Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
December 15, 2008

- 5026: *Proposed by Kenneth Korbin, New York, NY.*

  Given quadrilateral $ABCD$ with coordinates $A(-3,0)$, $B(12,0)$, $C(4,15)$, and $D(0,4)$. Point P has coordinates $(x,3)$. Find the value of x if

  $$\text{area }\triangle\text{PAD} + \text{area }\triangle\text{PBC} = \text{area }\triangle\text{PAB} + \text{area }\triangle\text{PCD}.$$

- 5027: *Proposed by Kenneth Korbin, New York, NY.*

  Find the x and y intercepts of

  $$y = x^7 + x^6 + x^4 + x^3 + 1.$$

- 5028: *Proposed by Michael Brozinsky, Central Islip, NY .*

  If the ratio of the area of the square inscribed in an isosceles triangle with one side on the base to the area of the triangle uniquely determine the base angles, find the base angles.

- 5029: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $x > 1$ be a non-integer number. Prove that

  $$\left(\frac{x+\{x\}}{[x]} - \frac{[x]}{x+\{x\}}\right) + \left(\frac{x+[x]}{\{x\}} - \frac{\{x\}}{x+[x]}\right) > \frac{9}{2},$$

  where $[x]$ and $\{x\}$ represents the entire and fractional part of x.

- 5030: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $A_1, A_2, \cdots, A_n \in M_2(\mathbf{C}), (n \geq 2)$, be the solutions of the equation $X^n = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$.

  Prove that $\displaystyle\sum_{k=1}^{n} Tr(A_k) = 0$.

- 5031: *Ovidiu Furdui, Toledo, OH.*

  Let x be a real number. Find the sum

$$\sum_{n=1}^{\infty}(-1)^{n-1}n\left(e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^n}{n!}\right).$$

Solutions

- 5008: *Proposed by Kenneth Korbin, New York, NY.*

  Given isosceles trapezoid $ABCD$ with $\angle ABD = 60^o$, and with legs $\overline{BC} = \overline{AD} = 31$.

  Find the perimeter of the trapezoid if each of the bases has positive integer length with $\overline{AB} > \overline{CD}$.

  Solution by David C. Wilson, Winston-Salem, N.C.

  Let the side lengths of $\overline{\text{AB}}= x$, $\overline{\text{BC}}$=31, $\overline{CD}= y$, $\overline{\text{DA}}$=31, and $\overline{\text{BD}}$=z.
  By the law of cosines

$$\begin{array}{rcl}
31^2 &=& x^2 + z^2 - 2xz\cos 60^o \text{ and} \\
31^2 &=& y^2 + z^2 - 2yz\cos 60^o \implies \\
961 &=& z^2 + x^2 - xz \text{ and} \\
961 &=& y^2 + z^2 - yz \implies \\
0 &=& (y^2 - x^2) - yz + xz \implies \\
0 &=& (y-x)(y+x) - z(y-x) = (y-x)(y+x-z) \implies \\
y - x &=& 0 \text{ or } y + x - z = 0.
\end{array}$$

  But $\overline{AB} > \overline{CD} \implies x > y \implies y - x \neq 0$. Thus, $y + x - z = 0 \implies z = x + y$. Thus,

$$961 = (x+y)^2 + x^2 - x(x+y) = x^2 + 2xy + y^2 + x^2 - x^2 - xy = x^2 + xy + y^2.$$

  Consider $x = 30, 29, \cdots, 18$. After trial and error with a calculator, when $x = 24$ then $y = 11 \implies z = 35$ and these check. Thus, the perimeter of $ABCD$ is $35 + 31 + 31 = 97$.

  Also solved by Dionne T. Bailey, Elsie M. Campbell, and Charles Diminnie (jointly), San Angelo, TX; Matt DeLong, Upland, IN; Lauren Christenson, Taylor Brennan, Ross Hayden, and Meaghan Haynes (jointly; students at Taylor University), Upland, IN; Charles McCracken, Dayton, OH; Amanda Miller (student, St.George's School), Spokane, WA; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; John Nord, Spokane, WA; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **5009:** *Proposed by Kenneth Korbin, New York, NY.*

Given equilateral triangle ABC with a cevian $\overline{CD}$ such that $\overline{AD}$ and $\overline{BD}$ have integer lengths. Find the side of the triangle $\overline{AB}$ if $\overline{CD} = 1729$ and if $(\overline{AB}, 1729) = 1$.

