Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
September 15, 2008

- 5020: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\begin{cases} x^7 - 13y = 21 \\ 13x - y^7 = 21 \end{cases}$$

- 5021: *Proposed by Kenneth Korbin, New York, NY.*

  Given

$$\frac{x + x^2}{1 - 34x + x^2} = x + 35x^2 + \cdots + a_n x^n + \cdots$$

  Find an explicit formula for a_n.

- 5022: *Proposed by Michael Brozinsky, Central Islip, NY .*

  Show that

$$\sin\left(\frac{x}{3}\right)\sin\left(\frac{\pi + x}{3}\right)\sin\left(\frac{2\pi + x}{3}\right)$$

  is proportional to $\sin(x)$.

- 5023: *Proposed by M.N. Deshpande, Nagpur, India.*

  Let $A_1A_2A_3\cdots A_n$ be a regular n-gon ($n \geq 4$) whose sides are of unit length. From A_k draw L_k parallel to $\mathrm{A}_{k+1}A_{k+2}$ and let L_k meet L_{k+1} at T_k. Then we have a "necklace" of congruent isosceles triangles bordering $A_1A_2A_3\cdots A_n$ on the inside boundary. Find the total area of this necklace of triangles.

- 5024: *Proposed by José Luis Díaz-Barrero and Josep Rubió-Massegú, Barcelona, Spain.*

  Find all real solutions to the equation

$$\sqrt{1+\sqrt{1-x}} - 2\sqrt{1-\sqrt{1-x}} = \sqrt[4]{x}.$$

- 5025: *Ovidiu Furdui, Toledo, OH.*

  Calculate the double integral
$$\int_0^1 \int_0^1 \{x-y\} dxdy,$$
  where $\{a\} = a - [a]$ denotes the fractional part of a.

Solutions

- 5002: *Proposed by Kenneth Korbin, New York, NY.*

  A convex hexagon with sides $3x, 3x, 3x, 5x, 5x$ and $5x$ is inscribed in a unit circle. Find the value of x.

  Solution by David E. Manes, Oneonta, NY.

  The value of x is $\dfrac{\sqrt{3}}{7}$.
  Note that each inscribed side of the hexagon subtends an angle at the center of the circle that is independent of its position in the circle The sides are subject to the constraint that the sum of the angles subtended at the center equals 360^o. Therefore the sides of the hexagon can be permuted from $3x, 3x, 3x, 5x, 5x, 5x$ to $3x, 5x, 3x, 5x, 3x, 5x$. In problem **4974** : (December 2007, Korbin, Lau) it is shown that the circumradius r is then given by
$$r = \sqrt{\frac{(3x)^2 + (5x)^2 + (3x)(5x)}{3}}.$$
  With $r = 1$, one obtains $x = \dfrac{\sqrt{3}}{7}$.

  Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; John Boncek, Montgomery, AL; M.N. Deshpande, Nagpur, India; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student at St George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Minerva P. Harwell (student at Auburn University), Montgomery, AL; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; Amanda Miller (student at St. George's School), Spokane, WA; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5003: *Proposed by Kenneth Korbin, New York, NY.*

  Find positive numbers x and y such that

$$\sqrt[3]{x+\sqrt{x^2-1}} + \sqrt[3]{x-\sqrt{x^2-1}} \quad = \quad \frac{7}{2} \text{ and}$$

$$\sqrt[3]{y+\sqrt{y^2-1}}+\sqrt[3]{y-\sqrt{y^2-1}} \;=\; \sqrt{10}$$

Solution by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX.

Let $A=\sqrt[3]{x+\sqrt{x^2-1}}$ and $B=\sqrt[3]{x-\sqrt{x^2-1}}$. Note that

$$\begin{aligned} A^3+B^3 &= 2x \quad \text{and} \\ AB &= 1. \end{aligned}$$

Since $A+B=\dfrac{7}{2}$,

$$\begin{aligned} \frac{343}{8} &= (A+B)^3 \\ &= A^3+3A^2B+3AB^2+B^3 \\ &= A^3+B^3+3AB(A+B) \\ &= 2x+\frac{21}{2}. \end{aligned}$$

Thus, $x=\dfrac{259}{16}$.

