Problems and Solutions

Albert Natian, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please email them to Prof. Albert Natian at Department of Mathematics, Los Angeles Valley College. Please present all proposed solutions and proposed problems according to formatting requirements delineated near the end of this document. Also, please make sure every proposed problem or proposed solution is provided in both ***LaTeX*** and pdf documents. *Thank you!*

To propose problems, email them to: problems4ssma@gmail.com

To propose solutions, email them to: solutions4ssma@gmail.com

Solutions to previously published problems can be seen at <www.ssma.org/publications>.

Solutions to the problems published in this issue should be submitted *before* June 1, 2024.

• **5769** *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

Evaluate the limit $L = \lim_{n\to\infty} n\, x_n$ where

$$x_n := \frac{\sin\frac{1}{n}}{\sin\frac{1}{n^2}} - n.$$

• **5770** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu" Drobeta Turnu - Severin, Romania.*

Suppose $a, b \in \mathbb{C}$ with $|a^2+1| \leqslant 1,\ |a^4+1| \leqslant 1,\ |b^3+1| \leqslant 1,\ |b^6+1| \leqslant 1$. Prove that

$$|a+b|^2 + |a-b|^2 \leqslant 4$$

• **5771** *Proposed by Goran Conar, Varaždin, Croatia.*

Suppose $x_1, x_2, \ldots, x_n \geqslant e$. Prove

$$\frac{n}{7} > \sum_{j=1}^{n} \frac{x_j}{1+x_j^3} > \frac{n}{2}\left(\prod_{j=1}^{n} x_j\right)^{-2/n}.$$

• **5772** *Proposed by by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

For the matrix A in $\mathcal{M}_2(\mathbb{R})$, solve the equation $A^3 = A - A^T$, where A^T denotes the transpose of A.

• **5773** *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the following integral

$$I = \int_0^1 \frac{x \ln^2 x}{x^3 + x\sqrt{x} + 1} dx.$$

• **5774** *Proposed by Toyesh Prakash Sharma (Student) St. C.F Andrews School, Agra, India.*

If $x \in \left[0, \pi/2\right]$, then show that

$$\left(\sin^2 x\right)^{\cos^2 x} + \left(\cos^2 x\right)^{\sin^2 x} \leqslant \frac{3}{2}.$$

Solutions

To Formerly Published Problems

• **5745** *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

For real x, solve the equation

$$2^{(2^x-1)^2} + 4^x = \sqrt{x} + 2^{x+1} + \log_2\left(1 + \sqrt{x}\right).$$

Solution 1 by Trey Smith, Angelo State University, San Angelo, TX.

Notice that both $x = 0$ and $x = 1$ are solutions to the equation. We will show that they are the only solutions.

Since $\sqrt{x}$ is a term in this equation, we may restrict out attention to non-negative numbers. Consider the following two functions:

$$f(x) = 4^x - 2^{x+1}, \quad \text{and} \quad g(x) = \sqrt{x} + \log_2(1 + \sqrt{x}) - 2^{(2^x-1)^2}.$$

Any solution to the original equation is a solution to the equation $f(x) = g(x)$. Now

$$\begin{aligned} f''(x) &= (\ln 4)^2 4^x - (\ln 2)^2 2^{x+1} \\ &= (\ln 4)^2 4^x - 2(\ln 2)^2 2^x \\ &= (\ln 4)^2 4^x - (\ln 4)(\ln 2) 2^x \\ &= (\ln 4)[(\ln 4)4^x - (\ln 2)2^x] > 0 \end{aligned}$$

for all $x > 0$. Hence f is concave up on $(0, \infty)$.

The function g is a bit more complicated.

The second derivative for $\sqrt{x}$ is $-\dfrac{1}{4x^{3/2}}$ which is negative on $(0, \infty)$.

The second derivative for $\log_2(1 + \sqrt{x})$ is

$$-\frac{1}{\ln 2} \cdot \frac{2\sqrt{x} + 1}{4x^{3/2}(1 + \sqrt{x})^2}$$

which is negative on $(0, \infty)$.

Finally, $2^{(2^x-1)^2}$ can be viewed as the composition $(a \circ b \circ c)(x)$ where $a(x) = 2^x$, $b(x) = x^2$, and $c(x) = 2^x - 1$. Notice that for all three of these functions, the function itself, the first derivative of the function, and the second derivative of the function are all positive on $(0, \infty)$. Since the second derivative of $(a \circ b \circ c)(x)$ will involve sums, products and compositions of a, b, c, and their first and second derivatives, the second derivative will be positive on $(0, \infty)$. Consequently, $g''(x)$ will be negative on $(0, \infty)$, so g is concave down on that interval.

Hence the functions f and g intersect at exactly two points, so 0 and 1 are the only two solutions.

Solution 2 by David A. Huckaby, Angelo State University, San Angelo, TX.

By inspection, $x = 0$ and $x = 1$ are solutions. We proceed to show that there are no other solutions.

Define $f(x) = 2^{(2^x-1)^2} + 4^x - 2^{x+1}$ and $g(x) = \sqrt{x} + \log_2(1 + \sqrt{x})$. Then $f'(x) = 2^x(2^x - 1)\left(2^{(2^x-1)^2}(\ln 2)^2 + \ln 4\right) > 0$ for $x > 0$. Similarly, $g'(x) = \dfrac{\sqrt{x}\ln 2 + 1 + \ln 2}{2(x + \sqrt{x})\ln 2} > 0$.

Clearly $g''(x) = -\dfrac{(\ln 2)x + 2\sqrt{x}(1 + \ln 2) + 1 + \ln 2}{4(\sqrt{x} + 1)^2 x^{3/2} \ln 2} < 0$.

Now

$$f''(x) = 2^{(2^x-1)^2+x+1}(2^x - 1)(\ln 2)^3 \left[2^{x+1}(2^x - 1)\ln 2 + 1\right] + 2^{(2^x-1)^2+2x+1}(\ln 2)^3 + 4^x(\ln 4)^2 - 2^{x+1}(\ln 2)^2.$$

The first two terms of $f''(x)$ are clearly positive (for $x > 0$). Note that the last two terms of $f''(x)$ can be rewritten as

$$4^x(\ln 4)^2 - 2^{x+1}(\ln 2)^2 = 2^{x+1}\left[2^{x-1}(\ln 4)^2 - (\ln 2)^2\right].$$

Since $\dfrac{1}{2}(\ln 4)^2 - (\ln 2)^2 \approx 0.48 > 0$, the last two terms of $f''(x)$ are also positive (for $x > 0$) so that $f''(x) > 0$ for $x > 0$. So $f''(x) > 0$ and $g''(x) < 0$ for $x > 0$, so that along with the boundary $x = 0$, there can be no more than one additional solution (since both f and g are continuous). So $x = 0$ and $x = 1$ are indeed the only solutions.

