Problems

Ted Eisenberg, Section Editor

****

This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
February 15, 2008

- 4990: *Proposed by Kenneth Korbin, New York, NY.*

  Solve
  $$40x + 42\sqrt{1-x^2} = 29\sqrt{1+x} + 29\sqrt{1-x}$$
  with $0 < x < 1$.

- 4991: *Proposed by Kenneth Korbin, New York, NY.*

  Find six triples of positive integers (a, b, c) such that
  $$\frac{9}{a} + \frac{a}{b} + \frac{b}{9} = c.$$

- 4992: *Proposed by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX.*

  A closed circular cone has integral values for its height and base radius. Find all possible values for its dimensions if its volume V and its total area (including its circular base) A satisfy $V = 2A$.

- 4993: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation
  $$126x^7 - 127x^6 + 1 = 0.$$

- 4994: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be three nonzero complex numbers lying on the circle $C = \{z \in \mathbf{C} : |\mathrm{z}| = \mathrm{r}\}$. Prove that the roots of the equation $az^2 + bz + c = 0$ lie in the ring shaped region $D = \left\{ z \in \mathbf{C} : \frac{1-\sqrt{5}}{2} \le |\mathrm{z}| \le \frac{1+\sqrt{5}}{2} \right\}$.

- 4995: *Proposed by K. S. Bhanu and M. N. Deshpande, Nagpur, India.*

Let A be a triangular array $a_{i,j}$ where $i = 1, 2, \cdots,$ and $j = 0, 1, 2, \cdots, i$. Let

$$a_{1,0} = 1, \ a_{1,1} = 2, \text{ and } a_{i,0} = T(i+1) - 2 \text{ for } i = 2, 3, 4, \cdots,$$

where $T(i+1) = (i+1)(i+2)/2$, the usual triangular numbers. Furthermore, let $a_{i,j+1} - a_{i,j} = j + 1$ for all j. Thus, the array will look like this:

$$\begin{array}{c} 1 \quad 2 \\ 4 \quad 5 \quad 7 \\ 8 \quad 9 \quad 11 \quad 14 \\ 13 \quad 14 \quad 16 \quad 19 \quad 23 \\ 19 \quad 20 \quad 22 \quad 25 \quad 29 \quad 34 \end{array}$$

Show that for every pair (i, j), $4a_{i,j} + 9$ is the sum of two perfect squares.

Solutions

- 4972:*Proposed by Kenneth Korbin, New York, NY.*

Find the length of the side of equilateral triangle ABC if it has a cevian $\overline{\mathrm{CD}}$ such that

$$\overline{\mathrm{AD}} = x, \quad \overline{\mathrm{BD}} = x + 1 \quad \overline{\mathrm{CD}} = \sqrt{y}$$

where x and y are positive integers with $20 < x < 120$.

Solution by Kee-Wai Lau, Hong Kong, China.

Applying the cosine formula to triangle CAD, we obtain

$$\overline{CD}^2 = \overline{AD}^2 + \overline{AC}^2 - 2\overline{AD} \cdot \overline{AC} \cos 60^o,$$

or

$$\begin{aligned} (\sqrt{y})^2 &= x^2 + (2x+1)^2 - 2x(2x+1)\cos 60^o \\ y &= 3x^2 + 3x + 1. \end{aligned}$$

For $20 < x < 120$, we find using a calculator that y is the square of a positive integer if $x = 104, y = 32761$. Hence the length of the side of equilateral triangle ABC is 209.

Comments:

1) Scott H. Brown, Montgomery, AL.

The list of pairs (x, y) that satisfy the equation $y = 3x^2 + 3x + 1$ is so large I will not attempt to name each pair...numerous triangles with the given conditions can be found.

2) David Stone and John Hawkins, Statesboro, GA.

The restriction on x seems artificial–every x produces a triangle. In fact, if we require the cevian length to be an integer, this becomes a Pell's Equation problem and we can generate nice solutions recursively in the usual fashion. The first few for $x,\ s = 2x+1,\ y = 3x^2+3x+1,\ \&$ cevian $= \sqrt{y}$ are:

7	15	169	13
104	209	32761	181
1455	2911	6355441	2521
20272	40545	1232922769	35113

Also solved by Peter E. Liley, Lafayette, IN, and the proposer.