Solution by David Stone and John Hawkins, Statesboro, GA.

The answer: $\overline{AB} = 1775, 1840, 1961, 1984$.

Let $x = \overline{AD}$ and $y = \overline{BD}$, with $s = x + y =$ the side length $\overline{AB}$. Applying the Law of cosines in each "subtriangle," we have

$$\begin{aligned} 1729^2 &= s^2 + x^2 - 2sx\cos\frac{\pi}{3} = s^2 + x^2 - sx \quad \text{and} \\ 1729^2 &= s^2 + y^2 - 2sy\cos\frac{\pi}{3} = s^2 + y^2 - sy. \end{aligned}$$

After adding equations and doing some algebra, we obtain the equation

$$y^2 + xy + x^2 = 1729^2.$$

Solving for y by the Quadratic Formula, we obtain

$$y = \frac{-x \pm \sqrt{4 \cdot 1729^2 - 3x^2}}{2} = \frac{-x \pm z}{2}$$

where $z = \sqrt{4 \cdot 1729^2 - 3x^2}$ must be an integer.

Because y must be positive, we have to choose $y = \dfrac{-x + z}{2}$.

Now we let Excel calculate, trying $x = 1, 2, \cdots, 1729$. We have 13 "solutions", but only four of them have $s = \overline{AB}$ relatively prime to 1729; hence only equilateral triangles of side length $AB = 1775, 1840, 1961$, and 1984 admit the cevian described in the problem.

x	$z = \sqrt{34586^2 - 3x^2}$	$y = (-x+z)/2$	$s = x + y$	$gcd(1729, s)$
96	3454	1679	1775	1
209	3439	1615	1824	19
249	3431	1591	1840	1
299	3419	1560	1859	13
361	3401	1520	1881	19
455	3367	1456	1911	91
504	3346	1421	1925	7
651	3269	1309	1960	7
656	3266	1305	1961	1
741	3211	1235	1976	247
799	3169	1185	1984	1
845	3133	1144	1989	13
931	3059	1064	1995	133

Note that we could let x run further, but the problem is symmetric in x and y, so we'd just recover these same solutions with x and y interchanged.

Comment by Kenneth Korbin, the proposer.

In the problem $\overline{CD} = (7)(13)(19)$ and there were exactly 4 possible answers. If $\overline{CD}$ would have been equal to (7)(13)(19)(31)then there would have been exactly 8 possible solutions.

Similarly, there are exactly 4 primitive Pythagorean triangles with hypotenuse (5)(13)(17) and there exactly 8 primitive Pythagorean triangles with hypotenuse (5)(13)(17)(29). And so on.

Also solved by Charles McCracken, Dayton, OH; David E. Manes, Oneonta, NY; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5010: *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona, Spain.*

Let α, β, and γ be real numbers such that $0 < \alpha \le \beta \le \gamma < \pi/2$. Prove that

$$\frac{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}{(\sin\alpha + \sin\beta + \sin\gamma)(\cos\alpha + \cos\beta + \cos\gamma)} \le \frac{2}{3}.$$

Solution by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy.

Proof After some simple simplification the inequality is

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \le \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha)$$

The concavity of $\sin(x)$ in the interval $[0, \pi]$ allows us to write $\sin(x+y) \ge (\sin(2x) + \sin(2y))/2$ thus

$$\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) \ge \sin 2\alpha + \sin 2\beta + \sin 2\gamma$$

concluding the proof.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; David E. Manes, Oneonta, NY; Boris Rays, Chesapeak, VA, and the proposers.

- 5011: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $\{a_n\}_{n\ge0}$ be the sequence defined by $a_0 = a_1 = 2$ and for $n \ge 2,\ a_n = 2a_{n-1} - \dfrac{1}{2}a_{n-2}$. Prove that

$$2^p a_{p+q} + a_{q-p} = 2^p a_p a_q$$

where $p \le q$ are nonnegative integers.

Solution 1 by R. P. Sealy, Sackville, New Brunswick, Canada.