Solution 3 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Note that the equation is only valid for $x \geqslant 0$. By simple inspection, it is observed that $x = 0$, and $x = 1$ are solutions of the equation. There are no more real roots of the equation. The equation may be written equivalently as

$$\begin{aligned} 2^{(2^x-1)^2} + 2^{2x} - 2 \cdot 2^x &= \sqrt{x} + \log_2(1+\sqrt{x}) \\ 2^{(2^x-1)^2} + 2^{2x} - 2 \cdot 2^x + 1 &= 1 + \sqrt{x} + \log_2(1+\sqrt{x}) \\ 2^{(2^x-1)^2} + (2^x-1)^2 - 1 &= \sqrt{x} + \log_2(1+\sqrt{x}). \end{aligned}$$

The conclusion follows now because function $f(x) = 2^{(2^x-1)^2} + (2^x-1)^2 - 1$ is convex, since $(2^x-1)^2 - 1$ is convex and $2^{(2^x-1)^2}$ is convex for $x \geqslant 0$, while function $g(x) = \sqrt{x} + \log_2(1+\sqrt{x})$ is concave for $x \geqslant 0$.

Solution 4 by Brian D. Beasley, Simpsonville, SC.

We show that the two real solutions are $x = 0$ and $x = 1$.

For $x \geqslant 0$, let $f(x) = 2^{(2^x-1)^2} + 4^x - \sqrt{x} - 2^{x+1} - \log_2(1+\sqrt{x})$. Then $f(0) = 0$ and $f(1) = 0$. For $x > 1$, $4^x > 2^{x+1}$ and $2^{(2^x-1)^2} > \sqrt{x} + \log_2(1+\sqrt{x})$, so $f(x) > 0$.

Next, we assume $0 < x < 1$ and show that $f(x) < 0$. Since x is in $(0,1)$, we have $2^x - 1 < x$ and thus

$$2^{(2^x-1)^2} < 2^{2^x-1} < 2^x.$$

We also observe that $\sqrt{x} > x$ on $(0,1)$ and hence

$$\log_2(1+\sqrt{x}) > \log_2(1+x) > x.$$

Then on $(0,1)$, we obtain $f(x) < g(x)$, where

$$g(x) = 2^x + 4^x - (x + 2^{x+1} + x) = 2^{2x} - 2^x - 2x.$$

Next, we calculate

$$g'(x) = (2\ln 2)(2^x)^2 - (\ln 2)2^x - 2.$$

Since $g'(x) = 0$ on $(0,1)$ if and only if $x \approx 0.5625$, with $g'(x) < 0$ on $(0, 0.5625)$ and $g'(x) > 0$ on $(0.5625, 1)$, we conclude that g is decreasing on the first interval and increasing on the second interval. But $g(0) = 0$ and $g(1) = 0$, so $g(x) < 0$ on $(0,1)$. Hence $f(x) < 0$ on $(0,1)$ as claimed.

Solution 5 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

Evidently $x = 0$ and $x = 1$ are solutions and we are done if we prove the convexity of the function

$$f(x) = 2^{(2^x-1)^2} + 4^x - \sqrt{x} - 2^{x+1} - \frac{\ln(1+\sqrt{x})}{\ln 2}$$

In this case indeed f would lie below the abscissa positive half axis for $0 \leqslant x \leqslant 1$ and above for $x \geqslant 1$. Now

$$-(\sqrt{x})'' = \frac{1}{4x^{3/2}} > 0, \quad \left(\frac{\ln(1+\sqrt{x})}{-\ln 2}\right)'' = \frac{(1+2\sqrt{x})}{4x^{3/2}(1+\sqrt{x})^2 \ln 2} > 0$$

and

$$\left(2^{(2^x-1)^2} + 4^x - 2^{x+1}\right)'' = 2(\ln 2)^3 2^{(2^x-1)^2} 2^x (2^{2x} - 1 + 2^{x+1}(2^x - 1)) \geqslant 0.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Bruno Salgueiro Fanego, Viveiro, Lugo, Spain; and the problem proposer.

• **5746** *Proposed by Problem proposed by Albert Stadler, Herrliberg, Switzerland.*

Let $m, n \geqslant 1$ and let $p_{m,n}(x)$ be the polynomial defined by

$$p_{m,n}(x) = e^{-x^m} \frac{d^n}{dx^n} e^{x^m}.$$

Prove that for all $n \geqslant 0$:

$$\left((x^m + y^m)^{n/m}\right) p_{m,n}\left((x^m + y^m)^{1/m}\right) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} p_{m,k}(x)\, p_{m,n-k}(y).$$

Solution 1 by Moti Levy, Rehovot, Israel.

The polynomials $p_{m,n}(x) = e^{-x^m} \dfrac{d^n}{dx^n} e^{x^m}$ are the subject of an article by E. T. Bell (1934).

Let us rewrite the right-hand-side of the equation in the problem statement

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-y} p_{m,k}(x)\, p_{m,n-k}(y) = n! \sum_{k=0}^{n} \frac{x^k p_{m,k}(x)}{k!} \frac{y^{n-y} p_{m,n-k}(y)}{(n-k)!}. \tag{1}$$

We define two sequences $(a_k)_{k\geqslant 0}$ and $(b_k)_{k\geqslant 0}$ as follows:

$$a_k := \frac{x^k p_{m,k}(x)}{k!}, \quad b_k = \frac{y^k p_{m,k}(y)}{k!}. \tag{2}$$

We observe that $\sum_{k=0}^{n} \frac{x^k p_{m,k}(x)}{k!} \frac{y^{n-y} p_{m,n-k}(y)}{(n-k)!}$ is convolution of the sequences $(a_k)_{k\geqslant 0}$ and $(b_k)_{k\geqslant 0}$,

$$\sum_{k=0}^{n} \frac{x^k p_{m,k}(x)}{k!} \frac{y^{n-y} p_{m,n-k}(y)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}. \tag{3}$$

