Problems

Ted Eisenberg, Section Editor

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This section of the Journal offers readers an opportunity to exchange interesting mathematical problems and solutions. Please send them to Ted Eisenberg, Department of Mathematics, Ben-Gurion University, Beer-Sheva, Israel or fax to: 972-86-477-648. Questions concerning proposals and/or solutions can be sent e-mail to <eisenbt@013.net>. Solutions to previously stated problems can be seen at <http://www.ssma.org/publications>.

Solutions to the problems stated in this issue should be posted before March 15, 2019

5523: *Proposed by Kenneth Korbin, New York, NY*

For every prime number P, there is a circle with diameter $4P^4+1$. In each of these circles, it is possible to inscribe a triangle with integer length sides and with area $(2P)(2P+1)(2P-1)(2P^2-1)$. Find the sides of the triangles if $P=2$ and if $P=3$.

5524: *Proposed by Michael Brozinsky, Central Islip, NY*

A billiard table whose sides obey the law of reflection is in the shape of a right triangle ABC with legs of length a and b where $a>b$ and hypotenuse c. A ball is shot from the right angle and rebounds off the hypotenuse at point P on a path parallel to leg CB that hits let CA at point Q. Find the ratio $\dfrac{AQ}{QC}$.

5525: *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu-Severin, Mehedinti, Romania*

Find real values for x and y such that:

$$4\sin^2(x+y)=1+4\cos^2 x+4\cos^2 y.$$

5526: *Proposed by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece*

The lengths of the sides of a triangle are 12, 16 and 20. Determine the number of straight lines which simultaneously halve the area and the perimeter of the triangle.

5527: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let a,b and c be positive real numbers such that $a+b+c=3$. Prove that for all real $\alpha>0$, holds:

$$\frac{1}{2}\left(\frac{1-a^{\alpha+1}b^{\alpha}}{a^{\alpha}b^{\alpha}}+\frac{1-b^{\alpha+1}c^{\alpha}}{b^{\alpha}c^{\alpha}}+\frac{1-c^{\alpha+1}a^{\alpha}}{c^{\alpha}a^{\alpha}}\right)$$

$$\leq \sqrt{\left(\frac{1-a^{\alpha+1}}{a^{\alpha}}+\frac{1-b^{\alpha+1}}{b^{\alpha}}+\frac{1-c^{\alpha+1}}{c^{\alpha}}\right)\left(\frac{1-a^{\alpha}b^{\alpha}c^{\alpha}}{a^{\alpha}b^{\alpha}c^{\alpha}}\right)}.$$

5528: *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Let $a > 0$. Calculate $\displaystyle\int_a^{\infty}\int_a^{\infty}\frac{\mathrm{d}x\mathrm{d}y}{x^6(x^2+y^2)}$.

Solutions

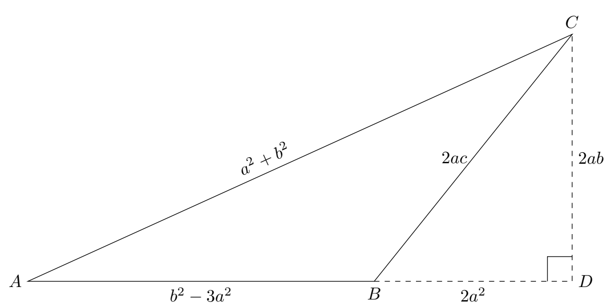
5505: *Proposed by Kenneth Korbin, New York, NY*

Given a Primitive Pythagorean Triple (a, b, c) with $b^2 > 3a^2$. Express in terms of a and b the sides of a Heronian Triangle with area $ab(b^2 - 3a^2)$.

(A Heronian Triangle is a triangle with each side length and area an integer.)

Solution 1 by Stanley Rabinowitz, Chelmsford, MA

One way of doing this would be to form an obtuse triangle ABC as shown with base of length $b^2 - 3a^2$ and altitude of length $2ab$, so that the area of $\triangle ABC$ is $ab(b^2 - 3a^2)$ as desired. If the line segment from B to D, the foot of the altitude from C, has length $2a^2$, then hypotenuse BC in $\triangle BDC$ would have length $2ac$, since this triangle would be similar to an a–b–c right triangle, scaled up by $2a$. Then AD would have length $b^2 - a^2$, and by the Pythagorean Theorem, AC would have length $a^2 + b^2$.
Thus, $\triangle ABC$ is the desired Heronian Triangle, with sides $b^2 - 3a^2$, $2a\sqrt{a^2 + b^2}$, and $a^2 + b^2$.



Solution 2 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Given a primitive Pythagorean triple (a, b, c) with $b^2 > 3a^2$, let

$$\begin{aligned} x &= b^2 - 3a^2, \\ y &= 2a\sqrt{a^2 + b^2}, \\ z &= a^2 + b^2. \end{aligned}$$

Note that $y = 2ac$ and $z = c^2$. Since $c^2 - 4a^2 = b^2 - 3a^2 > 0$ we have $c > 2a$.

We calculate

$$\begin{aligned} -x+y+z &= -(b^2-3a^2)+2ac+(a^2+b^2) \\ &= 4a^2+2ac>0, \end{aligned}$$

$$\begin{aligned} x-y+z &= (b^2-3a^2)-2ac+c^2 \\ &= (b^2-3a^2)+c(c-2a)>0, \end{aligned}$$

and

$$\begin{aligned} x+y-z &= (b^2-3a^2)+2ac-(a^2+b^2) \\ &= 2a(c-2a)>0. \end{aligned}$$

Thus $x+y>z$, $x+z>y$, and $y+z>x$ so (x,y,z) gives the sides of a Heronian triangle. Let s be the semiperimeter and A the area of this triangle.
By Heron's formula we have

$$A^2 = s(s-x)(s-y)(s-z).$$

We have

$$\begin{aligned} s &= \frac{x+y+z}{2} \\ &= b^2-a^2+ac, \end{aligned}$$

$$\begin{aligned} s-x &= b^2-a^2+ac-(b^2-3a^2) \\ &= ac+2a^2, \end{aligned}$$

$$\begin{aligned} s-y &= b^2-a^2+ac-2ac \\ &= b^2-a^2-ac, \end{aligned}$$

and

$$\begin{aligned} s-z &= b^2-a^2+ac-(a^2+b^2) \\ &= ac-2a^2. \end{aligned}$$

So

$$\begin{aligned} A^2 &= (b^2-a^2+ac)(ac+2a^2)(b^2-a^2-ac)(ac-2a^2) \\ &= [(b^2-a^2)^2-(ac)^2][(ac)^2-(2a^2)^2]. \end{aligned}$$

Now

$$\begin{aligned} (b^2-a^2)^2-(ac)^2 &= b^4-2a^2b^2+a^4-a^2(a^2+b^2) \\ &= b^4-3a^2b^2 \\ &= b^2(b^2-3a^2) \end{aligned}$$

and

$$\begin{aligned} (ac)^2-(2a^2)^2 &= a^2(c^2-4a^2) \\ &= a^2(b^2-3a^2) \end{aligned}$$

so $A^2 = a^2b^2(b^2 - 3a^2)^2$.