Similarly,

$$2y+3\sqrt{10}=10\sqrt{10}$$

and, thus, $y=\dfrac{7\sqrt{10}}{2}$.

Also solved by Brian D. Beasley, Clinton, SC; John Boncek, Montgomery, AL; M.N. Deshpande, Nagpur, India; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Amanda Miller (student at St. George's School), Spokane, WA; John Nord, Spokane, WA; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5004: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be nonnegative real numbers. Prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{\sqrt{ab}}{1+a+b}+\frac{\sqrt{bc}}{1+b+c}+\frac{\sqrt{ac}}{1+c+a}$$

Solution by John Boncek, Montgomery, AL.

We use the arithmetic-geometric inequality: If $x, y \geq 0$, then $x+y \geq 2\sqrt{xy}$. Now

$$\begin{aligned} \frac{a}{1+a} &\geq \frac{a}{1+a+b}, \text{ and} \\ \frac{b}{1+b} &\geq \frac{b}{1+a+b}, \text{ so} \end{aligned}$$

$$\frac{a}{1+a}+\frac{b}{1+b} \quad \geq \quad \frac{a+b}{1+a+b} \geq \frac{2\sqrt{ab}}{1+a+b}.$$

Similarly,

$$\frac{a}{1+a}+\frac{c}{1+c} \geq \frac{2\sqrt{ac}}{1+a+c}, \text{ and}$$

$$\frac{b}{1+b}+\frac{c}{1+c} \geq \frac{2\sqrt{bc}}{1+b+c}\ .$$

Summing up all three inequalities, we obtain

$$2\left(\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\right) \geq \frac{2\sqrt{ab}}{1+a+b}+\frac{2\sqrt{ac}}{1+a+c}+\frac{2\sqrt{bc}}{1+b+c}.$$

Divide both sides of the inequality by 2 to obtain the result.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; M.N. Deshpande, Nagpur, India; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; Boris Rays, Chesapeake, VA, and the proposers.

- **5005:** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{\sqrt{3}}{2}\left(a+b+c\right)^{1/2} \geq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

Solution 1 by Kee-Wai Lau, Hong Kong, China.

Since $a+b \geq 2\sqrt{ab} = \dfrac{2}{\sqrt{c}}$ and so on, and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\\
\leq\ & \frac{\sqrt{c}+\sqrt{a}+\sqrt{b}}{2}\\
=\ & \frac{1}{2}\Big((1)\sqrt{a}+(1)\sqrt{b}+(1)\sqrt{c}\Big)\\
\leq\ & \frac{1}{2}\sqrt{1+1+1}\sqrt{a+b+c}\\
=\ & \frac{\sqrt{3}}{2}\Big(a+b+c)^{1/2}
\end{aligned}$$

as required.

Solution 2 by Charles McCracken, Dayton, OH.

Suppose a=b=c=1. Then the original inequality reduces to $\dfrac{3}{2} \geq \dfrac{3}{2}$ which is certainly true.

Let L represent the left side of the original inequality and let R represent the right side. Allow a, b, and c to vary and take partial derivatives.

$$\frac{\partial L}{\partial a} = \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\Big(a+b+c\Big)^{-1/2} > 0. \text{ Similarly, } \frac{\partial L}{\partial b} > 0 \text{ and } \frac{\partial \mathrm{L}}{\partial \mathrm{c}} > 0.$$

$$\frac{\partial R}{\partial a} = -(a+b)^{-2} - c(a+b)^{-2} < 0. \text{ Similarly, } \frac{\partial R}{\partial b} < 0 \text{ and } \frac{\partial R}{\partial c} < 0.$$

So any change in a, b or c results in an increase in L and a decrease in R so that L is always greater than R.

Also solved by Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Paolo Perfetti (Department of Mathematics, University of Rome), Italy, and the proposer.

- 5006: *Proposed by Ovidiu Furdui, Toledo, OH.*

Find the sum

$$\sum_{k=2}^{\infty}(-1)^k \ln\left(1-\frac{1}{k^2}\right).$$

Solution 1 by Paul M. Harms, North Newton, KS.

