Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Proposals are always welcomed. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to: <*eisen@math.bgu.ac.il*> or to <*eisenbt@013.net*>.

Solutions to the problems stated in this issue should be posted before
December 15, 2007

- 4978: *Proposed by Kenneth Korbin, New York, NY.*

  Given equilateral triangle ABC with side $\overline{\text{AB}} = 9$ and with cevian $\overline{\text{CD}}$. Find the length of $\overline{\text{AD}}$ if $\triangle$ADC can be inscribed in a circle with diameter equal to 10.

- 4979: *Proposed by Kenneth Korbin, New York, NY.*

  Part I: Find two pairs of positive numbers (x, y) such that

  $$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{\sqrt{65}}{2},$$

  where x is an integer.

  Part II: Find four pairs of positive numbers (x, y) such that

  $$\frac{x}{\sqrt{1+y}-\sqrt{1-y}} = \frac{65}{2},$$

  where x is an integer.

- 4980: *J.P. Shiwalkar and M.N. Deshpande, Nagpur, India.*

  An unbiased coin is sequentially tossed until $(r+1)$ heads are obtained. The resulting sequence of heads (H) and tails (T) is observed in a linear array. Let the random variable X denote the number of double heads (HH's, where overlapping is allowed) in the resulting sequence. For example: Let $r = 6$ so the unbiased coin is tossed till 7 heads are obtained and suppose the resulting sequence of H's and T's is as follows:

  HHTTT**H**TTTT**HHH**TT**H**

Now in the above sequence, there are three double heads (HH's) at toss number $(1,2),(11,12)$ and $(12,13)$. So the random variable X takes the value 3 for the above observed sequence. In general, what is the expected value of X?

- 4981: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Find all real solutions of the equation

$$5^x + 3^x + 2^x - 28x + 18 = 0.$$

- 4982: *Proposed by Juan José Egozcue and José Luis Díaz-Barrero, Barcelona, Spain.*

  Calculate

$$\lim_{n\to\infty} \frac{1}{n+1}\left(\sum_{1\le i_1\le n+1}\frac{1}{i_1} + \sum_{1\le i_1<i_2\le n+1}\frac{1}{i_1 i_2} + \cdots + \sum_{1\le i_1<\ldots<i_n\le n+1}\frac{1}{i_1 i_2 \ldots i_n}\right).$$

- 4983: *Proposed by Ovidiu Furdui, Kalamazoo, MI.*

  Let k be a positive integer. Evaluate

$$\int_0^1 \left\{\frac{k}{x}\right\} dx,$$

where $\{a\}$ is the *fractional part* of a.

Solutions

- 4948: *Proposed by Kenneth Korbin, New York, NY.*

  The sides of a triangle have lengths $x_1, x_2,$ and x_3 respectively. Find the area of the triangle if

$$(x-x_1)(x-x_2)(x-x_3) = x^3 - 12x^2 + 47x - 60.$$

Solution by Jahangeer Kholdi and Robert Anderson (jointly), Portsmouth, VA.

The given equation implies that

$$\begin{aligned} x_1+x_2+x_3 &= 12 \\ x_1x_2+x_1x_3+x_2x_3 &= 47 \\ x_1x_2x_3 &= 60 \end{aligned}$$

from which by inspection, $x_1 = 3, x_2 = 4$ and $x_3 = 5$.

Editor's comment: At the time this problem was sent to the technical editor, the Journal was in a state of transition. A new editor- in-chief was coming on board and there was some question as to the future of the problem solving column. As such, I sent an advanced copy of the problem solving column to many of the regular contributors. In that advanced copy this polynomial was listed as $(x-x_1)(x-x_2)(x-x_3) = x^3-12x^2+47x-59$, and not with the constant term as listed above. Well, many of those who sent in solutions solved the problem in one of two ways: as above, obtaining the perimeter $x_1+x_2+x_3 = 12$; and then finding the area with Heron's formula. $A = \sqrt{6(6-x_1)(6-x_2)(6-x_3)}$.

