^{n}f(x_i) \geqslant f\left(\dfrac{\sum_{i=1}^{n}xi}{n}\right)$, that is,

$$\sum_{i=1}^{n}x_i\ln(x_i) - \sum_{i=1}^{n}x_i\ln(1+x_i) \geqslant n\cdot\frac{s}{n}\ln\left(\frac{s/n}{1+s/n}\right),$$

with equality if and only if $x_1 = \cdots = x_n$.
We first deduce that

$$\sum_{i=1}^{n}x_i\ln(x_i) \geqslant s\ln\left(\frac{s}{n+s}\right) + \sum_{i=1}^{n}x_i\ln(1+x_i)$$

and then, by exponantiation,

$$\prod_{i=1}^{n}x_i^{x_i} \geqslant \left(\frac{s}{n+s}\right)^s \prod_{i=1}^{n}(1+x_i)^{x_i}.$$

Equality holds if and only if $x_1 = \cdots = x_n$.

Solution 8 by Moti Levy, Rehovot, Israel.

The inequality may be rephrased as

$$\prod_{k=1}^{n}\left(1+\frac{1}{x_i}\right)^{x_i} \leqslant \left(1+\frac{n}{s}\right)^{s}.$$

or by taking logarithm of both sides and dividing by s

$$\sum_{i=1}^{n}\frac{x_i}{s}\ln\left(1+\frac{1}{x_i}\right) \leqslant \ln\left(1+\frac{n}{s}\right)$$

Let $w_i := \dfrac{x_i}{s}$, then $\displaystyle\sum_{i=1}^{n} w_i = 1$ and $w_i \leqslant 1$.

By Jensen inequality, when $f(u)$ is concave for $u > 0$,

$$\sum_{i=1}^{n} w_i f(u_i) \leqslant f\left(\sum_{i=1}^{n} w_i u_i\right).$$

Therefore, for $f(u) = \ln(1+u)$, and $u_i = \dfrac{1}{x_i}$, we have

$$\sum_{i=1}^{n} w_i \ln\left(1+\frac{1}{x_i}\right) \leqslant \ln\left(1+\sum_{i=1}^{n}\frac{w_i}{x_i}\right) = \ln\left(1+\sum_{i=1}^{n}\frac{1}{s}\right)$$
$$= \ln\left(1+\frac{n}{s}\right).$$

Equality occurs when $x_1 = \cdots = x_n = 1$.

Also solved by the problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
- LastName stands for YOUR last name.

Examples:

#1234_Max_Planck_Solution_SSMJ

#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

"Proposed Solution to #**** SSMJ"

where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]",

followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

Proposed solution to #1234 SSMJ

Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

FirstName_LastName_ProposedProblem_SSMJ_YourGivenNumber_ProblemTitle

If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

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1. On the top of first page of your proposal, begin with the phrase:

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3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