The generating functions of the sequences $(a_k)_{k\geqslant 0}$ and $(b_k)_{k\geqslant 0}$ are:

$$A(z) = \sum_{k=0}^{\infty} a_k z^k = \sum_{k=0}^{\infty} \frac{x^k p_{m,k}(x)}{k!} z^k = \sum_{k=0}^{\infty} \frac{p_{m,k}(x)}{k!} (xz)^k,$$
$$B(z) = \sum_{k=0}^{\infty} b_k z^k = \sum_{k=0}^{\infty} \frac{y^k p_{m,k}(y)}{k!} z^k = \sum_{k=0}^{\infty} \frac{p_{m,k}(y)}{k!} (yz)^k.$$

The generating function of the sequence $(c_n)_{n\geqslant 0}$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ is

$$C(z) = A(z) B(z).$$

Now let us find the closed form of the exponential generating function $P_m(z)$ of the sequence $\left(p_{m,k}(x)\right)_{k\geqslant 0}$,

$$P_m(z) = \sum_{k=0}^{\infty} \frac{p_{m,k}(x)}{k!} z^k = e^{-x^m} \sum_{k=0}^{\infty} \left(\frac{d^k}{dx^k} e^{x^m} \right) \frac{z^k}{k!}. \tag{4}$$

Since

$$\frac{d^k}{dz^k} e^{(z+x)^m} = \frac{d^k}{dx^k} e^{(z+x)^m},$$

then

$$\frac{d^k}{dx^k} e^{x^m} = \left. \frac{d^k}{dz^k} e^{(z+x)^m} \right]_{z=0}. \tag{5}$$

Plugging (5) into (4) we get

$$P_m(z) = e^{-x^m} \sum_{k=0}^{\infty} \left. \frac{d^k}{dz^k} e^{(z+x)^m} \right]_{z=0} \frac{z^k}{k!},$$

hence by Taylor's theorem, we obtain

$$P_m(z) = e^{-x^m} e^{(z+x)^m}. \tag{6}$$

It follows that

$$A(z) = P_m(xz) = e^{-x^m} e^{(z+x)^m} = e^{-x^m} e^{x^m (z+1)^m}, \tag{7}$$

and that

$$B(z) = P_m(yz) = e^{-y^m} e^{(z+y)^m} = e^{-y^m} e^{y^m (z+1)^m}. \tag{8}$$

By (7) and (8)

$$C(z) = e^{-(x^m+y^m)} e^{(x^m+y^m)(z+1)^m}.$$

The convolution implies that $[z^n] C(z)$, i.e., the coefficient of z^n of $C(z)$ is equal to $\sum_{k=0}^{n} \frac{x^k p_{m,k}(x)}{k!} \frac{y^{n-y} p_{m,n-k}(y)}{(n-k)!}$.

Now by Taylor's theorem applied to $C(z)$,

$$C(z) = e^{-(x^m+y^m)} \sum_{k=0}^{\infty} \left. \frac{d^k}{dz^k} e^{(x^m+y^m)(z+1)^m} \right]_{z=0} \frac{z^k}{k!}. \tag{9}$$

Since

$$\frac{d^k}{dz^k} e^{(\alpha z+\beta)^m}\Big]_{z=0} = \alpha^k \frac{d^k}{dz^k} e^{z^m}\Big]_{z=\beta},$$

then

$$\frac{d^k}{dz^k} e^{\left((x^m+y^m)^{\frac{1}{m}} z+(x^m+y^m)^{\frac{1}{m}}\right)^m}\Big]_{z=0} = (x^m+y^m)^{\frac{k}{m}} \frac{d^k}{dz^k} e^{z^m}\Big]_{z=(x^m+y^m)^{\frac{1}{m}}}$$

$$\begin{aligned}[z^n]\, C(z) &= \frac{1}{n!}\,(x^m+y^m)^{\frac{n}{m}}\, e^{-\left((x^m+y^m)^{\frac{1}{m}}\right)^m} \frac{d^n}{dz^n} e^{z^m}\Big]_{z=(x^m+y^m)^{\frac{1}{m}}} \\ &= \frac{1}{n!}\,(x^m+y^m)^{\frac{n}{m}}\, p_{n,m}\left((x^m+y^m)^{\frac{1}{m}}\right)\end{aligned}$$

We conclude that

$$\sum_{k=0}^{n} \frac{x^k p_{m,k}(x)}{k!} \frac{y^{n-y} p_{m,n-k}(y)}{(n-k)!} = \frac{1}{n!}\,(x^m+y^m)^{\frac{n}{m}}\, p_{n,m}\left((x^m+y^m)^{\frac{1}{m}}\right). \qquad \blacksquare$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Choose reals $x, y > 0$ with $x \neq y$. We start with the RHS of the claim. Application of the Cauchy integral formula yields that

$$\begin{aligned} e^{x^m+y^m}\mathrm{RHS} &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \frac{k!}{2\pi i} \int_{\partial D_1(x)} \frac{e^{-w^m}}{(w-x)^{k+1}} dw \frac{k!}{2\pi i} \int_{\partial D_2(y)} \frac{e^{-z^m}}{(z-y)^{n-k+1}} dz \\ &= \frac{n!}{(2\pi i)^2} \int_{\partial D_1(x)} \int_{\partial D_2(y)} \sum_{k=0}^{n} \frac{x^k}{(w-x)^{k+1}} \frac{y^{n-k}}{(z-y)^{n-k+1}} e^{-w^m-z^m} dz dw, \end{aligned}$$

where $D_1(x) = \{w \mid |w-x| < r_1\}$ and $D_2(y) = \{z \mid |z-y| < r_2\}$ with $r_1, r_2 > 0$ such that $D_1(x) \cap D_2(y) = \varnothing$. Evaluation of the geometric series yields

$$\sum_{k=0}^{n} \frac{x^k}{(w-x)^{k+1}} \frac{y^{n-k}}{(z-y)^{n-k+1}} = \frac{1}{(w-x)^{n+1}(z-y)^{n+1}} \frac{\left(x(z-y)\right)^{n+1} - \left(y(w-x)\right)^{n+1}}{x(z-y) - y(w-x)}.$$