Substituting 6 into $(x-x_1)(x-x_2)(x-x_3) = x^3-12x^2+47x-59$ gives $(6-x_1)(6-x_2)(6-x_3) = 7$. So, $A = \sqrt{(6)(7)} = \sqrt{42}$. But others noted that the equation $x^3-12x^2+47x-59$ has only one real root, and this gives the impossible situation of having a triangle with the lengths of two of its sides being complex numbers. The intention of the problem was that a solution should exist, and so the version of this problem that was posted on the internet had a constant term of -60. In the end I counted a solution as being correct if the solution path was correct, with special kudos going to those who recognized that the advanced copy version of this problem was not solvable.

Also solved by Brian D. Beasley, Clinton, SC; Mark Cassell (student, St. George's School), Spokane, WA; Pat Costello, Richmond, KY; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; José Luis Díaz-Barrero, Barcelona, Spain; Grant Evans (student, St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; David E. Manes, Oneonta, NY; Charles McCracken (two solutions as outlined above), Dayton, OH; John Nord (two solutions as outlined above), Spokane, WA; Boris Rays, Chesapeake, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- **4949:** *Proposed by Kenneth Korbin, New York, NY.*

A convex pentagon is inscribed in a circle with diameter d. Find positive integers a, b, and d if the sides of the pentagon have lengths a, a, a, b, and b respectively and if $a > b$. Express the area of the pentagon in terms of a, b, and d.

Solution by David Stone and John Hawkins, Statesboro, GA.

Note, that any solution can be scaled upward by any integer factor to produce infinitely many similar solutions.

We have three isosceles triangles with base a and equal sides $\frac{d}{2}$, and two isosceles triangles with base b and equal sides $\frac{d}{2}$. Let α be the measure of the angle opposite base a, and let β be the measure of the angle opposite the base b. Then $3\alpha + 2\beta = 2\pi$.

For each triangle with base a, the perimeter is $d + a$, and Heron's formula gives

$$A_n = \sqrt{\left(\frac{d+a}{2}\right)\left(\frac{d-a}{2}\right)\left(\frac{a}{2}\right)} = \frac{a}{4}\sqrt{d^2-a^2}.$$

We can also use the Law of Cosines to express the cosine of α as $\cos\alpha = \dfrac{a^2 - 2\left(\dfrac{d}{2}\right)^2}{-2\left(\dfrac{d}{2}\right)^2} = \dfrac{d^2-2a^2}{d^2}$.

From the Pythagorean Identity, it follows that

$$\sin\alpha = \sqrt{1-\left(\frac{d^2-2a^2}{d^2}\right)^2} = \frac{1}{d^2}\sqrt{d^4-d^4+4a^2d^2-4a^4} = \frac{2a}{d^2}\sqrt{d^2-a^2}.$$

Because the triangle is isosceles, with equal sides forming the angle α, an altitude through angle α divides the triangle into two equal right triangles. Therefore, $\cos\frac{\alpha}{2} = \frac{1}{d}\sqrt{d^2-a^2}$ and $\sin\frac{\alpha}{2} = \frac{a}{d}$.

For the triangles with base b, we can similarly obtain $A_b = \frac{b}{4}\sqrt{d^2-b^2}$ and $\cos\beta = \frac{d^2-2b^2}{d^2}$.

The area for the convex polygon is then

$$\begin{aligned} A_{polygon} &= 3A_a + 2A_b \\ &= \frac{3a}{4}\sqrt{d^2-a^2} + \frac{b}{2}\sqrt{d^2-b^2} \\ &= \frac{1}{4}\left(3a\sqrt{d^2-a^2} + 2b\sqrt{d^2-b^2}\right) \end{aligned}$$

in terms of a, b, and d.

Solving $3\alpha + 2\beta = 2\pi$, we find $\beta = \frac{2\pi - 3\alpha}{2} = \pi - \frac{3\alpha}{2}$.