- 4973: *Proposed by Kenneth Korbin, New York, NY.*

Find the area of trapezoid ABCD if it is inscribed in a circle with radius R=2, and if it has base $\overline{\text{AB}} = 1$ and $\angle\text{ACD} = 60^o$.

Solution by David E. Manes, Oneonta, NY.

The area A of the trapezoid is given by $A = \dfrac{3\sqrt{3}}{8}\left(15+\sqrt{5}\right)$.

Since the trapezoid is cyclic, it is isosceles so that $AD = BC$. Note that $\angle ACD = 60^o \Rightarrow \angle CAB = 60^o$ since alternate interior angles of a transversal intersecting two parallel lines are congruent. Therefore, from the law of sines in triangle ABC, $\dfrac{BC}{\sin 60^o} = 2R$ or $BC = 2\sqrt{3}$. Using the law of cosines in triangle ABC,

$$BC^2 = 1 + AC^2 - 2AC\cdot\cos 60^o,\ \text{ or } \text{AC}^2 - \text{AC} - 11 = 0.$$

Thus, AC is the positive root of this equation so that $AC = \dfrac{1+3\sqrt{5}}{2}$. Similarly, using the law of cosines in triangle ACD and recalling that $AD = BC$, one obtains

$$AD^2 = AC^2 + DC^2 - 2\cdot AC\cdot DC\cdot\cos 60^o$$

or $\text{DC}^2 - \left(\dfrac{1+3\sqrt{5}}{2}\right)DC + \dfrac{-1+3\sqrt{5}}{2} = 0$. Noting that $DC > 2$ and $\sqrt{6-2\sqrt{5}} = \sqrt{(1-\sqrt{5})^2} = \sqrt{5}-1$, it follows that $DC = 3\sqrt{5}-1$. Finally, let H be the point on line segment $\overline{DC}$ such that $\overline{AH}$ is perpendicular to $\overline{DC}$. Then the height h of the trapezoid is given by $h = AC\cdot\sin 60^o = \dfrac{\sqrt{3}}{4}\left(1+3\sqrt{5}\right)$. Hence,

$$A = \frac{1}{2}\left(AB+DC\right)\cdot h = \frac{1}{2}\left(1+3\sqrt{5}-1\right)\frac{\sqrt{3}}{4}\left(1+3\sqrt{5}\right) = \frac{3\sqrt{3}}{8}\left(15+\sqrt{5}\right).$$

Also solved by Robert Anderson, Gino Mizusawa, and Jahangeer Kholdi (jointly), Portsmouth, VA; Dionne Bailey, Elsie Campbell, and Charles Diminnie, (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Zhonghong Jiang, NY, NY; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4974: *Proposed by Kenneth Korbin, New York, NY.*

  A convex cyclic hexagon has sides a, a, a, b, b, and b. Express the values of the circumradius and the area of the hexagon in terms of a and b.

  Solution by Kee-Wai Lau, Hong Cong, China.

  We show that the circumradius R is $\sqrt{\dfrac{a^2+ab+b^2}{3}}$ and the area A of the hexagon is $\dfrac{\sqrt{3}(a^2+4ab+b^2)}{4}$.

  Denote the angle subtended by side a and side b at the center of the circumcircle respectively by θ and ϕ. Since $3\theta+3\phi=360^o$ so $\theta=120-\phi$ and

  $$\cos\theta=\cos(120^o-\phi)=\frac{-\cos\phi+\sqrt{3}\sin\phi}{2}. \text{ Hence,}$$

  $$(2\cos\theta+\cos\phi)^2=3(1-\cos^2\phi) \text{ or } 4\cos^2\theta+4\cos\theta\cos\phi+4\cos^2\phi-3=0.$$

  Now by the cosine formula $\cos\theta=\dfrac{2R^2-a^2}{2R^2}$ and $\cos\phi=\dfrac{2R^2-b^2}{2R^2}$.