The denominator of the latter fraction can be rewritten in the form $xz - yw$. We write $e^{x^m+y^m}\mathrm{RHS} = \frac{n!}{(2\pi i)^2}(I_1 + I_2)$, where

$$I_1 = \int_{\partial D_1(x)} \int_{\partial D_2(y)} \frac{1}{(w-x)^{n+1}(z-y)^{n+1}} \frac{\left(x(z-y)\right)^{n+1}}{xz-yw} e^{-w^m-z^m} dz dw$$

and

$$I_2 = \int_{\partial D_1(x)} \int_{\partial D_2(y)} \frac{1}{(w-x)^{n+1}(z-y)^{n+1}} \frac{-\left(y(w-x)\right)^{n+1}}{xz-yw} e^{-w^m-z^m} dz dw.$$

We can choose $r_2 > 0$ so small that there exists a region $G \supset D_1(x)$ such that $xz - yw \neq 0$, for all $w \in G$ and $z \in \partial D_2(y)$. After interchanging the order of both integrals, we conclude that

$$I_1 = x^{n+1} \int_{\partial D_2(y)} \frac{1}{(w-x)^{n+1}} \left(\int_{\partial D_1(x)} \frac{1}{xz - yw} e^{-w^m - z^m} dw \right) dz = 0$$

because the inner integral vanishes by the Cauchy Integral Theorem. Two applications of the Cauchy Integral Formula to the second integral yield

$$\begin{aligned} I_2 &= \int_{\partial D_2(y)} \frac{1}{(z-y)^{n+1}} \left(\int_{\partial D_1(x)} \frac{y^n}{w - \frac{x}{y} z} e^{-w^m - z^m} dw \right) dz \\ &= 2\pi i \int_{\partial D_1(x)} \frac{y^n}{(z-y)^{n+1}} e^{-\left(\frac{x}{y}z\right)^m - z^m} dz \\ &= \frac{(2\pi i)^2}{n!} y^n \left(\frac{d}{dz} \right)^n e^{-\left(\frac{x}{y}z\right)^m - z^m} \Bigg|_{z=y}. \end{aligned}$$

Hence,

$$\text{RHS} = y^n e^{-x^m - y^m} \left(\frac{d}{dz} \right)^n e^{-\left(\frac{x}{y}z\right)^m - z^m} \Bigg|_{z=y}.$$

Since

$$\left(\frac{d}{dz} \right)^n e^{-(cz)^m} \Bigg|_{z=y} = c^n \left(\frac{d}{dz} \right)^n e^{-z^m} \Bigg|_{z=cy} = c^n e^{(cy)^m} p_{m,n}(cy),$$

we conclude with $c = \left(1 + \left(\frac{x}{y} \right)^m \right)^{1/m}$ that

$$\begin{aligned} \text{RHS} &= y^n \left(1 + \left(\frac{x}{y} \right)^m \right)^{n/m} e^{-x^m - y^m} e^{x^m + y^m} p_{m,n} \left((x^m + y^m)^{1/m} \right) \\ &= (x^m + y^m)^{n/m} p_{m,n} \left((x^m + y^m)^{1/m} \right). \end{aligned}$$

By continuity this formula is valid also for $x = y$.

Also solved by the problem proposer.

• **5747** *Proposed by Raluca Maria Caraion, Călăraşi, Romania and Florică Anastase, Lehliu-Gară, Romania.*

Suppose $f : (2,3) \to (0,\infty)$ is a function with $f'(x) < 0$ and $f''(x) < 0$ for all x in $(2,3)$. Show that for a, b, c in $(1,2)$:

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1 + \sqrt{ab}} \right) \geqslant 2 \cdot \sqrt[4]{\prod_{cyc} f(a+1) \cdot \sum_{cyc} f(a+1)}.$$

Solution 1 by Michel Bataille, Rouen, France.

First, we remark that for $a, b > 1$, we have

$$\frac{(a+1)(b+1)}{1+\sqrt{ab}} \leqslant \frac{a+b+2}{2}. \quad (1)$$

Indeed, the inequality $2(a+1)(b+1) \leqslant (1+\sqrt{ab})(a+b+2)$ rewrites as $(\sqrt{ab}-1)(\sqrt{a}-\sqrt{b})^2 \geqslant 0$, which obviously holds.

Note in passing that for $a, b \in (1,2)$, we have $\frac{(a+1)(b+1)}{1+\sqrt{ab}} \in (2,3)$. This follows from (1) since $\frac{(a+1)(b+1)}{1+\sqrt{ab}} < \frac{2+2+2}{2} = 3$ and from $(a+1)(b+1)-2(1+\sqrt{ab}) = (\sqrt{a}-\sqrt{b})^2+(ab-1) > 0$.

Now, f being decreasing on $(2,3)$ (since $f'(x) < 0$), (1) gives

$$f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant f\left(\frac{a+b+2}{2}\right).$$

The function f being concave (since $f''(x) < 0$), we have

$$f\left(\frac{a+b+2}{2}\right) = f\left(\frac{(a+1)+(b+1)}{2}\right) \geqslant \frac{1}{2}f(a+1) + \frac{1}{2}f(b+1)$$

and we deduce that

$$\sum_{\text{cyc}} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant \frac{1}{2}\left(f(a+1)+f(b+1)+f(c+1)+\sum_{\text{cyc}} f(a+1)\right).$$

Lastly, from $AM-GM$, we obtain

$$f(a+1)+f(b+1)+f(c+1)+\sum_{\text{cyc}} f(a+1) \geqslant 4\sqrt[4]{\prod_{\text{cyc}} f(a+1)\cdot\sum_{\text{cyc}} f(a+1)}$$

and the required inequality follows.

Solution 2 by Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy.