Thus if (a, b, c) is a primitive Pythagorean triple with $b^2 > 3a^2$ then (x, y, z) with

$$x = b^2 - 3a^2, \quad y = 2a\sqrt{a^2 + b^2}, \quad z = a^2 + b^2$$

is a Heronian triangle with area $ab(b^2 - 3a^2)$.

N.B. For a particular (a, b, c) there can be other Heronian triangles with area $ab(b^2 - 3a^2)$. For example, for the primitive Pythagorean triple $(5, 12, 13)$ we are looking for a Heronian triangle with area 4140. The formulas above give the triangle $(69, 130, 169)$, but $(41, 202, 207)$ is another triangle with area 4140.

Solution 3 by Trey Smith, Angelo State University, San Angelo, TX

Let $x = b^2 - 3a^2$, $y = 2a\sqrt{a^2 + b^2}$, and $z = a^2 + b^2$ be the lengths of the three sides of the triangle. We first observe that all of these are positive integers; x and z obviously so, and y since $a^2 + b^2 = c^2$, so that

$$2a\sqrt{a^2 + b^2} \quad = \quad 2a\sqrt{c^2} \quad = \quad 2ac.$$

The perimeter of the triangle is

$$\begin{aligned} & x + y + z \\ = \; & (b^2 - 3a^2) + (2a\sqrt{a^2 + b^2}) + (a^2 + b^2) \\ = \; & (c^2 - 4a^2) + 2ac + c^2 \\ = \; & 2c^2 + 2ac - 4a^2. \end{aligned}$$

Then the semiperimeter is $s = c^2 + ac - 2a^2$. Applying Heron's formula to find the area A, we have

$$\begin{aligned} & A^2 \\ = \; & s(s - x)(s - y)(s - z) \\ = \; & s(s - (b^2 - 3a^2))(s - (2a\sqrt{a^2 + b^2}))(s - (a^2 + b^2)) \\ = \; & s(s - (c^2 - 4a^2))(s - 2ac)(s - c^2) \\ = \; & (c^2 + ac - 2a^2)((c^2 + ac - 2a^2) - (c^2 - 4a^2))((c^2 + ac - 2a^2) - 2ac)((c^2 + ac - 2a^2) - c^2) \\ = \; & (c^2 + ac - 2a^2)(ac + 2a^2)(c^2 - ac - 2a^2)(ac - 2a^2) \\ = \; & [(c + 2a)(c - a)][a(c + 2a)][(c + a)(c - 2a)][a(c - 2a)] \\ = \; & a^2(c^2 - a^2)(c^2 - 4a^2)^2 \\ = \; & a^2b^2(b^2 - 3a^2)^2. \end{aligned}$$

Thus $A = ab(b^2 - 3a^2)$.

Editor's Comment : David Stone and John Hopkins of Georgia Southern University added the following comment to their solution to this problem: "So how did we find x, y, z? We first tired the simplest possible example; $(a, b, c) = (3, 5, 12)$. After some algebra and some computer help, we found the triangle $(x, y, z) = (69, 169, 130)$ has the appropriate area. From this we conjectured the form for arbitrary x, y, z.

$$\begin{aligned} x &= 169 = 13^2 = c^2 \\ y &= 69 = 12^2 - 3 \cdot 5^2 = b^2 - 3a^2 \\ z &= 130 = 2 \cdot 5 \cdot 13 = 2ac. \end{aligned}$$

Then it only required simple algebra to verify this construction. Some Excel computations also lead us to the broader result (when $b^2 < 3a^2$). The perfect example of computing power assisting a person!"

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Ed Gray, Highland Beach, FL; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.

5506: *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu-Severin, Mehedinti, Romania*

Find $\Omega = \det \left[\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} \right]$.

Solution 1 by Michel Bataille, Ronen, France

Let $A = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$, $B = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$, $O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It is readily checked that $AB = BA = O_2$ and $A + B = 26I_2$.
Since $AB = BA$, the binomial theorem gives

$$(A+B)^{100} = \sum_{k=0}^{100} \binom{100}{k} A^k B^{100-k}. \qquad (1)$$

Now, if $k \in \{1, 2, \ldots, 50\}$, then

$$A^k B^{100-k} = A^k B^k B^{100-2k} = (AB)^k B^{100-2k} = O_2 \cdot B^{100-2k} = O_2$$

(note that $A^k B^k = (AB)^k$ since $AB = BA$) and similarly, if $k \in \{51, 52, \ldots, 99\}$, then $A^k B^{100-k} = A^{2k-100}(AB)^{100-k} = O_2$.

As a result, (1) gives $(A+B)^{100} = A^{100} + B^{100}$, that is, $26^{100} I_2 = A^{100} + B^{100}$. We can conclude:

$$\Omega = \det(26^{100} I_2) = 26^{200}.$$

Solution 2 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Observe

$$\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} = \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 5 \end{bmatrix} \right)^{100} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \left(\begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right)^{99} \begin{bmatrix} 1 & 5 \end{bmatrix} = 26^{99} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$$

and

$$\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} = \left(\begin{bmatrix} 5 \\ -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \end{bmatrix}\right)^{100} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}\left(\begin{bmatrix} 5 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}\right)^{99}\begin{bmatrix} 5 & -1 \end{bmatrix} = 26^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}.$$

It follows that

$$\Omega = \det\left[26^{99}\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + 26^{99}\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}\right] = \det\left[\begin{pmatrix} 26^{100} & 0 \\ 0 & 26^{100} \end{pmatrix}\right] = 26^{200}.$$

Solution 3 by David A. Huckaby, Angelo State University, San Angelo, TX

Let $A = \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}$ and $B = \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}$. Matrices A and B are each symmetric, hence orthogonally diagonalizable.

Solving the equation $\det(\lambda I - A) = 0$ yields $\lambda_1 = 0$ and $\lambda_2 = 26$ as the eigenvalues of A. Solving the equation $(\lambda I - A)\overrightarrow{x} = \overrightarrow{0}$ successively for $\lambda = 0$ and $\lambda = 26$ yields $\overrightarrow{x_1} = \begin{pmatrix} \frac{-5}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{pmatrix}$ and $\overrightarrow{x_2} = \begin{pmatrix} \frac{1}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} \end{pmatrix}$ as corresponding unit eigenvectors. So $A = \begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}$. Similarly, $B = \begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{-5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \\ \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}$.

