Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before June 15, 2008

- 5014: *Proposed by Kenneth Korbin, New York, NY.*

  Given triangle ABC with $a = 100, b = 105$, and with equal cevians $\overline{AD}$ and $\overline{BE}$. Find the perimeter of the triangle if $\overline{AE} \cdot \overline{BD} = \overline{CE} \cdot \overline{CD}$.

- 5015: *Proposed by Kenneth Korbin, New York, NY.*

  Part I: Find the value of
  $$\sum_{x=1}^{10} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$
  Part II: Find the value of
  $$\sum_{x=1}^{\infty} \text{Arcsin}\left(\frac{4x^2}{4x^4+1}\right).$$

- 5016: *Proposed by John Nord, Spokane, WA.*

  Locate a point (p, q) in the Cartesian plane with integral values, such that for any line through (p, q) expressed in the general form $ax + by = c$, the coefficients a, b, c form an arithmetic progression.

- 5017: *Proposed by M.N. Deshpande, Nagpur, India.*

  Let ABC be a triangle such that each angle is less than 90^0. Show that
  $$\frac{a}{c \cdot \sin B} + \frac{1}{\tan A} = \frac{b}{a \cdot \sin C} + \frac{1}{\tan B} = \frac{c}{b \cdot \sin A} + \frac{1}{\tan C}$$

where $a = l(\overline{BC}), b = l(\overline{AC})$, and $c = l(\overline{AB})$.

- 5018: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Write the polynomial $x^{5020} + x^{1004} + 1$ as a product of two polynomials with integer coefficients.

- 5019: *Michael Brozinsky, Central Islip, NY.*

  In a horse race with 10 horses the horse with the number one on its saddle is referred to as the number one horse, and so on for the other numbers. The outcome of the race showed the number one horse did not finish first, the number two horse did not finish second, the number three horse did not finish third and the number four horse did not finish fourth. However, the number five horse did finish fifth. How many possible orders of finish are there for the ten horses assuming no ties?

Solutions

- 4996: *Proposed by Kenneth Korbin, New York, NY.*

  Simplify:
$$\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right).$$

Solution by José Hernández Santiago, (student, UTM, Oaxaca, México.)

$$\begin{aligned}
\sum_{i=1}^{N}\binom{N}{i}\left(2^{i-1}\right)\left(1+3^{N-i}\right) &= \sum_{i=1}^{N}\binom{N}{i}2^{i-1}+\sum_{i=1}^{N}\binom{N}{i}2^{i-1}\cdot 3^{N-i}\\
&= \frac{1}{2}\sum_{i=1}^{N}\binom{N}{i}2^{i}+\frac{3^N}{2}\sum_{i=1}^{N}\binom{N}{i}\left(\frac{2}{3}\right)^{i}\\
&= \left(\frac{1}{2}\right)\left(\left(2+1\right)^{N}-1\right)+\left(\frac{3^N}{2}\right)\left(\left(\frac{2}{3}+1\right)^{N}-1\right)\\
&= \frac{3^N-1}{2}+\frac{3^N}{2}\left(\frac{5^N-3^N}{3^N}\right)\\
&= \frac{(3^N-1)3^N+3^N(5^N-3^N)}{2\cdot 3^N}\\
&= \frac{15^N-3^N}{2\cdot 3^N}\\
&= \frac{5^N-1}{2}
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Michael Brozinsky, Central

Islip, NY; John Boncek, Montgomery, AL; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Paul M. Harms, North Newton, KS; N. J. Kuenzi, Oshkosh, WI; Kee-Wai Lau, Hong Kong, China; Carl Libis, Kingston, RI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4997: *Proposed by Kenneth Korbin, New York, NY.*

Three different triangles with integer-length sides all have the same perimeter P and all have the same area K.
Find the dimensions of these triangles if $K = 420$.

Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX.

Let a, b, c be the sides of the triangle and, for convenience, assume that $a \leq b \leq c$. By Heron's Formula,

$$(420)^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{-a+b+c}{2}\right) \tag{1}$$