Solving the characteristic equation

$$r^2 - 2r + \frac{1}{2} = 0$$

and using the intitial conditions, we obtain the solution

$$a_n = \left(\frac{2+\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}}{2}\right)^n.$$

Note that

$$2^p a_{p+q} = \frac{(2+\sqrt{2})^{p+q} + (2-\sqrt{2})^{p+q}}{2^q} \text{ and}$$

$$
\begin{aligned}
a_{q-p} &= \frac{(2+\sqrt{2})^{q-p}+(2-\sqrt{2})^{q-p}}{2^{q-p}} \text{ while} \\
2^p a_p a_q &= \frac{(2+\sqrt{2})^{p+q}+(2-\sqrt{2})^{p+q}+2^p[(2+\sqrt{2})^{q-p}+(2-\sqrt{2})^{q-p}]}{2^q} \\
&= 2^p a_{p+q}+a_{q-p}.
\end{aligned}
$$

Solution 2 by Kee-Wai Lau, Hong Kong, China.

By induction, we obtain readily that for $n \geq 0$,

$$a_n = \left(\frac{2+\sqrt{2}}{2}\right)^n + \left(\frac{2-\sqrt{2}}{2}\right)^n.$$

Hence

$$
\begin{aligned}
a_p a_q &= \left(\left(\frac{2+\sqrt{2}}{2}\right)^p + \left(\frac{2-\sqrt{2}}{2}\right)^p\right)\left(\left(\frac{2+\sqrt{2}}{2}\right)^q + \left(\frac{2-\sqrt{2}}{2}\right)^q\right) \\
&= \left(\left(\frac{2+\sqrt{2}}{2}\right)^{p+q} + \left(\frac{2-\sqrt{2}}{2}\right)^{p+q}\right) + \left(\frac{2+\sqrt{2}}{2}\right)^p\left(\frac{2-\sqrt{2}}{2}\right)^q + \left(\frac{2-\sqrt{2}}{2}\right)^p\left(\frac{2+\sqrt{2}}{2}\right)^q \\
&= a_{p+q} + \left(\frac{2+\sqrt{2}}{2}\right)^p\left(\frac{2-\sqrt{2}}{2}\right)^q\left(\left(\frac{2-\sqrt{2}}{2}\right)^{q-p} + \left(\frac{2+\sqrt{2}}{2}\right)^q\right) \\
&= a_{p+q} + \frac{1}{2^p}a_{q-p},
\end{aligned}
$$

and the identity of the problem follows.

Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; David E. Manes, Oneonta, NY; Jose Hernández Santiago (student, UTM), Oaxaca, México; Boris Rays, Chesapeake, VA; David Stone and John Hawkins (jointly), Statesboro, GA; David C. Wilson, Winston-Salem, NC, and the proposer.

- 5012: *Richard L. Francis, Cape Girardeau, MO.*

Is the incenter of a triangle the same as the incenter of its Morley triangle?

Solution 1 by Kenneth Korbin, New York, NY.

The incenters are not the same unless the triangle is equilateral. For example, the isosceles right triangle with vertices at $(-6,0),(6,0)$ and $(0,6)$ has its incenter at $(0,6\sqrt{2}-6)$.

Its Morely triangle has vertices at $(0,12-6\sqrt{3}),(-6+3\sqrt{3},3)$, and $(6-3\sqrt{3},3)$ and has its incenter at $(0,6-2\sqrt{3})$.

Solution 2 by Kee-Wai Lau, Hong-Kong, China.

We show that the incenter I of a triangle ABC is the same as the incenter I_M of its Morley triangle if and only if ABC is equilateral.

In homogeneous trilinear coordinates, I is $1:1:1$ and I_M is

$$\cos\left(\frac{A}{3}\right)+2\cos\left(\frac{B}{3}\right)\cos\left(\frac{C}{3}\right):\cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right):\cos\left(\frac{C}{3}\right)+2\cos\left(\frac{A}{3}\right)\cos\left(\frac{B}{3}\right).$$

Clearly if ABC is equilateral, then $I = I_M$. Now suppose that $I = I_M$ so that

$$\cos\left(\frac{A}{3}\right)+2\cos\left(\frac{B}{3}\right)\cos\left(\frac{C}{3}\right) = \cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) \quad (1)$$

$$\cos\left(\frac{B}{3}\right)+2\cos\left(\frac{C}{3}\right)\cos\left(\frac{A}{3}\right) = \cos\left(\frac{C}{3}\right)+2\cos\left(\frac{A}{3}\right)\cos\left(\frac{B}{3}\right). \quad (2)$$