  Therefore,

  $$\begin{aligned}(2R^2-a^2)^2+(2R^2-a^2)(2R^2-b^2)+(2R^2-b^2)^2-3R^4 &= 0 \text{ or}\\ 9R^4-6(a^2+b^2)R^2+a^4+a^2b^2+b^4 &= 0.\end{aligned}$$

  Solving the equation we obtain $R^2=\dfrac{a^2+ab+b^2}{3}$ or $R^2=\dfrac{a^2-ab+b^2}{3}$. The latter result is rejected because if not, then for $a=b$, we have $\cos\theta=\cos\phi<0$ so that $\theta+\phi>180^o$, which is not true. Hence, $R=\sqrt{\dfrac{a^2+ab+b^2}{3}}$.

  To find A, we need to find the area of the triangles with sides R, R, a and R, R, b. The heights to bases a and b are respectively $\dfrac{\sqrt{4R^2-a^2}}{2}=\dfrac{\sqrt{3}(a+2b)}{6}$ and $\dfrac{\sqrt{4R^2-b^2}}{2}=\dfrac{\sqrt{3}(2a+b)}{6}$. Hence the area of the hexagon equals

  $$3\left(\frac{\sqrt{3}a(a+2b)}{12}+\frac{\sqrt{3}b(2a+b)}{12}\right)=\frac{\sqrt{3}}{4}\left(a^2+4ab+b^2\right) \text{ as claimed.}$$

  Also solved by Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Zhonghong Jiang, NY, NY; David E. Manes, Oneonta, NY; M. N. Deshpande, Nagpur, India; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; Jonathan Schrock, Seth Bird, and Jim Moore (jointly, students at Taylor University), Upland, IN; David Wilson, Winston-Salem, NC, and the proposer.

- 4975: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Solve in R the following system of equations

$$\left.\begin{array}{r} 2x_1 = 3x_2\sqrt{1+x_3^2} \\ 2x_2 = 3x_3\sqrt{1+x_4^2} \\ \cdots\cdots \\ 2x_n = 3x_1\sqrt{1+x_2^2} \end{array}\right\}$$

Solution by David Stone and John Hawkins, Statesboro, GA.

Squaring each equation and summing, we have

$$4(x_1^2+x_2^2+x_3^2+\cdots+x_n^2) = 9(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)+9(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+\cdots+x_{n-1}^2x_n^2).$$

So

$$0 = 5(x_1^2+x_2^2+x_3^2+\cdots+x_n^2)+9(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_4^2+\cdots+x_{n-1}^2x_n^2).$$

Because these squares are non-negative and the sum is zero, each term on the right-hand side must indeed equal 0. Therefore $x_1 = x_2 = x_3 = \cdots = x_n = 0$.

Alternatively, we could multiply the equations to obtain

$$2^n x_1x_2x_3x_4\cdots x_n = 3^n x_1x_2x_3x_{4n}\sqrt{1+x_1^2}\sqrt{1+x_2^2}\cdots\sqrt{1+x_n^2}.$$

If all x_k are non-zero, we'll have $\left(\dfrac{2}{3}\right)^n = \sqrt{1+x_1^2}\sqrt{1+x_2^2}\cdots\sqrt{1+x_n^2}$. The term on the left is < 1, while each term on the right is > 1, so the product is > 1. Thus we have reached a contradiction, forcing all x_k to be zero.

Also solved by Bethany Ballard, Nicole Gottier, and Jessica Heil (jointly, students, Taylor University), Upland, IN; Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX; Matt DeLong, Upland, IN; Paul M. Harms, North Newton, KS; Mandy Isaacson, Julia Temple, and Adrienne Ramsay (jointly, students, Taylor University), Upland, IN; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA, and the proposer.

- 4976: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers. Prove that

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \geq 27.$$

Solution by Matt DeLong, Upland, IN.

