Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
September 1, 2007

- 4972: *Proposed by Kenneth Korbin, New York, NY.*

  Find the length of the side of equilateral triangle ABC if it has a cevian $\overline{\mathrm{CD}}$ such that

  $$\overline{\mathrm{AD}} = x, \quad \overline{\mathrm{BD}} = x+1 \quad \overline{\mathrm{CD}} = \sqrt{y}$$

  where x and y are positive integers with $20 < x < 120$.

- 4973: *Proposed by Kenneth Korbin, New York, NY.*

  Find the area of trapezoid ABCD if it is inscribed in a circle with radius R=2, and if it has base $\overline{\mathrm{AB}} = 1$ and $\angle\mathrm{ACD} = 60^{\mathrm{o}}$.

- 4974: *Proposed by Kenneth Korbin, New York, NY.*

  A convex cyclic hexagon has sides $a, a, a, b, b,$ and b. Express the values of the circumradius and the area of the hexagon in terms of a and b.

- 4975: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Solve in R the following system of equations

  $$\left.\begin{array}{c} 2x_1 = 3x_2\,\sqrt{1+x_3^2} \\ 2x_2 = 3x_3\,\sqrt{1+x_4^2} \\ \cdots\cdots \\ 2x_n = 3x_1\,\sqrt{1+x_2^2} \end{array}\right\}$$

- 4976: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers. Prove that
$$\frac{a^2+3b^2+9c^2}{bc}+\frac{b^2+3c^2+9a^2}{ca}+\frac{c^2+3a^2+9b^2}{ab}\geq 27.$$

- 4977: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $1 < a < b$ be real numbers. Prove that for any $x_1, x_2, x_3 \in [a, b]$ there exist $c \in (a, b)$ such that
$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{3}{\log x_1x_2x_3}=\frac{4}{\log c}.$$

Solutions

- 4942: *Proposed by Kenneth Korbin, New York, NY.*

  Given positive integers a and b. Find the minimum and the maximum possible values of the sum $(a+b)$ if $\dfrac{ab-1}{a+b}=2007$.

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX.

If $\dfrac{ab-1}{a+b}=2007$, then
$$\begin{aligned} ab-1 &= 2007(a+b)\\ ab-2007a-2007b &= 1\\ ab-2007a-2007b+2007^2 &= 1+2007^2\\ (a-2007)(b-2007) &= 2\cdot 5^2\cdot 13\cdot 6197 \qquad (\mathbf{1}). \end{aligned}$$

Since **(1)** and the sum $(a+b)$ are symmetric in a and b, then we will assume that $a < b$. By the prime factorization in **(1)**, there are exactly 12 distinct values for $(a-2007)$ and $(b-2007)$ which are summarized below.

$a-2007$	$b-2007$	a	b	$a+b$
1	4,028,050	2,008	4,030,057	4,032,065
2	2,014,025	2,009	2,016,032	2,018,041
5	805,610	2,012	807,617	809,629
10	402,805	2,017	404,812	406,829
13	309,850	2,020	311,857	313,877
25	161,122	2,032	163,129	165,161
26	154,925	2,033	156,932	158,965
50	80,561	2,057	82,568	84,625
65	61,970	2,072	63,977	66,049
130	30,985	2,137	32,992	35,129
325	12,394	2,332	14,401	16,733
650	6,197	2,657	8,204	10,861

Thus, the minimum value is 10,861, and the maximum value is 4,032,065.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; John Nord, Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4943: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with $\overline{AB} = 19$, $\overline{BC} = 8$, $\overline{CD} = 6$, and $\overline{AD} = 17$. Find the area of the quadrilateral if both $\overline{AC}$ and $\overline{BD}$ also have integer lengths.

  Solution by Brian D. Beasley, Clinton, SC.