Let's show that $\frac{(a+1)(b+1)}{1+\sqrt{ab}} \leqslant 3$ by defining $a+b=2u$, $ab=v^2$ ($v \leqslant u$ by the AGM). The inequality becomes

$$1+v+\frac{2u-2v}{1+v} \leqslant 3 \iff v+\frac{2u-2v}{1+v} \leqslant 2, \quad 1 \leqslant v \leqslant u \leqslant 2$$

The function $v+\dfrac{2u-2v}{1+v} \doteq f(u)$ is linear increasing in u hence

$$f(u)\leqslant f(2)=v+\frac{4-2v}{1+v}\leqslant 2 \iff v^2-3v+2\leqslant 0 \iff 1\leqslant v\leqslant 2$$

which holds true because $v=\sqrt{ab}\leqslant 2$. It follows

$$\sum_{\text{cyc}} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right)\geqslant 3f(3)\geqslant 2\sqrt[4]{\prod_{\text{cyc}} f(2+1)\cdot\sum_{\text{cyc}} f(2+1)}\geqslant$$

$$\geqslant 2\sqrt[4]{\prod_{\text{cyc}} f(a+1)\cdot\sum_{\text{cyc}} f(a+1)}$$

$$3f(3)\geqslant 2\sqrt[4]{\prod_{\text{cyc}} f(2+1)\cdot\sum_{\text{cyc}} f(2+1)} \iff 81\geqslant 48$$

and the inequality is proven (apparently no need of the concavity of $f(x)$.)

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We will prove more precisely that

$$\sum_{cycl} f\left(\frac{(a+1)\,(b+1)}{1+\sqrt{ab}}\right)\geqslant\sqrt{3\sqrt{3}}\bullet\sqrt[4]{\prod_{cycl} f\,(a+1)\bullet\sum_{cycl} f\,(a+1)}$$

with equality if and only if a=b=c. (Note that $\sqrt{3\sqrt{3}}>2$).
We first note that $2<\dfrac{(a+1)\,(b+1)}{1+\sqrt{ab}}<3$ if a,b∈(1,2), for $2<\dfrac{(a+1)\,(b+1)}{1+\sqrt{ab}}$ is equivalent to $1+2\sqrt{ab}<a+b+ab$ which holds true, since $2\sqrt{ab}\leqslant a+b$ and ab>1, and $\dfrac{(a+1)\,(b+1)}{1+\sqrt{ab}}<3$ follows from $1+\dfrac{a+b}{2}-\dfrac{(a+1)\,(b+1)}{1+\sqrt{ab}}=\dfrac{\left(\sqrt{a}-\sqrt{b}\right)^2\left(\sqrt{ab}-1\right)}{2\left(1+\sqrt{ab}\right)}\geqslant 0$ so that $\dfrac{(a+1)\,(b+1)}{1+\sqrt{ab}}\leqslant 1+\dfrac{a+b}{2}$, and $\dfrac{a+b}{2}<2$.

The fact f'(x)<0 implies that f(x) is a (strictly) monotonically decreasing function. Hence

$$f\left(\frac{(a+1)\,(b+1)}{1+\sqrt{ab}}\right)\geqslant f\left(1+\frac{a+b}{2}\right).$$

The fact f''(x)<0 implies that f(x) is concave. Hence

$$\frac{1}{2}f\,(1+a)+\frac{1}{2}f\,(1+b)\leqslant f\left(1+\frac{a+b}{2}\right),$$

by Jensen's inequality. We conclude that

$$\sum_{cycl} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant \sum_{cycl} f\left(1+\frac{a+b}{2}\right) \geqslant \sum_{cycl}\left(\frac{1}{2}f(1+a)+\frac{1}{2}f(1+b)\right) =$$

$$= \sum_{cyc} f(1+a) = \sqrt[4]{\left(\sum_{cyc} f(1+a)\right)^3 \sum_{cyc} f(1+a)} \geqslant \sqrt[4]{27 \prod_{cyc} f(1+a) \sum_{cyc} f(1+a)} =$$

$$= \sqrt{3\sqrt{3}} \cdot \sqrt[4]{\prod_{cycl} f(a+1) \cdot \sum_{cycl} f(a+1)},$$

where in the second last step we have used the AM-GM inequality. Following the chain of inequalities we see that equality holds if and only if a=b=c.

Solution 4 by Moti Levy, Rehovot, Israel.

By AM-GM inequality

$$\prod_{cyc} f(a+1) \leqslant \frac{1}{27}\left(\sum_{cyc} f(a+1)\right)^3. \tag{10}$$

Plugging (10) in the right side of the original inequality, we get

$$2\sqrt[4]{\prod_{cyc} f(a+1) \sum_{cyc} f(a+1)} \leqslant \frac{2}{\sqrt[4]{27}} \sum_{cyc} f(a+1) \cong 0.877\,38 \sum_{cyc} f(a+1). \tag{11}$$

Claim: the following inequality holds for x, y in $(2,3)$

$$\frac{x+y}{2} \geqslant \frac{xy}{1+\sqrt{(x-1)(y-1)}}. \tag{12}$$

Proof of Claim: Inequality (12) is equivalent to

$$(x+y)\sqrt{(x-1)(y-1)} > 2xy - x - y. \tag{13}$$

It is easy to see that

$$xy > x + y \text{ for } x, y \text{ in } (2,3), \tag{14}$$

hence we may square both sides and get the equivalent inequality

$$(x+y)^2(x-1)(y-1) - (2xy - x - y)^2 > 0.$$

To complete the proof, we factor the left side

$$(x+y)^2(x-1)(y-1) - (2xy-x-y)^2 = (xy-x-y)(x-y)^2 > 0. \blacksquare$$

Setting $x = a + 1,\ y = b + 1$, then by claim,

$$\frac{(a+1)+(b+1)}{2} \geqslant \frac{(a+1)(b+1)}{1+\sqrt{ab}} \text{ for } a,b \text{ in } (1,2). \tag{15}$$

Since the function f is decreasing in $(1,2)$ then it follows that $f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant f\left(\frac{(a+1)+(b+1)}{2}\right)$, so that by symmetry,

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant \sum_{cyc} f\left(\frac{(a+1)+(b+1)}{2}\right). \tag{16}$$

Since the function f is concave in $(1,2)$ then by Jensen's inequality,

$$f\left(\frac{(a+1)+(b+1)}{2}\right) \geqslant \frac{1}{2} f(a+1) + \frac{1}{2} f(b+1). \tag{17}$$

It follows by symmetry that,

$$\sum_{cyc} f\left(\frac{(a+1)+(b+1)}{2}\right) \geqslant \sum_{cyc} f(a+1). \tag{18}$$

By (11), (17) and (18) we have

$$\sum_{cyc} f\left(\frac{(a+1)(b+1)}{1+\sqrt{ab}}\right) \geqslant \sum_{cyc} f(a+1) \geqslant \frac{2}{\sqrt[4]{27}} \sum_{cyc} f(a+1) \geqslant 2\sqrt[4]{\prod_{cyc} f(a+1) \sum_{cyc} f(a+1)}.$$

Also solved by the problem proposer.