Since a, b, c are positive integers, it is easily demonstrated that the quantities $(a+b-c)$, $(a-b+c)$, and $(-a+b+c)$ are all odd or all even. By (1), it is clear that in this case, they are all even. Therefore, there are positive integers x, y, z such that $a+b-c = 2x$, $a-b+c = 2y$, and $-a+b+c = 2z$. Then, $a = x+y$, $b = x+z$, $c = y+z$, $a+b+c = 2(x+y+z)$, and $a \leq b \leq c$ implies that $x \leq y \leq z$. With this substitution, (1) becomes

$$(420)^2 = xyz(x+y+z) \tag{2}$$

Since $x \leq y \leq z < x+y+z$, (2) implies that

$$x^4 < xyz(x+y+z) = (420)^2$$

and hence,

$$1 \leq x \leq \left\lfloor \sqrt{420} \right\rfloor = 20,$$

where $\lfloor m \rfloor$ denotes the greatest integer $\leq m$. Therefore, the possible values of x are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20$ (since x must also be a factor of $(420)^2$). Further, for each x, (2) implies that

$$y^3 < yz(x+y+z) = \frac{(420)^2}{x},$$

i.e.,

$$x \leq y \leq \left\lfloor \sqrt[3]{\frac{(420)^2}{x}} \right\rfloor,$$

(and y is a factor of $(420)^2 x$). Once we have assigned values to x and y, (2) becomes

$$z(x+y+z) = \frac{(420)^2}{xy}, \tag{3}$$

which is a quadratic equation in z. If (3) yields an integral solution $\geq y$, we have found a viable solution for x, y, z and hence, for a, b, c also. By finding all such solutions, we can find all Heronian triangles (triangles with integral sides and integral area) whose area is 420. Then, we must find three of these with the same perimeter to complete our solution.

The following two cases illustrate the typical steps encountered in this approach.

Case 1. If $x = 1$ and $y = 18$, (3) becomes

$$z^2 + 19z - 9800 = 0.$$

Since this has no integral solutions, this case does not lead to feasible values for a, b, c.

Case 2. If $x = 2$ and $y = 24$, (3) becomes

$$z^2 + 26z - 3675 = 0,$$

which has $z = 49$ as its only positive integral solution. These values of x, y, z yield $a = 26$, $b = 51$, $c = 73$, and $P = 150$.

The results of our approach are summarized in the following table.

x	y	z	a	b	c	P
1	6	168	7	169	174	350
1	14	105	15	106	119	240
1	20	84	21	85	104	210
1	25	72	26	73	97	196
1	40	49	41	50	89	180
2	24	49	26	51	73	150
2	35	35	37	37	70	144
4	21	35	25	39	56	120
5	9	56	14	61	65	140
5	21	30	26	35	51	112
6	15	35	21	41	50	112
8	21	21	29	29	42	100
9	20	20	29	29	40	98
10	15	24	25	34	39	98
12	12	25	24	37	37	98

Now, it is obvious that the last three entires constitute the solution of this problem.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4998: *Proposed by Jyoti P. Shiwalkar & M.N. Deshpande, Nagpur, India.*

  Let $A = [a_{i,j}]$, $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$ be a triangular array satisfying the following conditions:

$$\begin{aligned} 1)\ a_{i,1} &= L(i) \text{ for all } i \\ 2)\ a_{i,i} &= i \text{ for all } i \\ 3)\ a_{i,j} &= a_{i-1,j} + a_{i-2,j} + a_{i-1,j-1} - a_{i-2,j-1} \text{ for } 2 \le j \le (i-1). \end{aligned}$$

If $T(i) = \sum_{j=1}^{i} a_{i,j}$ for all $i \ge 2$, then find a closed form for $T(i)$, where $L(i)$ are the Lucas numbers, $L(1) = 1, L(2) = 3$, and $L(i) = L(i-1) + L(i-2)$ for $i \ge 3$.

Solution by Paul M. Harms, North Newton, KS.

Note that $a_{i-2,j}$ is not in the triangular array when $j = i-1$, so we set $a_{i-2,i-1} = 0$. From Lucas numbers $a_{i,1} = a_{i-1,1} + a_{i-2,1}$ for $i > 2$. For $i > 2$,

$$\begin{aligned} T(i) &= a_{i,1} + a_{i,2} + \cdots + a_{i,i-1} + i \\ &= (a_{i-1,1} + a_{i-2,1}) + (a_{i-1,2} + a_{i-2,2} + a_{i-1,1} - a_{i-2,1}) + \cdots \\ &\quad +(a_{i-1,i-1} + a_{i-2,i-1} + a_{i-1,i-2} - a_{i-2,i-2}) + i. \end{aligned}$$

Therefore we have

$$(a_{i-1,i-1} + a_{i-2,i-1} + a_{i-1,i-2} - a_{i-2,i-2}) = (i-1) + 0 + a_{i-1,i-2} - (i-2).$$

