Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before April 15, 2008

- 5002: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon with sides $3x, 3x, 3x, 5x, 5x$ and $5x$ is inscribed in a unit circle. Find the value of x.

- 5003: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\begin{aligned} \sqrt[3]{x+\sqrt{x^2-1}}+\sqrt[3]{x-\sqrt{x^2-1}} &= \frac{7}{2} \text{ and} \\ \sqrt[3]{y+\sqrt{y^2-1}}+\sqrt[3]{y-\sqrt{y^2-1}} &= \sqrt{10} \end{aligned}$$

- 5004: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{\sqrt{ab}}{1+a+b}+\frac{\sqrt{bc}}{1+b+c}+\frac{\sqrt{ca}}{1+c+a}$$

- 5005: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{\sqrt{3}}{2}\left(a+b+c\right)^{1/2} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

- 5006: *Proposed by Ovidiu Furdui, Toledo, OH.*

  Find the sum
$$\sum_{k=2}^{\infty}(-1)^k \ln\left(1-\frac{1}{k^2}\right).$$

- 5007: *Richard L. Francis, Cape Girardeau, MO.*

  Is the centroid of a triangle the same as the centroid of its Morley triangle?

Solutions

- 4984: *Proposed by Kenneth Korbin, New York, NY.*

  Prove that
$$\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{2009}+\sqrt{2011}} > \sqrt{120}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

The sum

$$\begin{aligned}
& \sum_{k=1}^{503}\frac{1}{\sqrt{4k-3}+\sqrt{4k-1}} \\
> \ & \frac{1}{2}\sum_{k=1}^{503}\left(\frac{1}{\sqrt{4k-3}+\sqrt{4k-1}}+\frac{1}{\sqrt{4k-1}+\sqrt{4k+1}}\right) \\
= \ & \frac{1}{2}\sum_{k=1}^{503}\left(\frac{\sqrt{4k-1}-\sqrt{4k-3}}{2}+\frac{\sqrt{4k+1}-\sqrt{4k-1}}{2}\right) \\
= \ & \frac{1}{4}\sum_{k=1}^{503}\left(\sqrt{4k+1}-\sqrt{4k-3}\right) \\
= \ & \frac{1}{4}\left(\sqrt{2013}-1\right) \\
= \ & \frac{1}{4}\sqrt{2013-2\sqrt{2013}+1} \\
> \ & \frac{1}{4}\left(\sqrt{2013-2(45)}+1\right) \\
> \ & \frac{1}{4}\sqrt{1920} \\
= \ & \sqrt{120}
\end{aligned}$$

as required.

Solution 2 by Kenneth Korbin, the proposer.

Let $K = \dfrac{1}{\sqrt{1}+\sqrt{3}} + \dfrac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \dfrac{1}{\sqrt{2009}+\sqrt{2011}}$.

Then, $K > \dfrac{1}{\sqrt{3}+\sqrt{5}} + \dfrac{1}{\sqrt{7}+\sqrt{9}} + \cdots + \dfrac{1}{\sqrt{2011}+\sqrt{2013}}$

and,

$$\begin{aligned}
2K &> \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{2011}+\sqrt{2013}} \\
&= \frac{\sqrt{3}-\sqrt{1}}{2} + \frac{\sqrt{5}-\sqrt{3}}{2} + \frac{\sqrt{7}-\sqrt{5}}{2} + \cdots + \frac{\sqrt{2013}-\sqrt{2011}}{2} \\
&= \frac{\sqrt{2013}-1}{2}. \text{ So,} \\
K &> \frac{\sqrt{2013}-1}{4} > \sqrt{120}.
\end{aligned}$$

Also solved by Brian D. Beasley, Clinton, SC; Charles R. Diminnie, San Angelo, TX; Paul M. Harms, North Newton, KS; Paolo Perfetti, Mathematics Department, U. of Rome, Italy, and David Stone & John Hawkins (jointly), Statesboro, GA.