Since for both A and B the matrix of eigenvectors is orthogonal, we have

$$A^{100} = \begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26^{100} \end{pmatrix}\begin{pmatrix} \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \end{pmatrix} = \begin{pmatrix} 26^{99} & 5\left(26^{99}\right) \\ 5\left(26^{99}\right) & 25\left(26^{99}\right) \end{pmatrix}, \text{ and}$$

$$B^{100} = \begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{-5}{\sqrt{26}} \\ \frac{5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 26^{100} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{26}} & \frac{5}{\sqrt{26}} \\ \frac{-5}{\sqrt{26}} & \frac{1}{\sqrt{26}} \end{pmatrix} = \begin{pmatrix} 25\left(26^{99}\right) & -5\left(26^{99}\right) \\ -5\left(26^{99}\right) & 26^{99} \end{pmatrix}.$$

So $\Omega = \det\left[A^{100} + B^{100}\right] = \det\begin{pmatrix} 26^{100} & 0 \\ 0 & 26^{100} \end{pmatrix} = 26^{200}$.

Solution 4 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

A way to calculate A^n for a 2×2 matrix is to use the Hamilton-Cayley Theorem:

$$A^2 - \text{Tr}(A) \cdot A + \det A \cdot I_2 = 0.$$

For example, if we have a 2×2 matrix $A = \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix}$ (or $A = \begin{pmatrix} a^2 & -a \\ -a & 1 \end{pmatrix}$) with $\det A = 0$ and $\text{Tr}(A) = a^2 + 1$, then the Hamilton-Cayley theorem becomes:

$$A^2 \quad = \quad \text{Tr}(A) = (a^2 + 1)^2 A.$$

$$
\begin{aligned}
A^3 &= (a^2+1)A^2 = (a^2+1)^2 A, \\
&\ldots \\
A^n &= (a^2+1)A^{n-1} = (a^2+1)^{n-1} A.
\end{aligned}
$$

So we have:

$$
\begin{aligned}
\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} &= (5^2+1)^{99} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = 26^{99} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}, \\
\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}^{100} &= (5^2+1)^{99} \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = 26^{99} \begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix}, \\
\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100} &= 26^{99} \left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix} \right) = 26^{100} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}
$$

and finally we have:

$$
\Omega = \det \left(\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 25 & -5 \\ 5 & 1 \end{pmatrix}^{100} \right) = \det \left(26^{100} \begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix}^{100} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 26^{100}.
$$

Solution 5 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

Let $c = \sqrt{26}$. We know that

$$
\begin{pmatrix} 1 & 5 \\ 5 & 25 \end{pmatrix} = \begin{pmatrix} -5/c & 1/c \\ 1/c & 5/c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix} \begin{pmatrix} -5/c & 1/c \\ 1/c & 5/c \end{pmatrix} \doteq A\Lambda A^{-1}
$$

$$
\begin{pmatrix} 25 & -5 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 1/c & -5/c \\ 5/c & 1/c \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 26 \end{pmatrix} \begin{pmatrix} 1/c & 5/c \\ -5/c & 1/c \end{pmatrix} \doteq B\Lambda B^{-1}
$$

$$
\Omega = A\Lambda^{100}A^{-1} + B\Lambda^{100}B^{-1}
$$

$$
A\Lambda^{100}A^{-1} = \begin{pmatrix} 26^{99} & 5\cdot 26^{99} \\ 5\cdot 26^{99} & 25\cdot 26^{99} \end{pmatrix}
$$

$$
B\Lambda^{100}B^{-1} = \begin{pmatrix} 25\cdot 26^{99} & -5\cdot 26^{99} \\ -5\cdot 26^{99} & 26^{99} \end{pmatrix}
$$

Thus

$$
\Omega = \det \begin{pmatrix} 26^{99}\cdot 26 & 0 \\ 0 & 26^{99}\cdot 26 \end{pmatrix} = 26^{200}.
$$

Also solved by Arkady Alt, San Jose, CA; Ashland University Undergraduate Problem Solving Group, Ashland University, Ashland, Ohio; Brian D. Beasley, Presbyterian College, Clinton, SC; Anthony J.

Bevelacqua, University of North Dakota, Grand Forks, ND; Dionne Bailey, Elsie Campbell and Charles Diminnie, Angelo Sate University, San Angelo, TX; Pat Costello, Eastern Kentucky University, Richmond, KY; David Diminnie, Texas Instruments Inc., Dallas, TX; Michael Faleski, University Center, MI; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Carl Libis, Columbia Southern University, Orange Beach, AL; Ismayil Mammadzada (student), ADA University, Baku, Azerbaijan; Pedro Pantoja, Natal/RN, Brazil; Ravi Prakash, Oxford University Press; New Delhi, India; Neculai Stanciu "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Henry Ricardo (four different proofs), Westchester Area Math Circle, NY; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Marian Ursărescu, "Roman Vodă" College, Roman, Romania; Daniel Văcaru, Pitesti, Romania, and the proposer.

5507: *Proposed by David Benko, University of South Alabama, Mobile, AL*

A car is driving forward on the real axis starting from the origin. Its position at time $0 \le t$ is $s(t)$. Its speed is a decreasing function: $v(t), 0 \le t$. Given that the drive has a finite path (that is $\lim\limits_{t\to\infty} s < \infty$), that $v(2t)/v(t)$ has a real limit c as $t \to \infty$, find all possible values of c.

Solution 1 by Moti Levy, Rehovot, Israel

We will show that the set of all possible values of c, is the interval $\left[0, \frac{1}{2}\right]$, i.e., $0 \le c \le \frac{1}{2}$.

Let us summarize the conditions on the speed function $v(t)$:
1) $v(t) \ge 0$,
2) $v(t)$ is decreasing function for all $t \ge 0$,
3) $\int_0^\infty v(t)\, dt < \infty$
4) $\lim_{t\to\infty} \frac{v(2t)}{v(t)} = c$, c is real number.

Since $v(t) \ge 0$, then clearly $c \ge 0$. Since $v(t)$ is decreasing function, then $c \le 1$. It follows that $0 \le c \le 1$.

Now we show that c can attain any value in the interval $\left[0, \frac{1}{2}\right]$.
Let r be a real number and $r > 1$. Then $v(t) = \frac{1}{1+t^r}$ satisfies all four requirements from the speed function, in particular

$$\int_0^\infty \frac{1}{1+t^r} dt < \infty, \quad \text{for} \;\; r > 1,$$

and

$$\lim_{t\to\infty} \frac{v(2t)}{v(t)} = \lim_{t\to\infty} \frac{1+t^r}{1+2^r t^r} = \frac{1}{2^r} = c.$$

It follows that $c \in \left(0, \frac{1}{2}\right)$.
To see that c can attain also the value zero, choose $v(t) = e^{-t}$.
To see that c can attain also the value $\frac{1}{2}$, choose $v(t) = \begin{cases} \frac{1}{2\ln^2 2}, & \text{for } 0 \le t \le 2, \\ \frac{1}{t\ln^2 t}, & \text{for } 2 < t. \end{cases}$

Then $v(t)$ satisfies all the four requirements from the speed function, in particular

$$\int_0^{\infty} v(t)\, dt = \frac{1}{\ln^2 2} + \frac{1}{\ln 2},$$

and

$$\lim_{t\to\infty} \frac{v(2t)}{v(t)} = \lim_{t\to\infty} \frac{t\ln^2 t}{2t\ln^2(2t)} = \frac{1}{2}.$$

To finish the proof, we have to show that $c \notin \left(\frac{1}{2}, 1\right]$.
Suppose $\lim_{t\to\infty} \frac{v(2t)}{v(t)} = c$, then for every $\varepsilon > 0$, there is a real number t_0 such that $t > t_0$ implies $\frac{v(2t)}{v(t)} > c - \varepsilon$.