From (1) we obtain

$$\left(\cos\left(\frac{A}{3}\right)-\cos\left(\frac{B}{3}\right)\right)\left(1-2\cos\left(\frac{C}{3}\right)\right)=0.$$

Since $0 < C < \pi$, so

$$1-2\cos\left(\frac{C}{3}\right)<0.$$

Thus,

$$\cos\left(\frac{A}{3}\right)=\cos\left(\frac{B}{3}\right) \text{ or } A = B.$$

Similarly from (2) we obtain $B = C$. It follows that ABC is equilateral and this completes the solution.

Also solved by David E. Manes, Oneonta, NY, and the proposer.

- 5013: *Proposed by Ovidiu Furdui, Toledo, OH.*

Let $k \geq 2$ be a natural number. Find the sum

$$\sum_{n_1,n_2,\cdots,n_k\geq 1}\frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k}.$$

Solution by Kee-Wai Lau, Hong Kong, China.

For positive integers $M_1, M_2, \cdots, M_k$, we have

$$\sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2}\cdots\sum_{n_k=1}^{M_k}\frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2\cdots+n_k}$$

$$= \sum_{n_1=1}^{M_1}\sum_{n_2=1}^{M_2}\cdots\sum_{n_k=1}^{M_k}(-1)^{n_1+n_2+\cdots+n_k}\int_0^1 x^{n_1+n_2+\cdots+n_k-1}dx$$

$$= \int_0^1\left(\sum_{n_1=1}^{M_1}(-1)^{n_1}x^{n_1}\right)\left(\sum_{n_2=1}^{M_2}(-1)^{n_2}x^{n_2}\right)\cdots\left(\sum_{n_k=1}^{M_k}(-1)^{n_k}x^{n_k}\right)x^{-1}dx$$

$$= \int_0^1\left(\frac{-x(1-(-x)^{M_1})}{1+x}\right)\left(\frac{-x(1-(-x)^{M_2})}{1+x}\right)\left(\frac{-x(1-(-x)^{M_k})}{1+x}\right)x^{-1}dx$$

$$
\begin{aligned}
&= (-1)^k \int_0^1 \frac{x^{k-1}(1-(-x)^{M_1})(1-(-x)^{M_2})\cdots(1-(-x)^{M_k}))}{(1+x)^k}dx \\
&= (-1)^k \int_0^1 \frac{x^{k-1}}{(1+x)^k}dx + O\Big(\int_0^1 x^{M_1}+x^{M_2}+\cdots+x^{M_k}\Big)dx \\
&= (-1)^k \int_0^1 \frac{x^{k-1}}{(1+x)^k}dx + O\Big(\frac{1}{M_1}+\frac{1}{M_2}+\cdots+\frac{1}{M_k}\Big)
\end{aligned}
$$

as $M_1, M_2, \cdots, M_k$ tend to infinity. Here the constants implied by the $O's$ depend at most on k.
It follows that the sum of the problem equals

$$
(-1)^k \int_0^1 \frac{x^{k-1}}{1+x)^k}dx = (-1)^k I_k, \quad \text{say.}
$$

Integrating by parts, we have for $k \geq 3$,

$$
\begin{aligned}
I_k &= \frac{1}{1-k}\int_0^1 x^{k-1} d((1+x)^{1-k}) \\
&= \frac{-1}{(k-1)2^{k-1}} + I_{k-1}.
\end{aligned}
$$

Since $I_2 = \ln 2 - \dfrac{1}{2}$, we obtain readily by induction that for $k \geq 2$.

$$
I_k = \ln 2 - \sum_{j=2}^{k} \frac{1}{(j-1)2^{j-1}}.
$$

we now conclude that for $k \geq 2$,

$$
\sum_{n_1,n_2,\cdots,n_k \geq 1} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1+n_2+\cdots+n_k} = (-1)^k\Big(\ln 2 - \sum_{j=1}^{k-1}\frac{1}{j(2^j)}\Big).
$$

Also solved by Paolo Perfetti, Mathematics Department, University "Tor Vergata", Rome, Italy; Paul M. Harms, North Newton, KS; Boris Rays, Chesapeake, VA, and the proposer.