Note that in $T(i)$ each term of row $(i-2)$ appears twice and subtracts out and each term of row $(i-1)$ except for the last term $(i-1)$, is added to itself. The term $(i-1)$ appears once. If we write the last term, i, of $T(i)$ as $i = (i-1)+1$, then $T(i) = 2T(i-1)+1$. The values of the row sums are:

$$\begin{aligned} T(1) &= 1 \\ T(2) &= 5 \\ T(3) &= 2(5)+1 \\ T(4) &= 2(2(5)+1)+1 = 2^2(5)+2+1 \\ T(5) &= 2\Big(2[2(5)+1]+1\Big)+1 = 2^3(5)+2^2+2+1, \text{ and in general} \\ T(i) &= 2^{i-2}(5) + (2^{i-3} + 2^{i-4} + \cdots + 1) \\ &= 2^{i-2}(5) + (2^{i-2} - 1) \\ &= 2^{i-2}(6) - 1 \\ &= 2^{i-1}(3) \text{ for } i \geq 2. \end{aligned}$$

Also solved by Carl Libis, Kingston, RI; N. J. Kuenzi, Oshkosh, WI; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 4999: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Find all real triplets (x, y, z) such that

$$\begin{aligned} x + y + z &= 2 \\ 2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2} &= 6\sqrt[9]{2} \end{aligned}$$

Solution by David E. Manes, Oneonta, NY.

The only real solution is $(x, y, z) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. Note that these values do satisfy each of the equations.

By the Arithmetic-Geometric Mean Inequality,

$$6\sqrt[9]{2} = 2^{x+y^2} + 2^{y+z^2} + 2^{z+x^2}$$

$$\geq \; 3\sqrt[3]{2^{x+y+z}\cdot 2^{x^2+y^2+z^2}} = 2\cdot 2^{2/3}\sqrt[3]{2^{x^2+y^2+z^2}}.$$

Therefore, $2^{x^2+y^2+z^2} \leq 2^{4/3}$ so that $x^2+y^2+z^2 \leq 4/3 \quad (\mathbf{1})$. Note that

$$\begin{aligned} 4 = (x+y+z)^2 &= x^2+y^2+z^2+2(xy+yz+zx), \text{ so that} \\ x^2+y^2+z^2 &= 4-2(xy+yz+zx). \end{aligned}$$

Substituting in ($\mathbf{1}$) yields the inequality $xy+yz+zx \geq \dfrac{4}{3}$. From $(x-y)^2+(y-z)^2+(z-x)^2 \geq 0$ with equality if and only if $x=y=z$, one now obtains the inequalities

$$\frac{4}{3} \geq x^2+y^2+z^2 \geq xy+yz+zx \geq \frac{4}{3}.$$

Hence

$$\begin{aligned} x^2+y^2+z^2 &= xy+yz+zx = \frac{4}{3} \\ (x-y)^2+(y-z)^2+(z-x)^2 &= 0, \text{ and } x=y=z=\frac{2}{3}. \end{aligned}$$

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Also solved by Dionne Bailey, Elsie Campbell, Charles Diminnie and Karl Havlak (jointly), San Angelo, TX; Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; Paolo Perfetti, Math Dept. U. of Rome, Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposers.

- 5000: *Proposed by Richard L. Francis, Cape Girardeau, MO.*

Of all the right triangles inscribed in the unit circle, which has the Morley triangle of greatest area?

Solution by Ken Korbin, New York, NY.

Given $\triangle ABC$ with circumradius $R=1$ and with $A+B=C=90^o$.
The side x of the Morley triangle is given by the formula

$$\begin{aligned} x &= 8\cdot R\cdot \sin(\frac{A}{3})\cdot\sin(\frac{B}{3})\cdot\sin(\frac{C}{3}) \\ &= 8\cdot 1\cdot \sin(\frac{A}{3})\cdot\sin(\frac{B}{3})\cdot\frac{1}{2} \\ &= 4\sin(\frac{A}{3})\sin(\frac{B}{3}). \end{aligned}$$

x will have a maximum value if

$$\frac{A}{3} = \frac{B}{3} = \frac{45^o}{3} = 15^o.$$

Then,

$$x = 4\sin^2(15^o)$$

$$\begin{aligned} &= 4\Big(\frac{1-\cos 30^o}{2}\Big) \\ &= 2-2\cos 30^o \\ &= 2-\sqrt{3}. \end{aligned}$$

The area of this Morley triangle is

$$\begin{aligned} & \frac{1}{2}\cdot(2-\sqrt{3})^2\cdot\sin 60^o \\ = & \frac{1}{2}(7-4\sqrt{3})\cdot\frac{\sqrt{3}}{2} = \frac{7\sqrt{3}-12}{4}. \end{aligned}$$

Comment by David Stone and John Hawkins: "It may be the maximum, but it is pretty small!"