Therefore,

$$\cos\beta = \cos\left(\pi - \frac{3\alpha}{2}\right) = -\cos\left(\frac{3\alpha}{2}\right) = -\cos\left(\alpha + \frac{\alpha}{2}\right) = -\cos\frac{\alpha}{2}\cos\alpha + \sin\frac{\alpha}{2}\sin\alpha.$$

Replacing the trig functions in this formula with the values computed above gives

$$\frac{d^2-2b^2}{d^2} = -\frac{\sqrt{d^2-a^2}}{d}\left(\frac{d^2-2a^2}{d^2}\right) + \frac{a}{d}\left(\frac{2a}{d^2}\right)\sqrt{d^2-a^2} = \frac{\sqrt{d^2-a^2}}{d}\left(4a^2-d^2\right).$$

Solving for b^2 in terms of a and d gives

$$b^2 = \frac{d^3 - \sqrt{d^2-a^2}\left(4a^2-d^2\right)}{2d}, \text{ or } b = \sqrt{\frac{d^3 - \sqrt{d^2-a^2}\left(4a^2-d^2\right)}{2d}}.$$

Note also that **(1)** $2b^2 = d^2 - \frac{\sqrt{d^2-a^2}\left(4a^2-d^2\right)}{d}$.

We can use this expression for b to compute the area of the polygon solely in terms of a and d.

$$A_{polygon} = \frac{3a}{4}\sqrt{d^2-a^2} + \frac{b}{2}\sqrt{d^2-b^2} = \frac{3a}{4}\sqrt{d^2-a^2} + \frac{a|3d^2-4a^2|}{4d}.$$

To find specific values which satisfy the problem, we use equation **(1).**

If $d^2 - a^2 = m^2$, then **(1)** becomes **(2)** $2b^2 = d^2 - \frac{m\left(4a^2-d^2\right)}{d} = d^2 - \frac{m\left(3a^2-m^2\right)}{d}$.

Then(a, m, d) is a Pythagorean triple, and thus a scalar multiple of a primitive Pythagorean triple (A, B, C). Using the standard technique, this triple is generated by two parameters, s and t:

$$\begin{cases} A = 2st \\ B = s^2 - t^2 \\ C = s^2 + t^2 \end{cases},$$

where $s > t$, s and t are relatively prime and have opposite parity. There are the two possibilities, where k is some scalar:

$$a = kA = 2kst,\ m = kB = k\left(s^2 - t^2\right), \text{and } d = kC = k\left(s^2 + t^2\right)$$

or

$$m = kA = 2kst,\ a = kB = k\left(s^2 - t^2\right), \text{and } d = kC = d\left(s^2 + t^2\right).$$

We'll find solutions satisfying the first set of conditions, recognizing that this will probably not produce all solutions of the problem. Substituting these in **(2)**,we find

$$2b^2 = d^2 - \frac{m(3a^2 - m^2)}{d} = k\left(s^2 + t^2\right)^2 - \frac{k(s^2 - t^2)\left(3(2ks)^2 - k^2\left(s^2 - t^2\right)^2\right)}{k(s^2 + t^2)}.$$

Simplifying, we find that $b^2 = \dfrac{k^2 s^2\left(s^2 - 3t^2\right)^2}{s^2 + t^2}$, and we want this b to be an integer. The simplest possible choice is to let $k^2 = s^2 + t^2$ (so that (s, t, k) is itself a Pythagorean triple); this forces $b = s\left(s^2 - 3t^2\right)$. We then have

$$a = 2kst = 2st\sqrt{s^2 + t^2},\ \ m = \sqrt{s^2 + t^2}\left(s^2 - t^2\right),\ \ d = k(s^2 + t^2) = k^3 = \left(s^2 + t^2\right)^{3/2} \text{ and}$$

$$b = s\left(s^2 - 3t^2\right).$$

That is, if (s, t, k) is a Pythagorean triple with $s^2 - 3t^2 > 0$, we have

$$\begin{cases} a = 2kst \\ b = s\left(s^2 - 3t^2\right) \\ d = k^3. \end{cases}$$

The restriction that $a > b$ imposes further conditions on s and t (roughly, $s < 3.08t$).