Now we define a staircase function $s(t)$, as follows:

$$s(t) := (c-\varepsilon)^k v(t_0), \quad for \quad 2^{k-1t_0}t_0 \le t < 2^k t_0, \qquad k = 1, 2, \ldots$$

Since the function $v(t)$ is positive decreasing function for all $t \ge 0$, then $v(t) \ge s(t)$, hence

$$\int_{t_0}^{\infty} v(t)\, dt \ge \int_{t_0}^{\infty} s(t)\, dt.$$

Integrating the staircase function, we get

$$\int_{t_0}^{\infty} s(t)\, dt = v(t_0) \sum_{k=1}^{\infty} (c-\varepsilon)^k 2^{k-1} = v(t_0)(c-\varepsilon) \sum_{k=0}^{\infty} (2(c-\varepsilon))^k.$$

If $c - \varepsilon \ge \frac{1}{2}$ then $\int_{t_0}^{\infty} s(t)\, dt$ diverges and so $\int_{t_0}^{\infty} v(t)\, dt$ diverges.
We conclude that if $c > \frac{1}{2}$ then $\int_{t_0}^{\infty} v(t)\, dt$ diverges, contradicting property 3) of the speed function.

Solution 2 by Kee-Wai Lau, Hong Kong, China

We show that

$$0 \le c \le \frac{1}{2} \tag{1}$$

Let $\lim\limits_{t\to\infty} s(t) = L < \infty$. Then $\lim\limits_{t\to\infty} s(2t) = L$ and $0 \le s(t) < s(2t) < L$ for $t > 0$. Hence, by L'Hôpital's rule, we have

$$1 \ge \lim_{t\to\infty} \frac{L - s(2t)}{L - s(t)} = \lim_{t\to\infty} \frac{\frac{ds(2t)}{dt}}{\frac{ds(t)}{dt}} = 2 \lim_{t\to\infty} \frac{v(2t)}{v(t)} = 2c.$$

Thus (1) holds.

By taking $s(t) = 1 - e^{-t},\ s(t) = 1 - (t+1)^{\frac{\ln(2c)}{\ln 2}},\ s(t) = 1 - \dfrac{1}{\ln(1+e)}$ according as $c = 0,\ 0 < c < \dfrac{1}{2},\ c = \dfrac{1}{2}$ we see that each c in (1) is in fact admissible.

Solution 3 by Albert Stadler, Herrliberg, Switzerland

We claim that the set C of possible values of c is the closed interval $\left[0, \dfrac{1}{2}\right]$.

Indeed, if $v(t)=v_0e^{-t}$, then $v(t)$ is a decreasing function , $\int_0^\infty v(t)dt<\infty$, and $\lim_{t\to\infty}\frac{v(2t)}{v(t)}=0$. So $0\in C$.

If $a\geq 1$ and $v(t)=\dfrac{v_0}{1+t^a\ln^2(1+t)}$ then $v(t)$ is a decreasing function, $\int_0^\infty v(t)dt<\infty$ and
$\lim_{t\to\infty}\frac{v(2t)}{v(t)}=\lim_{t\to\infty}\frac{1+t^a\ln^2(1+t)}{1+2^at^a\ln^2(1+2t)}=\frac{1}{2^a}$. So $\left(0,\frac{1}{2}\right]\subset C$. It remains to prove that if $c>\frac{1}{2}$ then c$\notin C$.

Suppose if possible that $\lim_{t\to\infty}\frac{v(2t)}{v(t)}=c$, where $c>\frac{1}{2}$. Let $\epsilon:=\frac{c-1/2}{2}>0$. Then there is a number $T=T(\epsilon)>0$ such that $-\epsilon<\frac{v(2t)}{v(t)}-c<\epsilon$, whenever $t>T$. We conclude that

$$\int_{2T}^\infty v(t)dt=2\int_T^\infty v(2t)dt>2(c-\epsilon)\int_T^\infty v(t)dt\geq 2(c-\epsilon)\int_{2T}^\infty v(t)dt=(1+2\epsilon)\int_{2T}^\infty v(t)dt>\int_{2T}^\infty v(t)dt,$$

which is a contradiction, and the proof is complete.

Also solved by the proposer.

5508: *Proposed by Pedro Pantoja, Natal RN, Brazil*

Let a,b,c be positive real numbers such that $a+b+c=1$. Find the minimum value of

$$f(a,b,c)=\frac{a}{3ab+2b}+\frac{b}{3bc+2c}+\frac{c}{3ca+2a}.$$

Solution 1 by Solution by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

To begin, we note that since $a,b,c>0$ and $a+b+c=1$, the Arithmetic - Geometric Mean Inequality implies that

$$\begin{aligned}a^2+b^2+c^2&=(a+b+c)\left(a^2+b^2+c^2\right)\\&=a^3+b^3+c^3+ab^2+bc^2+ca^2+a^2b+b^2c+c^2a\\&=\left(a^3+ab^2\right)+\left(b^3+bc^2\right)+\left(c^3+ca^2\right)+a^2b+b^2c+c^2a\\&\geq 2\sqrt{a^4b^2}+2\sqrt{b^4c^2}+2\sqrt{c^4a^2}+a^2b+b^2c+c^2a\\&=3\left(a^2b+b^2c+c^2a\right).\end{aligned}\tag{1}$$

As a result of (1), we have

$$\begin{aligned}1&=(a+b+c)^2\\&=a^2+b^2+c^2+2(ab+bc+ca)\\&\geq 3\left(a^2b+b^2c+c^2a\right)+2(ab+bc+ca)\\&=\left(3a^2b+2ab\right)+\left(3b^2c+2bc\right)+\left(3c^2a+2ca\right).\end{aligned}\tag{2}$$

Then, using property (2), the convexity of $g(x) = \frac{1}{x}$ on $(0, \infty)$, and Jensen's Theorem, we obtain

$$\begin{aligned} f(a,b,c) &= \frac{a}{3ab+2b} + \frac{b}{3bc+2c} + \frac{c}{3ca+2a} \\ &= ag(3ab+2b) + bg(3bc+2c) + cg(3ca+2a) \\ &\geq g[a(3ab+2b) + b(3bc+2c) + c(3ca+2a)] \\ &= g\left[\left(3a^2b+2ab\right) + \left(3b^2c+2bc\right) + \left(3c^2a+2ca\right)\right] \\ &= \frac{1}{(3a^2b+2ab)+(3b^2c+2bc)+(3c^2a+2ca)} \\ &\geq 1 \\ &= f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \end{aligned}$$

It follows that under the conditions $a, b, c > 0$ and $a+b+c=1$, the minimum value of $f(a,b,c)$ is $f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = 1$.