- 4985: *Proposed by Kenneth Korbin, New York, NY.*

  A Heron triangle is one that has both integer length sides and integer area. Assume Heron triangle ABC is such that $\angle B = 2\angle A$ and with (a,b,c)=1.

  PartI : Find the dimensions of the triangle if side a = 25.
  PartII : Find the dimensions of the triangle if 100 < a < 200.

Solution by Brian D. Beasely, Clinton, SC.

Using the Law of Sines, we obtain

$$\frac{\sin A}{a} = \frac{\sin(2A)}{b} = \frac{\sin(180^\circ - 3A)}{c} = \frac{\sin(3A)}{c},$$

where $\angle B = 2\angle A$ forces $0^\circ < A < 60^\circ$. Since $\sin(2A) = 2\sin A\cos A$ and $\sin(3A) = 3\sin A - 4\sin^3 A$, we have $b = 2a\cos A$ and $c = a(3-4\sin^2 A)$. In particular, $a < b < 2a$, and using $A = \cos^{-1}\left(\dfrac{b}{2a}\right)$ implies

$$c = 3a - 4a\left(1 - \left(\frac{b}{2a}\right)^2\right) = -a + \frac{b^2}{a}.$$

Then a divides b^2, so we claim that a must be a perfect square: Otherwise, if a prime p divides a but p^2 does not, then p divides b^2; thus p divides b, yet p^2 does not divide a, which would imply that p divides b^2/a and hence p divides c, a contradiction of $(a,b,c) = 1$.

Next, we note that the area of the triangle is $(1/2)bc\sin A$, which becomes

$$\frac{b(b+a)(b-a)}{2a}\sqrt{1-\left(\frac{b}{2a}\right)^2} = \frac{b(b+a)(b-a)}{4a^2}\sqrt{4a^2-b^2}.$$

I. Let $a = 25$. Then $25 < b < 50$ and $c = -25 + b^2/25$, so 5 divides b. Checking $b \in \{30, 35, 40, 45\}$ yields two solutions for which the area of the triangle is an integer:

$$(a, b, c) = (25, 30, 11) \text{ with area} = 132; \qquad (a, b, c) = (25, 40, 39) \text{ with area} = 468.$$

II. Let $100 < a < 200$. Then $a \in \{121, 144, 169, 196\}$.

If $a = 121$, then 11 divides b, so $b = 11d$ for $d \in \{12, 13, \ldots, 21\}$. Since the area formula requires $4a^2 - b^2 = 11^2(22^2 - d^2)$ to be a perfect square, we check that no such d produces a perfect square $22^2 - d^2$.

If $a = 144$, then 12 divides b, so $b = 12d$ for $d \in \{13, 14, \ldots, 23\}$. Since $4a^2 - b^2 = 12^2(24^2 - d^2)$ must be a perfect square, we check that no such d produces a perfect square $24^2 - d^2$.

If $a = 169$, then 13 divides b, so $b = 13d$ for $d \in \{14, 15, \ldots, 25\}$. Since $4a^2 - b^2 = 13^2(26^2 - d^2)$ must be a perfect square, we check that the only such d to produce a perfect square $26^2 - d^2$ is $d = 24$. This yields the triangle

$$(a, b, c) = (169, 312, 407) \text{ with area } 24{,}420.$$

If $a = 196$, then 14 divides b, so $b = 14d$ for $d \in \{15, 16, \ldots, 27\}$. Since $4a^2 - b^2 = 14^2(28^2 - d^2)$ must be a perfect square, we check that no such d produces a perfect square $28^2 - d^2$.

Comment: David Stone and John Hawkins of Statesboro, GA conjectured that in order to meet the conditions of the problem, a must equal p^2, where p is an odd prime congruent to 1 mod 4. With $p = m^2 + n^2$, there are one or two triangles, according to the ratio of m and n. If $\sqrt{3}n < m < (2 + \sqrt{3})n$, there are two solutions; if $m > (2 + \sqrt{3})n$, there is one solution; and if $n < m < \sqrt{3}n$, there is one solution.