  Let $x = \overline{AC}$ and $y = \overline{BD}$, where both x and y are positive integers. Let A_1 be the area of triangle ABC, A_2 be the area of triangle of ADC, A_3 be the area of triangle BAD, and A_4 be the area of triangle BCD. Then by Heron's formula, we have

  $$A_1 = \sqrt{s(s-19)(s-8)(s-x)} \qquad A_2 = \sqrt{t(t-17)(t-6)(t-x)},$$

  where $s = (19+8+x)/2$ and $t = (17+6+x)/2$. Similarly,

  $$A_3 = \sqrt{u(u-19)(u-17)(u-y)} \qquad A_4 = \sqrt{v(v-8)(v-6)(v-y)},$$

  where $u = (19+17+y)/2$ and $v = (8+6+y)/2$. Also, the lengths of the various triangle sides imply $x \in \{12, 13, \cdots, 22\}$ and $y \in \{3, 4, \cdots, 13\}$. We consider three cases for the area T of $ABCD$:

  Case 1: Assume $ABCD$ is convex. Then $T = A_1 + A_2 = A_3 + A_4$. But a search among the possible values for x and y yields no solutions in this case.

  Case 2: Assume $ABCD$ is not convex, with triangle BAD containing triangle BCD (i.e., C is interior to ABD). Then $T = A_1 + A_2 = A_3 - A_4$. Again, a search among the possible values for x and y yields no solutions in this case.

  Case 3: Assume $ABCD$ is not convex, with triangle ABC containing triangle ADC (i.e., D is interior to ABC). Then $T = A_1 - A_2 = A_3 + A_4$. In this case, a search among the possible values for x and y yields the unique solution $x = 22$ and $y = 4$; this produces $T = \sqrt{1815} = 11\sqrt{15}$.

  Due to the lengths of the quadrilateral, these are the only three cases for $ABCD$. Thus the unique value for its area is $11\sqrt{15}$.

  Also solved by Paul M. Harms, North Newton, KS; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 4944: *Proposed by James Bush, Waynesburg, PA.*

  Independent random numbers a and b are generated from the interval $[-1, 1]$ to fill the matrix $A = \begin{pmatrix} a^2 & a^2+b \\ a^2-b & a^2 \end{pmatrix}$. Find the probability that the matrix A has two real eigenvalues.

  Solution by Paul M. Harms, North Newton, KS.

  The characteristic equation is $(a^2-\lambda)^2-(a^4-b^2) = 0$. The solutions for λ are $a^2+\sqrt{a^4-b^2}$ and $a^2-\sqrt{a^4-b^2}$. There are two real eigenvalues when $a^4-b^2 > 0$ or $a^2 > |b|$. The region in the ab coordinate system which satisfies the inequality is between the parabolas $b = a^2$ and $b = -a^2$ and inside the square where a and b are both in $[-1, 1]$. From the symmetry of the region we see that the probability is the area in the first quadrant between the a-axis and $b = a^2$ from $a = 0$ to $a = 1$. Integrating gives a probability of $\dfrac{1}{3}$.

  Also solved by Tom Leong, Scotrun, PA; John Nord, Spokane, WA; David Stone and John Hawkins (jointly), Statesboro, GA; Boris Rays, Chesapeake, VA, and the proposer.

- 4945: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Prove that

$$17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+L_{k+2}^4\Big)^{1/2}=L_n^2+3L_{n+1}^2+5L_nL_{n+1}$$

where L_n is the n^{th} Lucas number defined by $L_0=2, L_1=1$ and for all $n\geq 2, L_n=L_{n-1}+L_{n-2}$.

Solution by Tom Leong, Scotrun, PA.