Some results, due to Excel:

s	t	k	b	a	d	*Area*
12	5	13	828	1,560	2,197	1,024,576
15	8	17	495	4,080	4,913	3,396,630
35	12	37	27,755	31,080	50,653	604,785,405
80	39	89	146,960	555,360	704,969	85,620,163,980
140	51	149	1,651,580	2,127,720	3,307,949	2,530,718,023,785
117	44	125	922,077	1,287,000	1,953,125	829,590,714,707
168	95	193	193,032	6,160,560	7,189,057	6,053,649,964,950
208	105	233	2,119,312	10,177,440	12,649,337	25,719,674,553,300
187	84	205	2,580,787	6,440,280	8,615,125	14,516,270,565,027
252	115	277	6,004,908	16,054,920	21,253,933	86,507,377,177,725
209	120	241	100,529	12,088,560	13,997,521	21,678,178,927,350
247	96	265	8,240,167	12,567,360	18,609,625	77,495,769,561,288
352	135	377	24,368,608	35,830,080	53,582,633	647,598,434,135,400

Also solved by the proposer

- 4950: *Proposed by Isabel Díaz-Iriberri and José Luis Díaz-Barrero, Barcelona, Spain.*

  Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}}+\frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}}+\frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \geq 3.$$

Solution by Kee-Wai Lau, Hong Kong, China

Since

$$\begin{aligned} a+b &= \frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a^3}+\sqrt[4]{b^3}) \ +(\sqrt[4]{a}-\sqrt[4]{b})^2(\sqrt{a}+\sqrt[4]{a}\sqrt[4]{b}+\sqrt{b})}{2} \\ &\geq \frac{(\sqrt[4]{a}+\sqrt[4]{b})(\sqrt[4]{a^3}+\sqrt[4]{b^3})}{2} \end{aligned}$$

with similar results for $b+c$ and $c+a$, so by the arithmetic mean-geometric mean inequality, we have

$$\begin{aligned} &\frac{a+b}{\sqrt[4]{a^3}+\sqrt[4]{b^3}}+\frac{b+c}{\sqrt[4]{b^3}+\sqrt[4]{c^3}}+\frac{c+a}{\sqrt[4]{c^3}+\sqrt[4]{a^3}} \\ \geq\ & \sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c} \\ \geq\ & 3\sqrt[12]{abc} \\ =\ & 3 \text{ as required.} \end{aligned}$$

Also solved by Michael Brozinsky (two solutions), Central Islip, NY; Dionne Bailey, Elsie Campbell, and Charles Diminnie (jointly), San Angelo, TX, and the proposer.

- 4951: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let α, β, and γ be the angles of an acute triangle ABC. Prove that

$$\pi \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}} \geq \alpha\sin\sqrt{\alpha}+\beta\sin\sqrt{\beta}+\gamma\sin\sqrt{\gamma}.$$

Solution by Elsie M. Campbell, Dionne T. Bailey and Charles Diminnie (jointly), San Angelo, TX.

Since α, β, and γ are the angles of an acute triangle,

$$\alpha, \beta, \gamma \in (0, \frac{\pi}{2}) \text{ and } \frac{\alpha}{\pi}+\frac{\beta}{\pi}+\frac{\gamma}{\pi}=1$$

Let $f(x) = \sin\sqrt{x}$ on $(0, \frac{\pi}{2})$. Then, since

$$f''(x) = -\frac{\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}}{4x^{3/2}} < 0$$

on $(0,\frac{\pi}{2})$, it follows that $f(x)$ is concave down on $(0, \frac{\pi}{2})$. Hence, by Jensen's Inequality and (1)

$$\frac{\alpha}{\pi}\sin\sqrt{\alpha}+\frac{\beta}{\pi}\sin\sqrt{\beta}+\frac{\gamma}{\pi}\sin\sqrt{\gamma} \quad\leq\quad \sin\sqrt{\frac{\alpha}{\pi}\cdot\alpha+\frac{\beta}{\pi}\cdot\beta+\frac{\gamma}{\pi}\cdot\gamma}$$

$$= \sin\sqrt{\frac{\alpha^2+\beta^2+\gamma^2}{\pi}},$$

with equality if and only if $\alpha = \beta = \gamma = \dfrac{\pi}{3}$.