Also solved by M.N. Deshpande, Nagpur, India; Grant Evans (student, Saint George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; David Stone & John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4986: *Michael Brozinsky, Central Islip, NY.*

  Show that if $0 < a < b$ and $c > 0$, that

$$\sqrt{(a+c)^2 + d^2} + \sqrt{(b-c)^2 + d^2} \le \sqrt{(a-c)^2 + d^2} + \sqrt{(b+c)^2 + d^2}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

Squaring both sides and simplifying, we reduce the desired inequality to

$$2c(b-a) + \sqrt{(a-c)^2 + d^2}\sqrt{(b+c)^2 + d^2} \ge \sqrt{(a+c)^2 + d^2}\sqrt{b-c)^2 + d^2}.$$

Squaring the last inequality and simplifying we obtain

$$\sqrt{(a-c)^2 + d^2}\sqrt{(b+c)^2 + d^2} \ge ab + ac - bc - c^2 - d^2. \qquad (1)$$

If $ab + ac - bc - c^2 - d^2 \le 0$, (1) is certainly true. If $ab + ac - bc - c^2 - d^2 > 0$, we square both sides of (1) and the resulting inequality simplifies to the trivial inequality $(a+b)^2 d^2 \ge 0$. This completes the solution.

Solution 2 by Paolo Perfetti, Mathematics Department, U. of Rome, Italy.
The inequality is

$$\sqrt{(b-c)^2+d^2}-\sqrt{(a-c)^2+d^2} \le \sqrt{(b+c)^2+d^2}-\sqrt{(a+c)^2+d^2}.$$

Defining $f(x)=\sqrt{(b+x)^2+d^2}-\sqrt{(a+x)^2+d^2},\ -c\le x\le c$, the inequality becomes $f(-c)\le f(c)$ so we prove that

$$f'(x)=\frac{b+x}{\sqrt{(b+x)^2+d^2}}-\frac{a+x}{\sqrt{(a+x)^2+d^2}}>0.$$

There are three possibilities: 1) $b+x>a+x\ge 0$, 2) $a+x<b+x<0$, and 3) $b+x>0,\ a+x<0$. It is evident that 3) implies $f'(x)>0$. With the condition 1), after squaring, we obtain

$$\begin{aligned}(b+x)^2((a+x)^2+d^2) &> (a+x)^2((b+x)^2+d^2) \text{ or}\\ (b+x)^2 &> (a+x)^2 \text{which is true.}\end{aligned}$$

As for 2) we have

$$\begin{aligned}\frac{|b+x|}{\sqrt{(b+x)^2+d^2}} &< \frac{|a+x|}{\sqrt{(a+x)^2+d^2}} \quad \text{or}\\ (b+x)^2 &< (a+x)^2\end{aligned}$$

and making the square root $-(b+x)<-(a+x)$ which is true as well.

Also solved by Angelo State University Problem Solving Group, San Angelo, TX; Paul M. Harms, North Newton, KS; Kenneth Korbin, New York, NY, and the proposer.

- 4987: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a,b,c be the sides of a triangle ABC with area S. Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+a^2)\le 64S^3\csc 2A\csc 2B\csc 2C.$$

Solution by José Luis Díaz-Barrero, the proposer.

Let $A'\in BC$ be the foot of h_a. We have,

$$h_a=c\sin B \quad \text{and} \quad BA'=c\cos B \tag{1}$$

and

$$h_a=b\sin C \quad \text{and} \quad A'C=b\cos C \tag{2}$$

Multiplying up and adding the resulting expressions yields

$$h_a(BA'+A'C)=\frac{b^2\sin 2C}{2}+\frac{c^2\sin 2B}{2}$$

or

$$c^2\sin 2B+b^2\sin 2C=4S$$

Likewise, we have

$$a^2 \sin 2C + c^2 \sin 2A = 4S,$$
$$a^2 \sin 2B + b^2 \sin 2A = 4S.$$

Adding up the above expressions yields

$$(a^2 + b^2) \sin 2C + (b^2 + c^2) \sin 2A + (c^2 + a^2) \sin 2B = 12S$$

Applying the AM-GM inequality yields

$$\sqrt[3]{(a^2 + b^2) \sin 2C (b^2 + c^2) \sin 2A (c^2 + a^2) \sin 2B} \leq 4S$$

from which the statement follows. Equality holds when $\triangle ABC$ is equilateral and we are done.