Using $\ln\left(1-\dfrac{1}{k^2}\right) = \ln\left(\dfrac{k-1}{k}\right) + \ln\left(\dfrac{k+1}{k}\right)$, the summation is

$$\begin{aligned}
&\left(\ln\frac{1}{2}+\ln\frac{3}{2}\right) - \left(\ln\frac{2}{3}+\ln\frac{4}{3}\right) + \left(\ln\frac{3}{4}+\ln\frac{5}{4}\right) - \ln\left(\frac{4}{5}+\ln\frac{6}{5}\right) + \cdots\\
= \quad & \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)^2 + \ln\left(\frac{3}{4}\right)^2 + \ln\left(\frac{5}{4}\right)^2 + \cdots\\
= \quad & \ln\left(\frac{1}{2}\right) + 2\Big[\ln\left(\frac{3}{2}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{5}{4}\right) + \ln\left(\frac{5}{6}\right) + \cdots\Big]\\
= \quad & \ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\cdots.
\end{aligned}$$

Wallis' product for $\dfrac{\pi}{2}$ is

$$\frac{\pi}{2} = \left(\frac{2}{1}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\left(\frac{6}{7}\right)\cdots.$$

Dividing both sides by 2 and taking the reciprocal yields

$$\frac{4}{\pi} = \left(\frac{3}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{7}{8}\right)\cdots.$$

The summation in the problem is then

$$\ln\left(\frac{1}{2}\right) + 2\ln\left(\frac{4}{\pi}\right) = \ln\left[\left(\frac{1}{2}\right)\left(\frac{16}{\pi^2}\right)\right] = \ln\left(\frac{8}{\pi^2}\right).$$

Solution 2 by Kee-Wai Lau, Hong Kong, China.

It can be proved readily by induction that for positive intergers n,

$$\sum_{k=2}^{2n}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 4\Big(\ln((2n)!) - 2\ln(n!)\Big) + \ln n + \ln(2n+1) - 2(4n-1)\ln 2.$$

By using the Stirling approximation $\ln(n!) = n\ln n - n + \dfrac{1}{2}\ln(2\pi n) + O\Big(\dfrac{1}{n}\Big)$ as $n \to \infty$, we obtain

$$\ln((2n)!) - 2\ln(n!) = 2n\ln 2 - \frac{\ln n}{2} - \frac{\ln \pi}{2} + O\Big(\frac{1}{n}\Big).$$

It follows that

$$\sum_{k=2}^{2n}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 3\ln 2 - 2\ln\pi + \ln\left(1+\frac{1}{2n}\right) + O\Big(\frac{1}{n}\Big) = 3\ln 2 - 2\ln\pi + O\Big(\frac{1}{n}\Big)$$

and that $\displaystyle\sum_{k=2}^{2n+1}(-1)^k \ln\left(1-\frac{1}{k^2}\right) = 3\ln 2 - 2\ln\pi + O\Big(\frac{1}{n}\Big)$ as well.

This shows that the sum of the problem equal $3\ln 2 - 2\ln\pi = \ln\left(\dfrac{8}{\pi^2}\right)$.

Also solved by Brian D. Beasley, Clinton, SC; Worapol Rattanapan (student at Montfort College (high school)), Chiang Mai, Thailand; Paolo Perfetti (Department of Mathematics, University of Rome), Italy; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 5007: *Richard L. Francis, Cape Girardeau, MO.*

  Is the centroid of a triangle the same as the centroid of its Morley triangle?

  Solution by Kenneth Korbin, New York, NY.

  The centroids are not the same unless the triangle is equilateral.

  For example, the isosceles right triangle with vertices at $(-6,0), (6,0)$ and $(0,6)$ has its centroid at $(0,2)$.

  Its Morley triangle has verticies at $(0, 12-6\sqrt{3}), (-6+3\sqrt{3}, 3)$, and $(6-3\sqrt{3}, 3)$ and has its centroid at $(0, 6-2\sqrt{3})$.

  Also solved by Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY, and the proposer.