Using the identity $a^4+b^4+(a+b)^4=2(a^2+ab+b^2)^2$ we have

$$\begin{aligned}
17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+L_{k+2}^4\Big)^{1/2} &= 17+\sqrt{2}\sum_{k=1}^{n}\Big(L_k^4+L_{k+1}^4+(L_k+L_{k+1})^4\Big)^{1/2}\\
&= 17+2\sum_{k=1}^{n}\Big(L_k^2+L_kL_{k+1}+L_{k+1}^2\Big)\\
&= 17+\sum_{k=1}^{n}L_k^2+\sum_{k=1}^{n}L_{k+1}^2+\sum_{k=1}^{n}(L_k+L_{k+1})^2\\
&= 17+\sum_{k=1}^{n}L_k^2+\sum_{k=1}^{n}L_{k+1}^2+\sum_{k=1}^{n}L_{k+2}^2\\
&= 17+L_{n+2}^2+2L_{n+1}^2-L_2^2-2L_1^2+3\sum_{k=1}^{n}L_k^2\\
&= 17+(L_n+L_{n+1})^2+2L_{n+1}^2-3^2-2\cdot 1^2+3\sum_{k=1}^{n}L_k^2\\
&= L_n^2+3L_{n+1}^2+2L_nL_{n+1}+6+3\sum_{k=1}^{n}L_k^2\\
&= L_n^2+3L_{n+1}^2+2L_nL_{n+1}+6+3\left(L_nL_{n+1}-2\right)\\
&= L_n^2+3L_{n+1}^2+5L_nL_{n+1}
\end{aligned}$$

where we used the identity $\sum_{k=1}^{n}L_k^2=L_nL_{n+1}-2$ which is easily proved via induction.

Comment: Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie started off their solution with

$$2(L_k^4+L_{k+1}^4+L_{k+2}^4)=(L_k^2+L_{k+1}^2+L_{k+2}^2)^2$$

and noted that this is a special case of Candido's Identity $2(x^4+y^4+(x+y)^4)=(x^2+y^2+(x+y)^2)^2$, for which Roger Nelsen gave a proof without words in *Mathematics Magazine* (vol. 78,no. 2). Candido used this identity to establish that $2(F_n^4+F_{n+1}^4+F_{n+2}^4)=(F_n^2+F_{n+1}^2+F_{n+2}^2)$, where F_n denotes the n^{th} Fibonacci number.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS, and the proposer.

- 4946: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let z_1, z_2 be nonzero complex numbers. Prove that

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|\right)\geq 4.$$

Solution by David Stone and John Hawkins (jointly), Statesboro, GA.

We note that for $a, b > 0$,

$$\begin{aligned}
a^2-2ab+b^2 &= (a-b)^2 \geq 0\\
\text{so } a^2+2ab+b^2 &\geq 4ab\\
\text{so } (a+b)(a+b) &\geq 4ab\\
\text{so } \frac{(a+b)}{ab}(a+b) &\geq 4\\
\text{or } \left(\frac{1}{a}+\frac{1}{b}\right)(a+b) &\geq 4
\end{aligned}$$

Therefore, **(1)** $\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)(|z_1|+|z_2|)\geq 4.$

For two complex numbers $w = a+bi$ and $v = c+di$, we have

$$\begin{aligned}
|(w-v)^2|+|(w+v)^2| &= |w-v|^2+|w+v|^2 = (a-c)^2+(b-d)^2+(a+c)^2+(b+d)^2\\
&= 2(a^2+b^2+c^2+d^2) = 2(|w|^2+|v|^2)
\end{aligned}$$

so, **(2)** $|(w-v)^2|+|(w+v)|^2 = 2(|w^2|+|v^2|)$.

Let w be such that $w^2 = z_1$ and v be such that $v^2 = z_2$. Substituting this into **(2),** we get $|w^2-2wv+v^2|+|w^2+2wv+v^2| = 2(|z_1|+|z_2|)$, hence

$$\left|\frac{z_1+z_2}{2}-wv\right|+\left|\frac{z_1+z_2}{2}+wv\right| = |z_1|+|z_2|.$$

Since $(wv)^2 = z_1z_2$, wv must equal $\sqrt{z_1z_2}$ or $-\sqrt{z_1z_2}$. Thus the preceding equation becomes

$$\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right| = |z_1|+|z_2|.$$

Multiplying by $\frac{1}{|z_1|}+\frac{1}{|z_2|}$, we get

$$\left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)\left(\left|\frac{z_1+z_2}{2}-\sqrt{z_1z_2}\right|+\left|\frac{z_1+z_2}{2}+\sqrt{z_1z_2}\right|\right) = \left(\frac{1}{|z_1|}+\frac{1}{|z_2|}\right)(|z_1|+|z_2|)\geq 4$$

by inequality **(1).**

Also solved by Tom Leong Scotrun, PA, and the proposers.