- 4988: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$3^{x^2-x-z} + 3^{y^2-y-x} + 3^{z^2-z-y} = 1.$$

Solution by Dionne Bailey, Elsie Campbell, Charles Diminnie, Karl Havlak, and Paula Koca (jointly), San Angelo, TX.

By the Arithmetic - Geometric Mean Inequality,

$$\begin{aligned} 1 &= 3^{x^2-x-z} + 3^{y^2-y-x} + 3^{z^2-z-y} \\ &\geq 3\sqrt[3]{3^{x^2-2x+y^2-2y+z^2-2z}} \\ &= \sqrt[3]{3^{(x-1)^2+(y-1)^2+(z-1)^2}} \end{aligned}$$

and hence,

$$3^{(x-1)^2+(y-1)^2+(z-1)^2} \leq 1.$$

It follows that

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 0,$$

i.e.,

$$x = y = z = 1.$$

Since it is easily checked that these values satisfy the original equation, the solution is complete.

Also solved by Kee-Wai Lau, Hong Kong, China; Charles McCracken, Dayton, OH; Paolo Perfetti, Mathematics Department, U. of Rome, Italy; Boris Rays, Chesapeake, VA, and the proposer.

- 4989: *Proposed by Tom Leong, Scotrun, PA.*

  The numbers $1, 2, 3, \cdots, 2n$ are randomly arranged onto $2n$ distinct points on a circle. For a chord joining two of these points, define its *value* to be the absolute value of the difference of the numbers on its endpoints. Show that we can connect the $2n$ points in disjoint pairs with n chords such that no two chords intersect inside the circle and the sum of the values of the chords is exactly n^2.

Solution 1 by Harry Sedinger, St. Bonaventure, NY.

First we show by induction that if there are n red points and n blue points (all distinct) on the circle, then there exist n nonintersecting chords, each connecting a read point an a blue point (with each point being used exactly once). This is obvious for $n = 1$.
Assume it is true for n and consider the case for $n + 1$. There obviously is a pair of adjacent points (no other points between them on one arc), one read and one blue.
Clearly they can be connected by a chord which does not intersect any chord connecting two other points. Removing this chord and the two end points then reduces the problem to the case for n, which can be done according to the induction hypothesis. The desired result is then true for $n + 1$ and by induction true for all n.
Now for the given problem, color the points numbered $1, 2, \cdots, n$ red and color the ones numbered $n + 1, n + 2, \cdots, 2n$ blue. From above there exists n nonintersecting chords and the sum of their values is

$$\sum_{k=n+1}^{2n} k - \sum_{k=1}^{n} k = \sum_{k=1}^{2n} k - 2\sum_{k=1}^{n} k = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2.$$

Solution 2 by Kenneth Korbin, New York, NY.

Arrange the numbers $1, 2, 3, \cdots, 2n$ randomly on points of a circle. Place a red checker on each point from 1 through n. Let

$$\sum R = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Place a black checker on each point numbered from $n + 1$ through $2n$. Let

$$\sum B = (n+1) + (n+2) + \cdots + (2n) = n^2 + \frac{n(n+1)}{2}.$$

Remove a pair of adjacent checkers that have different colors. Connect the two points with a chord. The value of this chord is $(B_1 - R_1)$.
Remove another pair of adjacent checkers with different colors. The chord between these two points will have value $(B_2 - R_2)$.
Continue this procedure until the last checkers are removed and the last chord will have value $(B_n - R_n)$.
The sum of the value of these n chords is

$$(B_1 - R_1) + (B_2 - R_2) + \cdots + (B_n - R_n) = \sum B - \sum R = n^2.$$

Also solved by N.J. Kuenzi, Oshkosh, WI; David Stone & John Hawkins (jointly), Statesboro,GA, and the proposer.