Solution 2 by David E. Manes, Oneonta, NY

We will show that the minimum value of f is 1.
By the Arithmetic Mean-Geometric Mean inequality, we get

$$f(a,b,c) \geq 3\sqrt[3]{\frac{a}{b(3a+2)} \cdot \frac{b}{c(3b+2)} \cdot \frac{c}{a(3c+2)}} = \frac{3}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}}.$$

We again use the AM-GM inequality to obtain

$$\begin{aligned} \sqrt[3]{(3a+2)(3b+2)(3c+2)} &\leq \frac{(3a+2)+(3b+2)+(3c+2)}{3} = \frac{3(a+b+c)+6}{3} \\ &= 3. \end{aligned}$$

Hence,

$$\frac{1}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}} \geq \frac{1}{3}$$

so that

$$f(a,b,c) \geq \frac{3}{\sqrt[3]{(3a+2)(3b+2)(3c+2)}} \geq 3 \cdot (1/3) = 1$$

with equality if and only if $a = b = c = \frac{1}{3}$.

Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

From Bergström's and the Arithmetic mean -Geometric mean inequalities,

$$f(a,b,c) = \frac{\left(\sqrt{\frac{a}{b}}\right)^2}{3a+2} + \frac{\left(\sqrt{\frac{b}{c}}\right)^2}{3b+2} + \frac{\left(\sqrt{\frac{c}{a}}\right)^2}{3c+2} \geq \frac{\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}\right)^2}{3a+2+3b+2+3c+2} = \left(\frac{\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{a}}}{3}\right)^2$$

$$\geq \sqrt[3]{\sqrt{\frac{a}{b}}\sqrt{\frac{b}{c}}\sqrt{\frac{c}{a}}} = 1.$$

Equality is attained iff it occurs in those two inequalities, that is, iff $\frac{\sqrt{\frac{a}{b}}}{3a+2} = \frac{\sqrt{\frac{b}{c}}}{3b+2} = \frac{\sqrt{\frac{c}{ba}}}{3c+2}$ and $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$. These last identities are true if and only if $a = b = c$, that is, if and only if $a = b = c = \frac{1}{3}$. In this case equality is also obtained in Bergström's inequality. So, the minimum value of $f(a, b, c)$ is 1, and this occurs if and only if $a = b = c = \frac{1}{3}$.

Solution 4 by Arkady Alt, San Jose,CA

Since $\left(\frac{a}{3a+2} - \frac{b}{3b+2}\right)\left(\left(-\frac{1}{a}\right) - \left(-\frac{1}{b}\right)\right) = \frac{2\,(a-b)^2}{ab\,(3b+2)\,(3a+2)} \geq 0$ then triples $\left(\frac{a}{3a+2}, \frac{b}{3b+2}, \frac{c}{3c+2}\right)$, $\left(-\frac{1}{a}, -\frac{1}{b}, -\frac{1}{c}\right)$ are agreed in order and, therefore, by the Rearrangement Inequality $\sum_{cyc} \frac{a}{3a+2} \cdot \left(-\frac{1}{a}\right) \geq \sum_{cyc} \frac{a}{3a+2} \cdot \left(-\frac{1}{b}\right) \iff$

$$\sum_{cyc} \frac{a}{(3a+2)\,b} \geq \sum_{cyc} \frac{a}{3a+2} \cdot \frac{1}{a} = \sum_{cyc} \frac{1}{3a+2}.$$

Also, by Cauchy Inequality $\sum_{cyc} (3a+2) \cdot \sum_{cyc} \frac{1}{3a+2} \geq 9 \iff 9 \cdot \sum_{cyc} \frac{1}{3a+2} \geq 9 \iff$ $\sum_{cyc} \frac{1}{3a+2} \geq 1$.Thus, $f(a,b,c) \geq 1$ and since $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = 1$ we may conclude that $\min f(a,b,c) = 1$.

Solution 5 by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece

Since $c = 1 - a - b$, then we have:

$$f(a,b,c) = \frac{a}{3ab+2b} + \frac{b}{3b(1-a-b)+2(1-a-b)} + \frac{1-a-b}{3(1-a-b)a+2a}.$$

That means that we may assume the function:

$$g(a,b) = \frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1)} + \frac{a+b+-1}{a(3a+3b-5)}.$$

To find the stationary points of $g(a,b)$, work out $\frac{\partial g}{\partial a}$ and $\frac{\partial g}{\partial b}$ and set both to zero . This gives two equations for two unknowns a and b. We may solve these equations for a and b (often there is more than one solution). Let (x,y) be a stationary point. If $g_{aa} > 0$ and $g_{bb} > 0$ at (x,y) then (x,y) is a minimum point . So,

$$\frac{\partial g}{\partial a} = -\frac{(a+b-1)(6a+3b-5)}{a^2(3a+3b-5)^2} - \frac{3a}{b(3a+2)^2} + \frac{b}{(3b+2)(a+b-1)^2} + \frac{1}{3ab+2b} + \frac{1}{a(3a+3b-5)}$$

$$\frac{\partial g}{\partial b} = -\frac{a}{(b^2(3a+2)} + \frac{b(3a+6b-1)}{(3b+2)^2(a+b-1)^2} - \frac{1}{(3b+2)(a+b-1)} + \frac{1}{a(3a+3b-5)} - \frac{3(a+b-1)}{a(3a+3b-5)^2},$$

and for $(a,b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have:

$$\min g(a,b) = \min\left[\frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1)} + \frac{a+b-1}{a(3a+3b-5)}\right] = 1.$$

and for $(a,b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have:

$$\min g(a,b) = \min\left[\frac{a}{3ab+2b} - \frac{b}{(3b+2)(a+b-1} + \frac{a+b-1}{a(3a+3b-5)}\right] = 1.$$

Solution 6 by Albert Stadler, Herrliberg, Switzerland

We will prove that the minimum value equals 1 and the minimum is assumed if and only if $a=b=c=1/3$. To that end we must prove that

$$f(a,b,c) = \frac{a(a+b+c)}{3ab+2b(a+b+c)} + \frac{a(a+b+c)}{3ab+2b(a+b+c)} + \frac{a(a+b+c)}{3ab+2b(a+b+c)} \geq 1.$$