• **5748** *Proposed by Narendra Bhandari, Bajura, Nepal.*

Let B_k denote the kth Bernoulli number. For positive integers m and n prove that

$$\int_{-\pi}^{\pi} \left[\sum_{k=1}^{n} k^m \sin(kx)\right]^2 dx = \frac{\pi}{2m+1} \sum_{k=0}^{2m} \binom{2m+1}{k} B_k n^{2m-n+1}.$$

Solution 1 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

By Parseval's theorem for Fourier series over $[-\pi, \pi]$ and Bernoulli's formula for the sum of powers, it follows that

$$\int_{-\pi}^{\pi} \left[\sum_{k=1}^{n} k^m \sin(kx)\right]^2 dx = \pi \sum_{k=1}^{n} k^{2m} = \frac{\pi}{2m+1} \sum_{k=0}^{2m} \binom{2m+1}{k} B_k n^{2m+1-k}.$$

This corrects a misprint in the statement of the problem.

Remark: Using the orthogonality of the trigonometric functions one can substitute Parseval's theorem by a direct proof as follows:

$$\begin{aligned}\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}k^m\sin(kx)\right]^2dx &= \sum_{k=1}^{n}\sum_{j=1}^{n}k^m j^m\int_{-\pi}^{\pi}\sin(kx)\sin(jx)\,dx\\ &= \sum_{k=1}^{n}k^{2m}\int_{-\pi}^{\pi}\sin^2(kx)\,dx=\pi\sum_{k=1}^{n}k^{2m}.\end{aligned}$$

Solution 2 by Moti Levy, Rehovot, Israel.

Since $\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx=\left\{\begin{matrix}\pi & if & m=n\\ 0 & if & m\neq n\end{matrix}\right\}$, for $m,n\geqslant 1$, then

$$\begin{aligned}\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n}k^m\sin(kx)\right]^2dx &= \int_{-\pi}^{\pi}\sum_{k=1}^{n}k^{2m}\sin^2(kx)\,dx\\ &= \sum_{k=1}^{n}k^{2m}\int_{-\pi}^{\pi}\sin^2(kx)\,dx=\pi\sum_{k=1}^{n}k^{2m}.\end{aligned}$$

The *Faulhaber's formula* expresses the sum of the p-th powers of the first n positive integers as polynomial in n.

$$\sum_{k=1}^{n}k^p=\frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k}B_k n^{p-k+1},$$

where B_k are the Bernoulli numbers with the convention that $B_1=+\frac{1}{2}$.

To complete the proof set $p=2m$ in the Faulhaber's formula. ■

Solution 3 by Michel Bataille, Rouen, France.

The well-known link between the sums of powers of the first $(n-1)$ positive integers and the Bernoulli numbers,

$$\sum_{k=1}^{n-1}k^m=\frac{1}{m+1}\sum_{j=0}^{m}\binom{m+1}{j}B_j n^{m+1-j}$$

(see for example, K.S. Williams, *Bernoulli's Identity Without Calculus*, Math. Magazine, Vol. 70, No 1, February 1997) makes me think that there are typos in the statement of the problem, so I prove the following identity instead:

$$\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n-1}k^m\sin(kx)\right]^2dx=\frac{\pi}{2m+1}\sum_{k=0}^{2m}\binom{2m+1}{k}B_k n^{2m-k+1}. \quad (1)$$

For positive integers k, ℓ, with $k \neq \ell$, we have

$$\int_{-\pi}^{\pi} \sin^2(kx)\, dx = \int_{-\pi}^{\pi} \frac{1 - \cos(2kx)}{2}\, dx = \pi - \left[\frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} = \pi$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(kx)\sin(\ell x)\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos((k-\ell)x) - \cos((k+\ell)x))\, dx \\ &= \frac{1}{2}\left[\frac{\sin((k-\ell)x)}{k-\ell} - \frac{\sin((k+\ell)x)}{k+\ell}\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

It follows that

$$\int_{-\pi}^{\pi}\left[\sum_{k=1}^{n-1} k^m \sin(kx)\right]^2 dx = \sum_{k=1}^{n-1}\int_{-\pi}^{\pi} k^{2m}\sin^2(kx)\, dx = \pi\sum_{k=1}^{n-1} k^{2m}$$

The identity (1) follows since

$$\sum_{k=1}^{n-1} k^{2m} = \frac{1}{2m+1}\sum_{k=0}^{2m}\binom{2m+1}{k} B_k n^{2m-k+1}.$$

Also solved by Yunyong Zhang, Chinaunicom, Yunnan, China; Albert Stadler, Herrliberg, Switzerland; and the problem proposer.

• **5749** *Proposed by Prakash Pant, Mathematics Initiatives in Nepal(MIN), Bardiya, Nepal.*

Let x, y, z be positive real numbers with $x + y + z = 3$. Prove that:

$$\prod_{x,y,z}(x^{1/x}e^e) \leqslant e^{\sum_{x,y,z} e^x}$$

For what values of x, y and z does equality hold?

Solution 1 by Sudip Rokaya, Gandaki Boarding School, Humla, Nepal.

We will show an equivalent inequality for

$$\prod_{x,y,z}(x^{1/x}e^e) \leqslant e^{\sum_{x,y,z} e^x}.$$

Taking log on both sides of the latter, we get

$$\sum_{x,y,z}(\frac{1}{x}(logx) + e) \leqslant \sum_{x,y,z} e^x$$

$$3e \leqslant \sum_{x,y,z}(e^x - \frac{logx}{x})$$

$$\frac{1}{3}\sum x, yz(e^x - \frac{logx}{x}) \geqslant e$$

which is equivalent to the original given inequality and which is what we will prove.