Also solved by the proposer

- 4952: *Proposed by Michael Brozinsky, Central Islip, NY & Robert Holt, Scotch Plains, NJ.*

  An archeological expedition discovered all dwellings in an ancient civilization had $1, 2$, or 3 of each of k independent features. Each plot of land contained three of these houses such that the k sums of the number of each of these features were all divisible by 3. Furthermore, no plot contained two houses with identical configurations of features and no two plots had the same configurations of three houses. Find **a)** the maximum number of plots that a house with a given configuration might be located on, and **b)** the maximum number of distinct possible plots.

  Solution by Paul M. Harms, North Newton, KS

  Let $\binom{n}{r}$ be the combination of n things taken r at a time. With k independent features there are $\binom{k}{1} = k$ number of different "groups" containing one feature, $\binom{k}{2}$ different "groups" containing two features, etc. To have the sum of independent features in a plot of three houses be divisible by three, there are four possibilities. **I.** Each house in a plot has one feature. **II.** Each house in a plot has two features. **III.** Each house in a plot has three features. **IV.** One house in a plot has one feature, another house has two features, and the third house has three features.

  The maximum number of distinct plots can be found by summing the number of plots for each of the four possibilities above. The sum is

$$\binom{\binom{k}{1}}{3} + \binom{\binom{k}{2}}{3} + \binom{\binom{k}{3}}{3} + \binom{k}{1}\binom{k}{2}\binom{k}{3}$$

  This is the result for part **b).**

  For part **a),** first consider a house with one fixed feature. There are plots in possibilities I and IV. In possibility I the other two houses can have any combination of the other $(k-1)$ single features so there are $\binom{k-1}{2}$ plots. In possibility IV the number of plots with a house with one fixed feature is $\binom{k}{1}\binom{k}{2}\binom{k}{3}$. The number of plots with houses with different features is the following: For a house with one fixed feature there are $\binom{k-1}{2} + \binom{k}{2}\binom{k}{3}$ plots. For a house with two fixed features there are $\binom{\binom{k}{2}-1}{2} + \binom{k}{1}\binom{k}{3}$ plots. For a house with three fixed features there are $\binom{\binom{k}{2}-1}{2} + \binom{k}{1}\binom{k}{2}$ plots.

Also solved by the proposer.

- 4953: *Proposed by Tom Leong, Brooklyn, NY.*

  Let$\pi(x)$ denote the number of primes not exceeding x. Fix a positive integer n and define sequences by $a_1 = b_1 = n$ and

  $$a_{k+1} = a_k - \pi(a_k) + n, \qquad b_{k+1} = \pi(b_k) + n + 1 \quad \text{for k} \geq 1.$$

  a) Show that $\lim\limits_{k \to \infty} a_k$ is the n^{th} prime.

  b) Show that $\lim\limits_{k \to \infty} b_k$ is the n^{th} composite.

  Solution by Paul M. Harms, North Newton, KS.

  Any positive integer m is less than the m^{th} prime since 1 is not a prime. In part **a)** with $a_1 = n$, we have $\pi(n)$ primes less than or equal to n. We need $n - \pi(n)$ more primes than n has in order to get to the n^{th} prime. Note that a_2 is greater than a_1 by $n - \pi(n)$. If **all** of the integers from $a_1 + 1$ to a_2 are prime, then a_2 is the n^{th} prime. If not all of the integers indicated in the last sentence are primes, we see that a_3 is greater than a_2 by the number of non-primes from $a_1 + 1$ to a_2. This is true in general from a_k to a_{k+1} since $a_{k+1} = a_k + (n - \pi(a_k))$. If a_k is not the n^{th} prime, then a_{k+1} will increase by the quantity of integers to get to the n^{th} prime provided all integers a_{k+1} will increase by the quantity of integers to get to the n^{th} prime provided all integers $a_k + 1$,to a_{k+1}. We see that the sequence increases until some $a_m = N$, the n^{th} prime. Then $a_{m+1} = a_m + (n - \pi(a_m)) = a_m + 0 = a_m$. In this same way it is seen that $a_k = a_m$ for all k greater that m. Thus the limit for the sequence in part **a)** is the n^{th} prime.