We clear denominators and get the equivalent inequality

$$10\sum_{cycl} a^4b^2 + 24\sum_{cycl} a^3b^3 + 18\sum_{cycl} a^4c^2 + 4\sum_{cycl} a^5c \geq 2\sum_{cycl} a^4bc + 15\sum_{cycl} a^3b^2c + 11\sum_{cycl} a^3bc^2 + 28\sum_{cycl} a^2b^2c^2. \quad (1)$$

By the (weighted)AM-GM inequality,

$$\sum_{cycl} a^4b^2 + \sum_{cycl} a^4c^2 \geq 2\sum_{cycl} a^4bc,$$

$$15\sum_{cycl} a^3b^3 = 15\sum_{cycl}\left(\frac{2}{3}a^3b^3 + \frac{1}{3}c^3a^3\right) \geq 15\sum_{cycl} a^3b^2c,$$

$$11\sum_{cycl} a^4c^2 = 11\sum_{cycl}\left(\frac{2}{3}a^4c^2 + \frac{1}{6}b^4a^2 + \frac{1}{6}c^4b^2\right) \geq 11\sum_{cycl} a^3bc^2,$$

$$9\sum_{cycl} a^4b^2 \geq 27a^2b^2c^2,$$

$$9\sum_{cycl} a^3b^3 \geq 27a^2b^2c^2,$$

$$6\sum_{cycl} a^4c^2 \geq 18a^2b^2c^2,$$

$$4\sum_{cycl} a^5c \geq 12a^2b^2c^2,$$

and (1) follows if we add the last seven inequalities. In all seven inequalities equality holds if and only if $a=b=c$.

Comment by Stanley Rabinowitz of Chelmsford, MA. Problems such as this are easily solvable by computer algebra systems these days. For example; the Mathematica command
$\text{Minimize}\left[\{\text{a}/(3\text{a}*\text{b}+2\text{b})+\text{b}/(3\text{b}*\text{c}+2\text{c})+\text{c}/(3\text{c}*\text{a}+2\text{a}),\ \text{a}>0\ \&\&\ \text{b}>0\ \&\&\ \text{c}>0\ \&\&\ a+b+c=1\},\{a,b,c\}\right]$ responds by saying that the minimum value is 1 and occurs when $a=b=c=\dfrac{1}{3}$.

Also solved by Konul Aliyeva (student), ADA University, Baku, Azerbaijan; Michel Bataille, Rouen, France; Ed Gray, Highland Beach, FL; Tran Hong (student), Cao Lanh School, Dong Thap, Vietnam; Sanong Huayrerai, Rattanakosinsomphothow School, Nakon, Pathom, Thailand; Seyran Ibrahimov, Baku State University, Maasilli, Azerbaijan; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Stanley Rabinowitz of Chelmsford, MA; Neculai Stanciu "George Emil Palade" School, Buză, Romania and Titu Zvonaru, Comănesti, Romania; Daniel Văcaru, Pitesti, Romania; Nicusor Zlota "Traian Vuia Technical College, Focsani, Romania, and the proposer.

5509: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let x, y, z be positive real numbers that add up to one and such that $0 < \dfrac{x}{y}, \dfrac{y}{z}, \dfrac{z}{x} < \dfrac{\pi}{2}$. Prove that

$$\sqrt{x\cos\left(\frac{y}{z}\right)}+\sqrt{y\cos\left(\frac{z}{x}\right)}+\sqrt{z\cos\left(\frac{x}{y}\right)}<\frac{3}{5}\sqrt{5}.$$

Solution 1 by Michel Bataille, Rouen, France

The Cauchy-Schwarz inequality provides

$$\sqrt{x}\sqrt{\cos\left(\frac{y}{z}\right)}+\sqrt{y}\sqrt{\cos\left(\frac{z}{x}\right)}+\sqrt{z}\sqrt{\cos\left(\frac{x}{y}\right)}\le(x+y+z)^{1/2}\left(\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\right)^{1/2}.$$

Since $x+y+z=1$, it follows that the left-hand side L of the proposed inequality satisfies

$$L\le\left(\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\right)^{1/2}.$$

Thus, it suffices to show that

$$\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)<\frac{9}{5}. \qquad (1)$$

Now, Jensen's inequality applied to the cosine function, which is concave on $(0,\frac{\pi}{2})$, yields

$$\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)\le 3\cos\left(\frac{y/z+z/x+x/y}{3}\right). \qquad (2)$$

But we have $1 = \sqrt[3]{\frac{y}{z}\cdot\frac{z}{x}\cdot\frac{x}{y}} \le \frac{y/z+z/x+x/y}{3}$ (by AM-GM) and
$0 < \frac{y/z+z/x+x/y}{3} < \frac{3\cdot\frac{\pi}{2}}{3} = \frac{\pi}{2}$, hence

$$\cos\left(\frac{y/z+z/x+x/y}{3}\right) \le \cos(1)$$

(since the cosine function is decreasing on $(0, \frac{\pi}{2})$).
Then (2) gives $\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right) \le 3\cos(1)$. There just remains to remark that $\cos(1) < 0.6 = \frac{3}{5}$ to obtain the desired inequality (1).

Solution 2 by Tran Hong (student), Cao Lanh School, Dong Thap, Vietnam

$$LHS \overset{BCS}{\le} \sqrt{x+y+z}\sqrt{\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)}$$

$$= \sqrt{1}\cdot\sqrt{\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)} \qquad (1)$$

$$Let\ f(t)=\cos t, t\in\left(0,\frac{\pi}{2}\right) \Rightarrow f''(t) = -\cos t < 0$$

Using Jensen's we have:

$$f\left(\frac{y}{z}\right)+f\left(\frac{z}{x}\right)+f\left(\frac{x}{y}\right) \le 3\cdot f\left(\frac{\frac{y}{z}+\frac{z}{x}+\frac{x}{y}}{3}\right)$$

$$= 3\cos\left(\frac{\frac{y}{z}+\frac{z}{x}+\frac{x}{y}}{3}\right) \le 3\cos(1).$$

$$\Rightarrow \{(1) \le \sqrt{3\cos(1)} \approx 1,2731 < 3\cdot\frac{\sqrt{5}}{5} \approx 1,3416.$$

Solution 3 by David E. Manes, Oneonta, NY

Let $J = \sqrt{x\cos\left(\frac{y}{z}\right)}+\sqrt{y\cos\left(\frac{z}{x}\right)}+\sqrt{z\cos\left(\frac{x}{y}\right)}$. We will show that $J \le \sqrt{3\cos 1} < \frac{3}{5}\sqrt{5}$.
By the Cauchy-Schwarz inequality, one obtains

$$J = \sqrt{x\cos\left(\frac{y}{z}\right)}+\sqrt{y\cos\left(\frac{z}{x}\right)}+\sqrt{z\cos\left(\frac{x}{y}\right)} \le \sqrt{x+y+z}\sqrt{\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)}$$
$$= \sqrt{\cos\left(\frac{y}{z}\right)+\cos\left(\frac{z}{x}\right)+\cos\left(\frac{x}{y}\right)}.$$