In fact, I will prove that the sum is at least 39. Rewrite the sum

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab} \text{ as}$$

$$\frac{a^2}{bc}+3\frac{b}{c}+9\frac{c}{b}+\frac{b^2}{ca}+3\frac{c}{a}+9\frac{a}{c}+\frac{c^2}{ab}+3\frac{a}{b}+9\frac{b}{a}.$$

Rearranging the terms gives

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)+6\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)$$

Now, repeatedly apply the Arithmetic Mean-Geometric Mean inequality.

$$\begin{aligned}
\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab} &\geq 3\left(\frac{a^2b^2c^2}{bccaab}\right)^{1/3}=3\\
\frac{b}{c}+\frac{c}{b} &\geq 2\left(\frac{bc}{cb}\right)^{1/2}=2\\
\frac{c}{a}+\frac{a}{c} &\geq 2\left(\frac{ac}{ca}\right)^{1/2}=2\\
\frac{a}{b}+\frac{b}{a} &\geq 2\left(\frac{ab}{ba}\right)^{1/2}=2\\
\frac{c}{b}+\frac{a}{c}+\frac{b}{a} &\geq 3\left(\frac{cab}{bca}\right)^{1/3}=3.
\end{aligned}$$

Thus, we have

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+\frac{a}{b}+\frac{b}{a}\right)+6\left(\frac{c}{b}+\frac{a}{c}+\frac{b}{a}\right)\geq 3+3(2+2+2)+6(3).$$

In other words

$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab}\geq 39$$

.

Also solved by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie, San Angelo, TX; Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly, students, Taylor University), Upland, IN; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4977: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $1 < a < b$ be real numbers. Prove that for any $x_1, x_2, x_3 \in [a, b]$ there exist $c \in (a, b)$ such that

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log c}.$$

Solution by Solution 1 by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie, San Angelo, TX .

Strictly speaking, the conclusion is incorrect as stated. If $a = x_1 = x_2 = x_3$, then

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log a}.$$

Similarly,

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log b}$$

when $b = x_1 = x_2 = x_3$.

The statement is true when $1 < a \leq x_1 \leq x_2 \leq x_3 \leq b$ with $x_1 \neq x_3$. Since

$$\frac{3}{\log x_1x_2x_3}=\frac{3}{\log x_1+\log x_2+\log x_3},$$

then

$$\frac{4}{\log x_3} < \frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{3}{\log x_1x_2x_3} < \frac{4}{\log x_1}.$$

By the Intermediate Value Theorem, there exists $c \in (a, b)$ such that

$$\frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{3}{\log x_1x_2x_3} = \frac{4}{\log c}.$$

Solution 2 by Paul M. Harms, North Newton, KS.

Assume $x_1 < x_3$ with x_2 in the interval $[x_1, x_3]$. For $x > 1$, we note that $f(x) = \log(x)$ and $g(x) = 1/\log(x)$ are both continuous, one-to-one, positive functions with $f(x)$ strictly increasing and $g(x)$ strictly decreasing.

Consider

$$\frac{3}{\log(x_1x_2x_3)} = \frac{1}{\dfrac{\log(x_1) + \log(x_2) + \log(x_3)}{3}}.$$

The denominator is the average of the 3 log values which means this average value is between the extremes $\log x_1$ and $\log x_3$. Since $f(x)$ is one-to-one and continuous there is a value x_4 where $x_1 < x_4 < x_3$ and $\log x_4 = \dfrac{(\log x_1 + \log x_2 + \log x_3)}{3}$ with $\log x_4$ between $\log x_1$ and $\log x_3$.

The equation in the problem can now be written

$$\frac{\frac{1}{\log x_1} + \frac{1}{\log x_2} + \frac{1}{\log x_3} + \frac{1}{\log x_4}}{4} = \frac{1}{\log c} \text{ or}$$

$$\frac{g(x_1) + g(x_2) + g(x_3) + g(x_4)}{4} = \frac{1}{\log c}.$$

The average of the four $g(x)$ values is between the extremes $g(x_1)$ and $g(x_3)$. Since $g(x)$ is continuous and one-to-one there is a value $x = c$ such that

$$g(c) = \frac{1}{\log c} = \frac{g(x_1) + g(x_2) + g(x_3) + g(x_4)}{4}$$

where $x_1 < c < x_3$ and, thus, $a < c < b$.

Note that if $x_1 = x_2 = x_3$, then we obtain $c = x_1 = x_2 = x_3$. If we want $a < c < b$, then it appears that we need to keep x_1, x_2 and x_3 away from a and b when these three x-values are equal to each other.

Also solved by Michael Brozinsky, Central Islip, NY; Matt DeLong, Upland, IN; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.