  For part **b)** note that n is less than the n^{th} composite. Since the integer 1 and integers $\pi(n)$ are not composite, the n^{th} composite must be at least $1+\pi(n)$ greater than n. With $b_1 = n$ we see that $b_2 = n + (1 + \pi(n))$. Then b_2 will be the n^{th} composte provided all integers $n + 1, n + 2, \cdots, n + 1 + \pi(n)$ are composites. If some of the integers in the last sentence are prime, then b_3 is greater than b_2 by the number of primes in the integers from $b_1 + 1$ to b_2. In general, b_{k+1} is greater than b_k by the number of primes in the integers from $b_{k-1} + 1$ to b_k and the sequence will be an increasing sequence until the n^{th} composite is reached. If $b_m = N$, the n^{th} composite, then all integers from $b_{m-1} + 1$ to b_m are composite. Then $\pi(b_{m-1}) = \pi(b_m)$ and $b_{m+1} = \pi(b_{m-1}) + 1 + n = b_m = N$. We see that $b_k = N$ for all k at least as great as m. Thus the limit of the sequence in part **b)** is the n^{th} composite.

  Also solved by David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4954: *Proposed by Kenneth Korbin, New York, NY.*

  Find four pairs of positive integers (a, b) that satisfy

  $$\frac{a+i}{a-i} \cdot \frac{b+i}{b-i} = \frac{111+i}{111-i}$$

  with $a < b$.

  Solution by David E. Manes, Oneonta, NY.

  The only solutions (a, b) with $a < b$ are $(112, 12433), (113, 6272), (172, 313)$, and $(212, 233)$.

  Expanding the given equation and clearing fractions, one obtains $[2(111)(a + b) - 2(ab - 1)]i = 0$. Therefore, $\dfrac{ab-1}{a+b} = 111$. Let $b = a + k$ for some positive integer k. Then the above equation reduces to a quadratic in a; namely $a^2 + (k-222)a - (111k+1) = 0$ with roots given by

$$a = \frac{(222-k) \pm \sqrt{k^2 + 49288}}{2}.$$

Since a is a positive integer, it follows that $k^2 + 49288 = n^2$ or

$$n^2 - k^2 = (n+k)(n-k) = 49288 = 2^3 \cdot 61 \cdot 101.$$

Therefore, $n+k$ and $n-k$ are positive divisors of 49288. The only such divisors yielding solutions are

$n+k$	$n-k$
24644	2
12322	4
404	122
244	202

Solving these equations simultaneously gives the following values for(n,k) :

$$(12323, 12321), (6163, 6159), (263, 141), \text{ and } (223, 21)$$

from which the above cited solutions for a and b are found.

Also solved by Brian D. Beasley, Clinton, SC; Elsie M. Campbell, Dionne T. Bailey, and Charles Diminnie (jointly), San Angelo, TX; Daniel Copeland (student at St. George's School), Spokane, WA; Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly; students at Taylor University), Upland, IN; Grant Evans (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; John Nord, Spokane, WA; Homeira Pajoohesh, David Stone, and John Hawkins (jointly), Statesboro, GA, and the proposer.

- 4955: *Proposed by Kenneth Korbin, New York, NY.*

Between 100 and 200 pairs of red sox are mixed together with between 100 and 200 pairs of blue sox. If three sox are selected at random, then the probability that all three are the same color is 0.25. How many pairs of sox were there altogether?

Solution by Brian D. Beasley, Clinton, SC.