At the risk of being redundant, note that

$$J \le \sqrt{\sum_{cyc}\cos\left(\frac{y}{z}\right)} = \sqrt{(x+y+z)\sum_{cyc}\cos\left(\frac{y}{z}\right)}$$
$$= \sqrt{(x+y+z)\cos\left(\frac{y}{z}\right)+(x+y+z)\cos\left(\frac{z}{x}\right)+(x+y+z)\cos\left(\frac{x}{y}\right)}.$$

Since the cosine function is concave on the interval $(0, \pi/2)$, it follows by Jensen's inequality that for each of the following terms in the cyclic sum under the square root sign, we get

$$x\cos\left(\frac{y}{z}\right) + y\cos\left(\frac{z}{x}\right) + z\cos\left(\frac{x}{y}\right) \leq \cos\left(\frac{xy}{z} + \frac{yz}{x} + \frac{xz}{y}\right)$$
$$y\cos\left(\frac{y}{z}\right) + z\cos\left(\frac{z}{x}\right) + x\cos\left(\frac{x}{y}\right) \leq \cos\left(\frac{y^2}{z} + \frac{z^2}{x} + \frac{x^2}{y}\right)$$
$$z\cos\left(\frac{y}{z}\right) + x\cos\left(\frac{z}{x}\right) + y\cos\left(\frac{x}{y}\right) \leq \cos(y+z+x) = \cos 1.$$

Therefore, $J \leq \sqrt{\cos\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) + \cos\left(\frac{y^2}{z} + \frac{z^2}{x} + \frac{x^2}{y}\right) + \cos 1}$. For the first term, $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}$, in parentheses above, observe that using the Arithmetic Mean-Geometric Mean inequality, one obtains

$$\frac{1}{2}\left(\frac{xy}{z} + \frac{yz}{x}\right) \geq \sqrt{\frac{xy^2z}{xz}} = y,$$
$$\frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y}\right) \geq \sqrt{\frac{xyz^2}{xy}} = z,$$
$$\frac{1}{2}\left(\frac{zx}{y} + \frac{xy}{z}\right) \geq \sqrt{\frac{x^2yz}{yz}} = x.$$

Summing the above terms yields

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \geq x + y + z = 1. \quad \mathbf{(1)}$$

Using the Cauchy-Schwarz inequality in the Engel-Titu form for the second term in parentheses in J above, one immediately obtains

$$\frac{y^2}{z} + \frac{z^2}{x} + \frac{x^2}{y} \geq \frac{(y+z+x)^2}{z+x+y} = 1. \quad \mathbf{(2)}$$

Since the cosine function is decreasing on the interval $[0, \pi/2]$ and as a result of inequalities **(1)** and **(2)**, it follows that $\cos\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \leq \cos 1$ and $\cos\left(\frac{y^2}{z} + \frac{z^2}{x} + \frac{x^2}{y}\right) \leq \cos 1$. Therefore,

$$J = \sqrt{x\cos\left(\frac{y}{z}\right)} + \sqrt{y\cos\left(\frac{z}{x}\right)} + \sqrt{z\cos\left(\frac{x}{y}\right)} \leq \sqrt{3\cos 1}.$$

Finally, note that for each of the above steps the inequalities become equalities if and only if $x = y = z = \frac{1}{3}$.

Solution 4 by Daniel Văcaru, Pitesti, Romania

One has

$$\sqrt{x\cos\frac{y}{z}} + \sqrt{y\cos\frac{z}{x}} + \sqrt{z\cos\frac{x}{y}} \leq \underbrace{\sqrt{x+y+z}} \cdot \sqrt{\cos\frac{y}{z} + \cos\frac{z}{x} + \cos\frac{x}{y}} = \sqrt{\cos\frac{y}{z} + \cos\frac{z}{x} + \cos\frac{x}{y}} =$$

$$\sqrt{\sin\left(\frac{\pi}{2}-\frac{y}{z}\right)+\sin\left(\frac{\pi}{2}-\frac{z}{x}\right)+\sin\left(\frac{\pi}{2}-\frac{x}{y}\right)} \overbrace{<} \sqrt{\left(\frac{\pi}{2}-\frac{y}{z}\right)+\left(\frac{\pi}{2}-\frac{z}{x}\right)+\left(\frac{\pi}{2}-\frac{x}{y}\right)}$$

$$= \sqrt{3\frac{\pi}{2}-\left(\frac{y}{z}+\frac{z}{x}+\frac{x}{y}\right)}.$$

The inequality under the brace is true because $\sin x < x$, $\forall x \in \left(0, \frac{\pi}{2}\right)$. On the other hand, one knows that $\frac{y}{z}+\frac{z}{x}+\frac{x}{y} \geq 3$ by the MA-MG inequality. Therefore one has

$$\sqrt{x\cos\frac{y}{z}}+\sqrt{\cos\frac{z}{x}}+\sqrt{\cos\frac{x}{y}} < \sqrt{3\frac{\pi}{2}-3} = \sqrt{3\cdot\left(\frac{\pi}{2}-1\right)} < \sqrt{3\cdot\left(\frac{32}{20}-1\right)} = \sqrt{\frac{3\cdot 12}{20}} = \frac{3}{\sqrt{(5}} = \frac{3}{5}\sqrt{5}.$$

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland and the proposer.

5510: *Proposed by Ovidiu Furdui and Alina Sîntămărian both at the Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\sum_{n=1}^{\infty}\left[4^n\left(\zeta(2n)-1\right)-1\right],$$

where ζ denotes the Riemann zeta function.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

$$\sum_{n=1}^{\infty}(4^n(\zeta(2n)-1)-1) = \sum_{n=1}^{\infty}\left(4^n\left(\sum_{m=2}^{\infty}\frac{1}{m^{2n}}\right)-1\right) = \sum_{n=1}^{\infty}\sum_{m=3}^{\infty}\left(\frac{2}{m}\right)^{2n} =$$

$$= \sum_{m=3}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2}{m}\right)^{2n} = \sum_{m=3}^{\infty}\frac{\left(\frac{2}{m}\right)^2}{1-\left(\frac{2}{m}\right)^2} = \sum_{m=3}^{\infty}\frac{4}{m^2-4} = \sum_{m=3}^{\infty}\left(\frac{1}{m-2}-\frac{1}{m+2}\right) =$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} = \frac{25}{12}.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy

$$\sum_{n=1}^{\infty}[4^n(\zeta(2n)-1)-1]=\sum_{n=1}^{\infty}[4^n\sum_{k=2}^{\infty}\frac{1}{k^{2n}}-1]=\sum_{n=1}^{\infty}\sum_{k=3}^{\infty}\frac{4^n}{k^{2n}}=$$
$$=\sum_{k=3}^{\infty}\sum_{n=1}^{\infty}\frac{4^n}{k^{2n}}=\sum_{k=3}^{\infty}\frac{4}{k^2}\frac{1}{(1-\frac{4}{k^2})}=\sum_{k=3}^{\infty}\frac{4}{k^2-4}=$$
$$=\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k-2}-\frac{1}{k-1}\right]+\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k-1}-\frac{1}{k}\right]+\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k}-\frac{1}{k+1}\right]+$$
$$+\lim_{n\to\infty}\sum_{k=3}^{n}\left[\frac{1}{k+1}-\frac{1}{k+2}\right]=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}$$