- 4947: *Proposed by Tom Leong, Brooklyn, NY.*

Define a set S of positive integers to be *among composites* if for any positive integer n, there exists an $x \in S$ such that all of the $2n$ integers $x \pm 1, x \pm 2, \ldots, x \pm n$ are composite. Which of the following sets are among composites? (a) The set $\{a + dk | k \in N\}$ of terms of any given arithmetic progression with $a, d \in N, d > 0$. (b) The set of squares. (c) The set of primes. (d)* The set of factorials.

Remarks and solution by the proposer, (with a few slight changes made in the comments by the editor).

This proposal arose after working Richard L. Francis's problems 4904 and 4905; it can be considered a variation on the idea in problem 4904. My original intention was to propose parts (c) and (d) only; however, I couldn't solve part (d) and, after searching the MAA journals, I later found that the question posed by part (c) is not original at all. An article in (*The Two-Year College Mathematics Journal*, Vol. 12, No. 1, Jan 1981, p. 36) solves part (c). However it appears that the appealing result of part (c) is not well-known and the solution I offer differs from the published one. Parts (a) and (b), as far as I know, are original.

Solution. The sets in (a), (b) and (c) are all among composites. In the solutions below, let n be any positive integer.

(a) Choose $m \geq n$ and $m > d$. Clearly the consecutive integers $(3m)! + 2, (3m)! + 3, \ldots, (3m)! + 3m$ are all composite. Furthermore since $d \leq m - 1$, one of the integers $(3m)! + m + 2, (3m)! + m + 3, \ldots, (3m)! + 2m$ belongs to the arithmetic progression and we are done.

(b) By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes congruent to 1 mod 4. Let $p > n$ be prime with $p \equiv 1 \pmod 4$. From the theory of quadratic residues, we know -1 is a quadratic residue mod p, that is, there is a positive integer r such that $r^2 \equiv -1 \pmod p$. Also by Wilson's theorem, $(p-1)! \equiv -1 \pmod p$. Put $x = [r(p-1)!]^2$. Then $x \pm 2, x \pm 3, \ldots, x \pm (p-1)$ are all composite. Furthermore, $x - 1 = [r(p-1)!]^2 - 1 = [r(p-1)! + 1][r(p-1)! - 1]$ is composite and $x \equiv r^2[(p-1)!]^2 \equiv -1(-1)^2 \equiv -1 \pmod p$, that is, $x + 1$ is composite.

(c) Let $p > n + 1$ be an odd prime. First note $p!$ and $(p-1)! - 1$ are relatively prime. Indeed, the prime divisors of $p!$ are all primes not exceeding p while none of those primes divide $(p-1)!-1$ (clearly primes less than p do not divide $(p-1)!-1$, while $(p-1)!-1 \equiv -2 \pmod p$ by Wilson's theorem). Appealing to Dirichlet's theorem again, there are infinitely many primes x of the form $x = kp! + (p-1)! - 1$. So $x - 1, x - 2, \ldots, x - (p-2)$ and $x+1, x+3, x+4, \ldots, x+p$ are all composite. By Wilson's theorem, $(p-1)!+1$ is divisible by p; hence $x + 2$ is divisible by p, that is, composite.

Remarks. (b) In fact, it can similarly be shown that the set of nth powers for any positive integer n is among composites.

(d) For any prime p, let $x = (p-1)!$. Then $x \pm 2, x \pm 3, \ldots, x \pm (p-1)$ are all composite and by Wilson's theorem, $x + 1$ is also composite. It remains: is $x - 1 = (p-1)! - 1$ composite? I don't know; however it's unlikely to be prime for all primes p.