Also solved by Michael Brozinsky, Central Islip, NY; Kee-Wai Lau, Hong Kong, China; David Stone an John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5001:** *Proposed by Ovidiu Furdui, Toledo, OH.*

Evaluate:

$$\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx.$$

Solution by Kee-Wai Lau, Hong Kong, China.

We show that $\displaystyle\int_0^\infty \ln^2\left(\frac{x^2}{x^2+3x+2}\right)dx = 2\ln^2 2+\frac{11\pi^2}{6}$.

Denote the integral by I. Replacing x by $1/x$, we obtain

$$\begin{aligned} I &= \int_0^\infty \frac{\ln^2\Big((x+1)(2x+1)\Big)}{x^2}dx = \int_0^\infty \frac{\ln^2(x+1)}{x^2}dx + \int_0^\infty \frac{\ln^2(2x+1)}{x^2}dx \\ & \qquad +2\int_0^\infty \frac{\ln(x+1)\ln(2x+1)}{x}dx \\ &= I_1+I_2+2I_3, \text{ say.} \end{aligned}$$

Integrating by parts, we obtain

$$I_1 = \int_0^\infty \ln^2(x+1)d(\frac{-1}{x}) = 2\int_0^\infty \frac{\ln(x+1)}{x(x+1)}dx = 2\int_1^\infty \frac{\ln x}{x(x-1)}dx.$$

Replacing x by $/(1-x)$,we obtain $I_1 = -2\displaystyle\int_0^1 \frac{\ln(1-x)}{x}dx = 2\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{3}$.

Replacing x by $x/2$ in I_2, we see that $I_2 = 2I_1 = \dfrac{2\pi^2}{3}$. Next note that

$$I_3 = \int_0^\infty \ln(x+1)\ln(2x+1)d(\frac{-1}{x}) = \int_0^\infty \frac{\ln(2x+1)}{x(x+1)}dx + 2\int_0^\infty \frac{\ln(x+1)}{x(2x+1)}dx = J_1+2J_2, \text{ say.}$$

Replacing x by $x/2$, then x by $x-1$ and then x by $1/x$, we have

$$J_1 = 2\int_0^\infty \frac{\ln(x+1)}{x(x+2)}dx \;=\; 2\int_1^\infty \frac{\ln x}{(x-1)(x+1)}dx$$

$$= \; -2\int_0^1 \frac{\ln x}{(1-x)(1+x)}dx = -\int_0^1 \ln x\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx.$$

Integrating by parts, we have,

$$J_1 = \int_0^1 \frac{-\ln(1-x)+\ln(1+x)}{x}dx = \sum_{n=1}^\infty \frac{1+(-1)^{n-1}}{n^2} = \frac{\pi^2}{6}+\frac{\pi^2}{12} = \frac{\pi^2}{4}.$$

We now evaluate J_2. Replacing $x+1$ by x and then x by $1/x$, we have

$$J_2 = \int_1^\infty \frac{\ln x}{(x-1)(2x-1)}dx = -\int_0^1 \frac{\ln x}{(1-x)(2-x)}dx = -\int_0^1 \frac{\ln x}{1-x}+\int_0^1 \frac{\ln x}{2-x}dx = \frac{\pi^2}{6}+K, \text{ say.}$$

Replacing x by $1-x$

$$K = \int_0^1 \frac{\ln(1-x)}{1+x}dx \;=\; \int_0^1 \frac{\ln(1+x)}{1+x}dx + \int_0^1 \frac{\ln(1-x)-\ln(1+x)}{1+x}dx$$

$$= \; \frac{1}{2}\ln^2 2 + \int_0^1 \frac{\ln\left(\dfrac{1-x}{1+x}\right)}{1+x}dx.$$

By putting $y = \dfrac{1-x}{1+x}$, we see that the last integral reduces to $\displaystyle\int_0^1 \frac{\ln y}{1+y}dy = -\frac{\pi^2}{12}$.

Hence, $K = \dfrac{1}{2}\ln^2 2 - \dfrac{\pi^2}{12}$, $J_2 = \dfrac{1}{2}\ln^2 2 + \dfrac{\pi^2}{12}$, $I_3 = \ln^2 2 + \dfrac{5\pi^2}{12}$ and finally

$$I = I_1 + I_2 + 2I_3 = \frac{\pi^2}{3} + \frac{2\pi^2}{3} + 2\left(\ln^2 2 + \frac{5\pi^2}{12}\right) = 2\ln^2 2 + \frac{11\pi^2}{6} \text{ as desired.}$$

Also solved by Paolo Perfetti, Math. Dept., U. of Rome, Italy; Worapol Rattanapan (student at Montfort College (high school)), Chiang Mai, Thailand, and the proposer.