Solution 3 by Moti Levy, Rehovot, Israel

Let

$$S:=\sum_{n=1}^{\infty}\left(4^n\left(\zeta\left(2n\right)-1\right)-1\right),\qquad S_N:=\sum_{n=1}^{N}\left(4^n\left(\zeta\left(2n\right)-1\right)-1\right).$$

Then

$$S_N=\sum_{n=1}^{N}\left(\left(2^{2n}\sum_{k=2}^{\infty}\frac{1}{k^{2n}}\right)-1\right)=\left(\sum_{n=1}^{N}\sum_{k=2}^{\infty}\frac{2^{2n}}{k^{2n}}\right)-N$$
$$=\sum_{k=3}^{\infty}\sum_{n=1}^{N}\frac{2^{2n}}{k^{2n}}=\sum_{k=3}^{\infty}\frac{4}{k^2-4}\left(1-\left(\frac{2}{k}\right)^{2N}\right)$$

$$S=\lim_{N\to\infty}S_N=\sum_{k=3}^{\infty}\frac{4}{k^2-4}=\sum_{k=3}^{\infty}\left(\frac{1}{k-2}-\frac{1}{k+2}\right)$$
$$=\sum_{k=1}^{\infty}\frac{1}{k}-\sum_{k=5}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}.$$

Now, as a bonus, let us evaluate parametrized version of the above sum:

$$S\left(t\right):=\sum_{n=1}^{\infty}\left(t^{2n}\left(\zeta\left(2n\right)-1\right)-\frac{t^{2n}}{2^{2n}}\right),\quad S_N\left(t\right):=\sum_{n=1}^{N}\left(t^{2n}\left(\zeta\left(2n\right)-1\right)-\frac{t^{2n}}{2^{2n}}\right)$$

Then

$$S\left(t\right)_N=\sum_{n=1}^{N}\left(\left(t^{2n}\sum_{k=2}^{\infty}\frac{1}{k^{2n}}\right)-\frac{t^{2n}}{2^{2n}}\right)=\left(\sum_{n=1}^{N}\sum_{k=2}^{\infty}\frac{t^{2n}}{k^{2n}}\right)-\sum_{n=1}^{N}\frac{t^{2n}}{2^{2n}}$$
$$=\left(\sum_{k=2}^{\infty}\sum_{n=1}^{N}\frac{t^{2n}}{k^{2n}}\right)-\sum_{n=1}^{N}\frac{t^{2n}}{2^{2n}}=\sum_{k=3}^{\infty}\sum_{n=1}^{N}\frac{t^{2n}}{k^{2n}}=\sum_{k=3}^{\infty}\frac{t^2}{k^2-t^2}\left(1-\left(\frac{t}{k}\right)^{2N}\right)$$

$$S\left(t\right)=\lim_{N\to\infty}S\left(t\right)_N=\sum_{k=3}^{\infty}\frac{t^2}{k^2-t^2}$$

Let us assume that t is not a positive integer and satisfies the inequality $t > -1$, then

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - t^2} = \frac{\psi(t+1) - \psi(-t+1)}{2t},$$

where $\psi(t)$ is the Digamma function.

$$\psi(-z+1) = \psi(z) + \pi \cot(\pi z)$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - t^2} = \frac{1}{2t}\left(\frac{1}{t} - \cot(\pi t)\right)$$

$$\begin{aligned} S(t) &= t^2 \frac{1}{2t}\left(\frac{1}{t} - \pi \cot(\pi t)\right) - \frac{t^2}{1^2 - t^2} - \frac{t^2}{2^2 - t^2} \\ &= \frac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)}. \end{aligned}$$

We summarize our result as follows,

$$S(t) = \begin{cases} \dfrac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)}, & |t| < 3 \\ \lim\limits_{t\to 1} \dfrac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)} = \dfrac{5}{12}, & |t| = 1, \\ \lim\limits_{t\to 2} \dfrac{5t^4 - 15t^2 + 4 - \pi t\left(t^4 - 5t^2 + 4\right)\cot(\pi t)}{2\left(t^4 - 5t^2 + 4\right)} = \dfrac{25}{12}, & |t| = 2. \end{cases}$$

Remark: The function $S(t)$, as defined above, is continuous in the interval $|t| < 3$.

Reference:

Borwein, Jonathan; Bradley, David M.; Crandall, Richard (2000). "*Computational Strategies for the Riemann Zeta Function*". J. Comp. App. Math. 121 (1–2): 247–296.

Solution 4 by Kee-Wai Lau, Hong Kong, China

Denote the sum of the problem by S so that $S = \sum_{n=1}^{\infty}\sum_{k=3}^{\infty}\left(\frac{2}{k}\right)^{2n}$.

Since the summands are positive, so interchanging the order of summation, we have

$$S = \sum_{k=3}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2}{k}\right)^{2n} = 4\sum_{k=3}^{\infty}\frac{1}{k^2 - 4}.$$

For any integer $M \geq 3$, we have

$$4\sum_{k=3}^{\infty}\frac{1}{k^2 - 4} = \sum_{k=3}^{M}\left(\frac{1}{k-2} - \frac{1}{k+2}\right) = \frac{25}{12} - \sum_{k=M-1}^{M+2}\frac{1}{k}.$$

It follows that $S = \dfrac{25}{12}$.

Comment by Editor : **Ed Gray of Highland Beach, FL** wrote: "I didn't know any recursive formula that would help, so I did the sum by brute force, computing the sum of the first 10 terms, getting a result of 2.0828.....This aroused my curiosity, so I went to Wolfram-alpha and sought the sum for a great number of terms, like 100 and 300. It became clear that the answer is $2.0833333333\cdots$ forever. Converting this to a fraction, we get a beautiful answer of 25/24". He continued on saying that he did not actually solve the problem. This is being mentioned here as a very useful heuristic for getting a feel for the problem, and as a caveat that there are an infinite number of different ways to express a closed form representation for a specific decimal.

Also solved by Michel Bataille, Rouen, France; Bruno Salgueiro Fanego, Viveiro, Spain; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece, and the proposer.

Mea Culpa

Mary Wagner-Krankel of St. Mary's University in San Antonio, TX should have been credited with having solved problem 5500.