Let R be the number of pairs of red sox and B be the number of pairs of blue sox. Then $200 \le R + B \le 400$ and

$$\frac{2R(2R-1)(2R-2) + 2B(2B-1)(2B-2)}{(2R+2B)(2R+2B-1)(2R+2B-2)} = \frac{1}{4}.$$

Thus $4[R(2R-1)(R-1) + B(2B-1)(B-1)] = (R+B)(2R+2B-1)(R+B-1)$, or equivalently

$$4(2R^2 + 2B^2 - R - B - 2RB)(R+B-1) = (2R^2 + 2B^2 - R - B + 4RB)(R+B-1).$$

This yields $6R^2 + 6B^2 - 3R - 3B - 12RB = 0$ and hence $2(R-B)^2 = R+B$. Letting $x = R - B$, we obtain $R = x^2 + \frac{1}{2}x$ and $B = x^2 - \frac{1}{2}x$, so x is even. In addition, the size of $R+B$ forces $|x| \in \{10, 12, 14\}$. A quick check shows that only $|x| = 12$ produces values for R and B between 100 and 200, giving the unique solution $\{R, B\} = \{138, 150\}$. Thus $R + B = 288$.

Also solved by Pat Costello, Richmond, KY; Paul M. Harms, North Newton, KS, and the proposer.

- 4956: *Proposed by Kenneth Korbin, New York, NY.*

  A circle with radius $3\sqrt{2}$ is inscribed in a trapezoid having legs with lengths of 10 and 11. Find the lengths of the bases.

  Solution by Eric Malm, Stanford, CA.

  There are two different solutions: one when the trapezoid is shaped like /O\, and the other when it is configured like /O/. In fact, by reflecting the right-hand half of the plane about the x-axis, we can interchange between these two cases. Anyway, in the first case, the lengths of the bases are $7-\sqrt{7}$ and $14+\sqrt{7}$, and in the second case they are $7+\sqrt{7}$ and $14-\sqrt{7}$.

  Also solved by Michael Brozinsky, Central Islip, NY; Daniel Copeland (student at St. George's School), Spokane, WA; Paul M. Harms, North Newton, KS; Peter E. Liley, Lafayette, IN; Charles McCracken, Dayton, OH; Boris Rays, Chesapeake, VA; Nate Wynn (student at St. George's School), Spokane, WA, and the proposer.

- 4957: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

  Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_0=1, a_1=2, a_2=1$ and for all $n\geq 3$, $a_n^3=a_{n-1}a_{n-2}a_{n-3}$. Find $\lim\limits_{n\to\infty} a_n$.

  Solution by Michael Brozinsky, Central Islip, NY.

  If we write $a_n=2^{b_n}$ we have $b_n=\dfrac{b_{n-1}+b_{n-2}+b_{n-3}}{3}$ where $b_0=0, b_1=1$, and $b_2=0$. The characteristic equation is

$$\begin{aligned} x^3 &= \frac{x^2}{3}+\frac{x}{3}+\frac{1}{3} \text{ with roots} \\ r_1 &= 1,\ r_2=\frac{-1+i\sqrt{2}}{3}, \text{ and } r_3=\frac{-1-i\sqrt{2}}{3}. \end{aligned}$$

  The generating function $f(n)$ for $\{b_n\}$ is (using the initial conditions) found to be

$$\begin{aligned} f(n) &= A+B\left(\frac{-1+i\sqrt{2}}{3}\right)^n+C\left(\frac{-1-i\sqrt{2}}{3}\right)^n \text{ where} \\ A &= \frac{1}{3},\ B=-\frac{1}{6}-\frac{5i\sqrt{2}}{12}, \text{ and } C=-\frac{1}{6}+\frac{5i\sqrt{2}}{12}. \end{aligned}$$

  Since $|r_2|=|r_3|=\dfrac{\sqrt{6}}{4}<1$ we have the last two terms in the expression for $f(n)$ approach 0 as n approaches infinity, and hence $\lim\limits_{n\to\infty} b_n=\dfrac{1}{3}$ and so $\lim\limits_{n\to\infty} a_n=\sqrt[3]{2}$.

  Also solved by Brian D. Beasley, Clinton, SC; Paul M. Harms, North Newton, KS; Kee-Wai Lau, Hong Kong, China; Boris Rays and Jahangeer Khold (jointly), Chesapeake, VA & Portsmouth, VA; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins, Statesboro, GA, and the proposer.