Consider the function $f(x) = e^x - \dfrac{logx}{x}$:

$$f''(x) = e^x + \frac{3 - 2logx}{x^3} \geqslant x + \frac{3 - 2logx}{x^3}$$

$$= \frac{x^4 - 2logx + 3}{x^3}$$

$$\geqslant \frac{x^4 - 2x^2 + 3}{x^3}$$

$$= \frac{(x^2 - 1)^2 + 2}{x^3} \geqslant 0 \ \ (x > 0)$$

Thus, $f''(x)$ is always positive when $x \in R^+$. Using Jensen's Inequality,

$$\frac{f(x) + f(y) + f(z)}{3} \geqslant f(\frac{x + y + z}{3})$$

$$\frac{\sum_{x,y,z} e^x - \frac{logx}{x}}{3} \geqslant e^{\frac{x+y+z}{3}} - \frac{log(\frac{x+y+z}{3})}{\frac{x+y+z}{3}}$$

$$= e^1 - \frac{log(1)}{1}$$

$$= e - 0 = e.$$

Solution 2 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

The inequality is equivalent to

$$1 \leqslant e^{f(x)+f(y)+f(z)},$$

where $f(x) = e^x - e - (\log x)/x$. Since $f''(x) = e^x + (3 - 2\log x)\, x^{-3} \geqslant e^x + (3 - 2\log 3)\, x^{-3} \geqslant e^x > 0$, for $0 < x \leqslant 3$, the function f is convex on $(0, 3]$. By Jensen's inequality, we conclude that $0 = f(1) = f\left(\dfrac{x+y+z}{3}\right) \leqslant \dfrac{f(x) + f(y) + f(z)}{3}$ which implies that $e^{f(x)+f(y)+f(z)} \geqslant 1$.

Remark: Since $f''(x) > 0$, for $x > 0$, the more general inequality

$$\prod_{i=1}^{n}\left(x_k^{1/x_k} e^e\right) \leqslant \exp\left(\sum_{k=1}^{n} e^{x_k}\right),$$

is valid, for positive real numbers $x_1, \ldots, x_n$ with $\sum_{k=1}^{n} x_k = n$.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Let $P = x^{1/x}y^{1/y}z^{1/z}$. Assume wlog that $x \leqslant y \leqslant z$, so that $1/x \geqslant 1/y \geqslant 1/z$ and $\ln x \leqslant \ln y \leqslant \ln z$. Taking the log of P and applying Chebyshev's sum inequality to the two oppositely ordered sequences, we see that

$$\begin{aligned}
\ln P &= \frac{1}{x}\ln x + \frac{1}{y}\ln y + \frac{1}{z}\ln z \\
&\leqslant \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)\left(\frac{\ln x + \ln y + \ln z}{3}\right) \\
&= \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \cdot \ln \sqrt[3]{xyz} \\
&\leqslant \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \cdot \ln\left(\frac{x+y+z}{3}\right) = 0,
\end{aligned}$$

which implies that $P = x^{1/x}y^{1/y}z^{1/z} \leqslant 1$ and $\prod_{x,y,z}(x^{1/x}e^e) \leqslant e^{3e}$. Noting that

$$e^x + e^y + e^z \geqslant 3\sqrt[3]{e^x e^y e^z} = 3e^{\frac{x+y+z}{3}} = 3e,$$

we conclude that $\prod_{x,y,z}(x^{1/x}e^e) \leqslant e^{\sum_{x,y,z} e^x}$.

Solution 4 by Michel Bataille, Rouen, France.

The exponential function is strictly convex, hence $e^x + e^y + e^z \geqslant 3e^{(x+y+z)/3} = 3e$ (with equality if and only if $x = y = z (= 1)$), and strictly increasing, hence $e^{e^x+e^y+e^z} \geqslant e^{3e}$ with equality if and only if $e^x + e^y + e^z = 3e$. Therefore, we have

$$e^{\sum_{x,y,z} e^x} \geqslant e^{3e}$$

with equality if and only if $x = y = z = 1$.

The left-hand side of the inequality being $e^{3e}(x^{1/x}y^{1/y}z^{1/z})$, there just remains to prove that $x^{1/x}y^{1/y}z^{1/z} \leqslant 1$, that is, $\frac{\ln x}{x} + \frac{\ln y}{y} + \frac{\ln z}{z} \leqslant 0$.

Now, the function $f : x \mapsto f(x) = \frac{\ln x}{x}$ satisfies $f''(x) < 0$ for $x \in (0, e^{3/2})$ (since $f''(x) = (2\ln x - 3)x^{-3}$), hence is strictly concave in $(0, 3)$. It follows that

$$\frac{\ln x}{x} + \frac{\ln y}{y} + \frac{\ln z}{z} \leqslant 3 \cdot \frac{\ln((x+y+z)/3)}{(x+y+z)/3} = 0$$

with equality if and only if $x = y = z = 1$.

In conclusion, the required inequality holds, with equality if and only if $x = y = z = 1$.

Solution 5 by Moti Levy, Rehovot, Israel.

Taking logarithm of both sides, the inequality in the problem statement is equivalent to

$$\frac{1}{x}\ln(x) + \frac{1}{y}\ln(x) + \frac{1}{z}\ln(x) \leqslant (e^x - e) + (e^y - e) + (e^z - z) \tag{19}$$

The function $f(u) := \frac{1}{u}\ln(u)$ is concave for $0 < u < 3$ since

$$f''(u) = \frac{1}{u^3}\left(2\ln(u) - 3\right) < 0 \quad for \;\; 0 < u < 3.$$

Then by Jensen's inequality

$$\frac{1}{3}\left(\frac{1}{x}\ln(x) + \frac{1}{y}\ln(x) + \frac{1}{z}\ln(x)\right) \leqslant \frac{1}{\frac{x+y+z}{3}}\ln\left(\frac{x+y+z}{3}\right) = \ln(1) = 0. \tag{20}$$

The function $g(u) := e^u - u$ is convex for $0 < u < 3$ since

$$g''(u) = e^u > 0 \quad for \;\; 0 < u < 3.$$

Then by Jensen's inequality

$$\frac{1}{3}\left((e^x - e) + (e^y - e) + (e^z - z)\right) \geqslant e^{\frac{x+y+z}{3}} - e = e - e = 0. \tag{21}$$

Inequalities (20) and (21) imply (19). ■

Also solved by Toyesh Prakash Sharma, Agra College, Agra, India; Albert Stadler, Herrliberg, Switzerland; Perfetti Paolo, dipartimento di matematica Università di "Tor Vergata", Roma, Italy; and the problem proposer.

• **5750** *Proposed by Albert Natian, Problem Section Editor.*

Assuming all the radicands are non-negative, solve the system of equations for real x and y:

$$\left\{\begin{array}{c} \sqrt[3]{xy + 3x - y + 3} + \sqrt[|x|]{5xy - x + y + 4} = \sqrt[|x|]{3xy - 7x + 3y - 2} \\ \\ \sqrt{3x + 9y + 15} + \sqrt[|yx|]{5y - 2x + 34} = \sqrt[|yx|]{2y - 3x + 29} \end{array}\right\}.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland.