- 4958: *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*

Let $f : [a, b] \to R$ $(0 < a < b)$ be a continuous function on $[a, b]$ and derivable in (a, b). Prove that there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{1}{c\sqrt{ab}} \cdot \frac{\ln(ab/c^2)}{\ln(c/a) \cdot \ln(c/b)}.$$

Solution by the proposer.
Consider the function $F : [a, b] \to R$ defined by

$$F(x) = (\ln x - \ln a)(\ln x - \ln b) \exp\left[\sqrt{ab}\, f(x)\right]$$

Since F is continuous function on $[a, b]$, derivable in (a, b) and $F(a) = F(b) = 0$, then by Rolle's theorem there exists $c \in (a, b)$ such that $F'(c) = 0$. We have

$$F'(x) = \left[\frac{1}{x}(\ln x - \ln b) + \frac{1}{x}(\ln x - \ln a)\right.$$

$$\left.+\sqrt{ab}(\ln x - \ln a)(\ln x - \ln b)f'(x)\right] \exp\left[\sqrt{ab}\, f(x)\right]$$

and

$$\frac{1}{c}\ln\left(\frac{c^2}{ab}\right) + \sqrt{ab}\ln\left(\frac{c}{a}\right)\ln\left(\frac{c}{b}\right)\, f'(c) = 0$$

From the preceding immediately follows

$$\sqrt{ab}\ln(c/a)\ln(c/b)\; f'(c) = \frac{1}{c}\ln(ab/c^2)$$

and we are done.

- 4959: *Proposed by Juan-Bosco Romero Márquez, Valladolid, Spain.*

Find all numbers $N = ab$, were $a, b = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, such that

$$[S(N)]^2 = S(N^2),$$

where S(N)=a+b is the sum of the digits. For example:

$$\begin{array}{rcll} N &=& 12 & N^2 = 144 \\ S(N) &=& 3 & S(N^2) = 9 \quad \text{and } [\text{S(N)}]^2 = \text{S(N}^2). \end{array}$$

Solution by Jeremy Erickson, Matthew Russell, and Chad Mangum (jointly, students at Taylor University), Upland, IN.
We start by considering the possibilities that exist for N. Since there are 10 possibilities for a and for b, there are 100 possibilities for N. It would not be incorrect to check all 100 cases, however we need not do so.
We can eliminate the majority of these 100 cases without directly checking them. If we assume that $S(N) \geq 6$, then $[S(N)] \geq 36$, which means that for the property to hold, $S(N^2) \geq 36$ as well. This would require $N^2 \geq 9999$. However, this leads us to a contradiction because the largest possible value for N by our definition is $99, and$N^2 in that case is only $N^2 = 99^2 = 9801 < 9999$. Therefore, we need not check any number N such $S(N) > 6$. More precisely, any number N in the intervals$[6, 9]; [15, 19]; [24, 29]; [33, 39]; [42, 49]; [51, 99]$ need not be checked. This leaves us with 21 cases that can easily be checked.

After checking each of these cases separately, we find that for 13 of them, the property $[S(N)]^2 = S(N^2)$ does in fact hold. These 13 solutions are

$$N = 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 30, 31.$$

We show the computation for $N = 31$ as an example:

$$\begin{aligned} N &= 31 & N^2 = 31^2 = 961 \\ S(N) &= 3+1=4 & S(N^2) = 9+6+1 = 16 \\ [S(N)]^2 &= 4^2 = 16 & \\ [S(N)]^2 &= S(N^2) = 16 \text{ for } N = 31. & \end{aligned}$$

The other 12 solutions can be checked similarly.

Also solved by Paul M. Harms, North Newton, KS; Jahangeer Kholdi, Robert Anderson and Boris Rays (jointly), Portsmouth, Portsmouth, & Chesapeake, VA; Peter E. Liley, Lafayette, IN; Jim Moore, Seth Bird and Jonathan Schrock (jointly, students at Taylor University), Upland, IN; R. P. Sealy, Sackville, New Brunswick, Canada; David Stone and John Hawkins (jointly), Statesboro, GA, and the proposer.

Late Solutions

Late solutions by **David E. Manes of Oneonta, NY** were received for problems 4942 and 4944.

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