Since all radicands are assumed to be non-negative we deduce from the first equation that $xy + 3x - y + 3 \geqslant 0$ and $5xy - x + y + 4 \leqslant 3xy - 7x + 3y - 2$, and from the second equation that $3x + 9y + 15 \geqslant 0$ and $5y - 2x + 34 \leqslant 2y - 3x + 29$.
So

$$\left\{ \begin{array}{c} xy + 3x - y + 3 = 0 \\ x + 3y + 5 = 0 \end{array} \right\}.$$

We solve this system of equations and find $x = -3y - 5$ and $(-3y - 5)\, y + 3\,(-3y - 5) - y + 3 = -3\,(y + 1)\,(y + 4) = 0$.

So $(x, y) \in \{(-2, -1), (7, -4)\}$. For $(x, y) = (7, -4)$, not all radicands are non-negative. $(x, y) = (-2, -1)$ is indeed a solution that satisfies the given system of equations, and it is the only one where all the radicands are non-negative.

Solution 2 by Michel Bataille, Rouen, France.

We show that the pair $(x, y) = (-2, -1)$ is the unique solution to the system.
It is readily checked that $(-2, -1)$ is a solution. Conversely, let (x, y) be a solution.
We remark that for any x, y, we have

$$(5xy - x + y + 4) - (3xy - 7x + 3y - 2) = 2(xy + 3x - y + 3) \tag{1}$$

and

$$(5y - 2x + 34) - (2y - 3x + 29) = \frac{1}{3}(3x + 9y + 15). \tag{2}$$

Using the fact that the function $t \mapsto t^{1/r}$ is increasing on $(0, \infty)$ when $r > 0$, (1), and $xy+3x-y+3 \geqslant 0$, we obtain

$$\sqrt[|x|]{5xy - x + y + 4} \geqslant \sqrt[|x|]{3xy - 7x + 3y - 2}.$$

The first equation of the system then yields $\sqrt[3]{xy + 3x - y + 3} \leqslant 0$ so that $xy + 3x - y + 3 = 0$.
In a similar way, the second equation of the system gives $x + 3y + 5 = 0$. Therefore we have $x = -3y - 5$ and $y(-3y - 5) + 3(-3y - 5) - y + 3 = 0$, that is, $y^2 + 5y + 4 = 0$. Thus, $y = -1$ or -4 and $(x, y) = (-2, -1)$ or $(7, -4)$. However, the latter leads to $5xy - x + y + 4 = -147 < 0$, hence has to be rejected and we must have $(x, y) = (-2, -1)$.

Also solved by problem proposer.

Editor's Statement: It goes without saying that the problem proposers, as well as the solution proposers, are the *élan vital* of the Problems/Solutions Section of SSMJ. As the editor of this Section of the Journal, I consider myself fortunate to be in a position to receive, compile and organize a wealth of proposed ingenious problems and solutions intended for online publication. My unwavering gratitude goes to all the amazingly creative contributors. We come together from across continents because we find intellectual value, joy and satisfaction in mathematical problems, both in their creation as well as their solution. So that our collective efforts serve us well, I kindly ask all contributors to adhere to the following guidelines. As you peruse below, you may construe that the guidelines amount to a lot of work. But, as the samples show, there's not much to do. Your cooperation is much appreciated!

Keep in mind that the examples given below are your best guide!

Formats, Styles and Recommendations

When submitting proposed problem(s) or solution(s), please send both **LaTeX** document and **pdf** document of your proposed problem(s) or solution(s). There are ways (discoverable from the internet) to convert from Word to proper LaTeX code. Porposals without a *proper* **LaTeX** document will not be published regrettably.

Regarding Proposed Solutions:

Below is the FILENAME format for all the documents of your proposed solution(s).

#ProblemNumber_FirstName_LastName_Solution_SSMJ

- FirstName stands for YOUR first name.
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Examples:

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#9876_Charles_Darwin_Solution_SSMJ

Please note that every problem number is *preceded* by the sign # .

All you have to do is copy the FILENAME format (or an example below it), paste it and then modify portions of it to your specs.

Please adopt the following structure, in the order shown, for the presentation of your solution:

1. On top of the first page of your solution, begin with the phrase:

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where the string of four astrisks represents the problem number.

2. On the second line, write

"Solution proposed by [your First Name, your Last Name]", followed by your affiliation, city, country, all on the same linear string of words. Please see the example below. Make sure you do the same for your collaborator(s).

3. On a new line, state the problem proposer's name, affiliation, city and country, just as it appears published in the Problems/Solutions section.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of the problem.

Here is a sample for the above-stated format for proposed solutions:

> *Proposed solution to #1234 SSMJ*
>
> *Solution proposed by Emmy Noether, University of Göttingen, Lower Saxony, Germany.*
>
> *Problem proposed by Isaac Newton, Trinity College, Cambridge, England.*
>
> **Statement of the problem:**
>
> Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
>
> **Solution of the problem:** . . . . . .

Regarding Proposed Problems:

For all your proposed problems, please adopt for all documents the following FILENAME format:

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If you do not have a ProblemTitle, then leave that component as it already is (i.e., ProblemTitle).

The component YourGivenNumber is any UNIQUE 3-digit (or longer) number you like to give to your problem.

Examples:

Max_Planck_ProposedProblem_SSMJ_314_HarmonicPatterns

Charles_Darwin_ProposedProblem_SSMJ_358_ProblemTitle

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3. On a new line state the title of the problem, if any.

4. On a new line below the above, write in bold type: "**Statement of the Problem**".

5. Below the latter, state the problem. Please make sure the statement of your problem (unlike the preceding item) is not in bold type.

6. Below the statement of the problem, write in bold type: "**Solution of the Problem**".

7. Below the latter, show the entire solution of your problem.

Here is a sample for the above-stated format for proposed problems:

Problem proposed to SSMJ

Problem proposed by Isaac Newton, Trinity College, Cambridge, England.

Principia Mathematica (⟵ You may choose to not include a title.)

Statement of the problem:

Compute $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Solution of the problem: . . . . . .

♣ ♣ ♣ Thank You! ♣ ♣ ♣